ON COMMUTATIVE BE-ALGEBRAS

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Abstract. In this paper we investigate the relationship between BE-algebras, implicative algebras, and J-algebras. Moreover, we define commutative BE-algebras and state that these algebras are equivalent to the commutative dual BCK-algebras.

1. Introduction

In 1967 J. C. Abbot introduced in [1] the concept of implication algebras as algebras connected with a propositional calculus. In [5] K. Iséki introduced a wide class of abstract algebras: BCK-algebras. Recently, R. A. Borzooei and S. Khosravi Shoar ([2]) showed that the implication algebras are equivalent to the dual implicative BCK-algebras. W. H. Cornish ([4]) introduced the condition (J) and proved the BCK-algebras satisfying (J) form a variety. In [7], as a generalization of a BCK-algebra, H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra.

In this paper we show that any implication algebra is a BE-algebra and that every BE-algebra satisfies (J). Moreover, we define commutative BE-algebras and state that these algebras are equivalent to the commutative dual BCK-algebras.

2. Preliminaries

Definition 2.1. ([7]) An algebra \((X; *, 1)\) of type \((2, 0)\) is called a BE-algebra if for all \(x, y, z \in X\) the following identities hold:

\[
\begin{align*}
(BE1) & \quad x * x = 1, \\
(BE2) & \quad x * 1 = 1, \\
(BE3) & \quad 1 * x = x, \\
(BE4) & \quad x * (y * z) = y * (x * z). 
\end{align*}
\]

Lemma 2.2. ([7]) If \((X; *, 1)\) is a BE-algebra, then \(x * (y * x) = 1\) for any \(x, y \in X\).

Definition 2.3. ([8]) A dual BCK-algebra is an algebra \((X; *, 1)\) of type \((2, 0)\) satisfying \((BE1), (BE2),\) and the following axioms:

\[
\begin{align*}
(dBCK1) & \quad x * y = y * x = 1 \implies x = y, \\
(dBCK2) & \quad (x * y) * ((y * z) * (x * z)) = 1, \\
(dBCK3) & \quad x * ((x * y) * y) = 1. 
\end{align*}
\]

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Lemma 2.4. ([8], Theorem 2.5) Let $(X; *, 1)$ be a dual BCK-algebra and $x, y, z \in X$. Then:
(a) $x * (y * z) = y * (x * z)$,
(b) $1 * x = x$.

From Lemma 2.4 we have

Proposition 2.5. Any dual BCK-algebra is a BE-algebra.

Example 2.6. Let $\mathbb{N}$ be the set of all natural numbers and $*$ be the binary operation on $\mathbb{N}$ defined by

$x * y = \begin{cases} y & \text{if } x = 1 \\ 1 & \text{if } x \neq 1. \end{cases}$

It is easy to see that $(\mathbb{N}; *, 1)$ is a BE-algebra, but it is not a dual BCK-algebra.

Definition 2.7. ([1]) An algebra $(X; *)$ of type (2) is called an implication algebra if for all $x, y, z \in X$ the following identities hold:
(I1) $(x * y) * x = x$,
(I2) $(x * y) * y = (y * x) * x$,
(I3) $x * (y * z) = y * (x * z)$.

In any implication algebra $(X; *)$, $x * x = y * y$ for all $x, y \in X$. This was proved by W. Y. Chen and J. S. Oliveira [3]. Let 1 stand for the constant $x * x$. R. A. Borzooei and S. Khosravi Shoar proved the following result:

Proposition 2.8. ([2]) If $(X; *)$ is an implication algebra, then $(X; *, 1)$ is a dual BCK-algebra.

Propositions 2.8 and 2.5 give

Proposition 2.9. Any implication algebra is a BE-algebra.

Definition 2.10. ([6]) An algebra $(X; *)$ consisting of a set $X$ with a binary operation $*$ on $X$ is said to be a J-algebra if
(J) $x * (x * (y * (y * x))) = y * (y * (x * (x * y)))$
for all $x, y \in X$.

Proposition 2.11. Let $(X; *, 1)$ be a BE-algebra. Then $(X; *)$ is a J-algebra.

Proof. Let $x, y \in X$. By (BE4), Lemma 2.2, and (BE2) we have

$x * (x * (y * (y * x))) = x * (y * (x * (y * x))) = x * (y * 1) = x * 1 = 1.$

Similarly,

$y * (y * (x * (x * y))) = y * (x * (y * (x * y))) = y * (x * 1) = y * 1 = 1.$

Hence (J) holds, and therefore $X$ is a J-algebra.
3. Commutative BE-algebras

**Definition 3.1.** Let \((X; *, 1)\) be a BE-algebra or a dual BCK-algebra. We say that \(X\) is commutative if
\[
(C) \quad (x * y) * y = (y * x) * x
\]
for all \(x, y \in X\).

**Example 3.2.** Let \(N_0 = \mathbb{N} \cup \{0\}\) and let \(*) be the binary operation of \(N_0\) defined by
\[
x * y = \begin{cases} 
0 & \text{if } x \geq y \\
y - x & \text{if } y > x.
\end{cases}
\]
Observe that \((N_0; *, 0)\) is a commutative BE-algebra. Obviously, \(x * x = 0, x * 0 = 0,\) and \(0 * x = x\) for all \(x \in N_0\). Thus \((BE1)–(BE3)\) hold. Let \(x, y, z \in N_0\). To prove \((BE4)\) we consider two cases.

Case 1: \(x + y < z\).
Then \(x < z\) and \(y < z\). Hence \(x * z = z - x\) and \(y * z = z - y\). Therefore
\[
x * (y * z) = x * (z - y) = z - y - x = (z - x) - y = y * (z - x) = y * (x * z).
\]

Case 2: \(x + y \geq z\).
Then \(x \geq z - y \geq y * z\). From this we obtain \(x * (y * z) = 0\). Similarly, since \(y \geq z - x \geq x * z\), we conclude that \(y * (x * z) = 0\). Consequently, \(x * (y * z) = y * (x * z)\). Thus \((N_0; *, 0)\) is a BE-algebra.

Now we shall prove that \((N_0; *, 0)\) is commutative. Without loss of generality we can assume that \(x \geq y\). Then \((x * y) * y = 0 * y = y\) and \((y * x) * x = (x - y) * x = x - (x - y) = y\). Hence \((x * y) * y = (y * x) * x\) and we see that \((N_0; *, 0)\) is a commutative BE-algebra.

**Proposition 3.3.** If \((X; *, 1)\) is a commutative BE-algebra, then for all \(x, y \in X\),
\[
x * y = 1 \text{ and } y * x = 1 \text{ imply } x = y.
\]

**Proof.** Let \(x, y \in X\) and suppose that \(x * y = y * x = 1\). Then
\[
x = 1 * x = (y * x) * x = (x * y) * y = 1 * y = y.
\]

**Theorem 3.4.** If \((X; *, 1)\) is a commutative BE-algebra, then \((X; *, 1)\) is a dual BCK-algebra.

**Proof.** Proposition 3.3 yields \((dBCK1)\). Now let \(x, y, z \in X\). Applying \((BE4)\) and \((C)\) we have
\[
(y * z) * (x * z) = x * [(y * z) * z] = x * [(z * y) * y] = (z * y) * (x * y).
\]
Hence
\[
(x * y) * [(y * z) * (x * z)] = (x * y) * [(z * y) * (x * y)].
\]
Lemma 2.2 now shows that \((x * y) * [(y * z) * (x * z)] = 1\), and therefore \((dBCK2)\) holds. Moreover, by \((BE4)\) and \((BE1)\), \(x * ((x * y) * y) = (x * y) * (x * y) = 1\). From this we have \((dBCK3)\), and consequently, \(X\) is a dual BCK-algebra.

By Proposition 2.5 and Theorem 3.4 we have

**Corollary 3.5.** \((X; *, 1)\) is a commutative BE-algebra if and only if it is a commutative dual BCK-algebra.
Definition 3.6. Let \((X; *, 1)\) be a BE-algebra. We define the binary operation " + " on \(X\) as the following: for any \(x, y \in X\)
\[
x + y = (x * y) * y.
\]
Clearly, \(X\) is a commutative BE-algebra if and only if \(x + y = y + x\) for all \(x, y \in X\).

Lemma 3.7. Let \((X; *, 1)\) be a commutative BE-algebra. Then for all \(x, y, z \in X\):
\begin{align*}
(a) & \quad x * (x + y) = 1, \\
(b) & \quad x * y = y * z = 1 \implies x * z = 1, \\
(c) & \quad x * y = 1 \implies (x + z) * (y + z) = 1, \\
(d) & \quad x * z = y * z = 1 \implies (x + y) * z = 1.
\end{align*}

Proof. (a) By Theorem 3.4, \(X\) is a dual BCK-algebra. From (dBCK3) we obtain (a).
(b) Applying (dBCK2) and Lemma 2.4 (b) we have (b).
(c) To prove (c), let \(x * y = 1\). From (dBCK2) we deduce that \((y * z) * (x * z) = 1\). Again using (dBCK2) we get \([(x * z) * z] * [(y * z) * z] = 1\), i.e. \((x + z) * (y + z) = 1\).
(d) To prove (d), let \(x * y = y * z = 1\). From (c) we conclude that \((x + y) * (y + z) = 1\) and \((y + z) * (z + z) = 1\). By (b), \((x + y) * (z + z) = 1\), and hence \((x + y) * z = 1\).

Proposition 3.8. If \((X; *, 1)\) is a commutative BE-algebra, then \((X; +)\) is a semilattice.

Proof. Obviously \(x + x = x\) and \(x + y = y + x\) for all \(x, y \in X\). We will now prove that + is associative. Let \(x, y, z \in X\). From Lemma 3.7 (a) we have \(x * (x + y) = 1\) and \((x + y) * [(x + y) + z] = 1\). Therefore
\[
(1) \quad x * [(x + y) + z] = 1.
\]
Since \(y * (x + y) = 1\), Lemma 3.7 (c) shows that
\[
(2) \quad (y + z) * [(x + y) + z] = 1.
\]
By Lemma 3.7 (d), from (1) and (2) we obtain
\[
(3) \quad [(x + (y + z)] * [(x + y) + z] = 1.
\]
Similarly,
\[
(4) \quad [(x + y) + z] * [x + (y + z)] = 1.
\]
From (3) and (4) it follows by (dBCK1) that \((x + y) + z = x + (y + z)\).

References


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