COMPARISON OF WHITTLE TYPE PORTMANTEAU TESTS

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ABSTRACT. For an ARMA adequacy test, Box and Pierce (1970) proposed a portman-
teau test $T_{BP}$. However, because the accuracy of $T_{BP}$ by $\chi^2$-approximation is not good,
various modifications of $T_{BP}$ have been introduced by many authors. Taniguchi and
Amano (2008) proposed an important portmanteau test $T_{WLR}$ of natural Whittle type
which is always asymptotically $\chi^2$ distributed under the null hypothesis that ARMA
model is adequate. This paper compares $T_{WLR}$ with another famous portmanteau tests
Ljung-Box’s $T_{LB}$, Li-McLeod’s $T_{LM}$ and Monti’s $T_{MN}$ and proves its accuracy by sim-
ulation. Empirical powers of those portmanteau tests are also compared numerically.

1. Introduction

One of the most important stages of building a model in time series
is to verify the adequacy of a fitted model. In particular, sample residual autocorrelations
are usually used. For ARMA adequacy test, Box and Pierce (1970) proposed a test statistic
$T_{BP}$ which is the squared sum of $m$ sample autocorrelations of the estimated residual pro-
cess of ARMA(p,q). Under the null hypothesis that the ARMA(p,q) model is adequate, it is
suggested that $T_{BP}$ is approximately distributed as $\chi^2_{m-p-q}$. However, Davies et al. (1977)
claimed that the $\chi^2_{m-p-q}$-approximation is not adequate and Ljung and Box (1978) and
Li-McLeod (1981) proposed test statistics $T_{LB}$ and $T_{LM}$ as a modification of $T_{BP}$. Recently
Monti (1994) proposed a portmanteau test $T_{MN}$ using the residual partial autocorrelations.
Various modified versions of $T_{BP}$ (see Li (2004)) have been proposed. Under the null hy-
pothesis that ARMA(p,q) is adequate, these test statistics are much closer to chi-square
distribution than $T_{BP}$.

The test statistic $T_{BP}$ and modifications of $T_{BP}$ are called the portmanteau test and have
been widely used. Taniguchi and Amano (2008) proved that $T_{BP}$ does not converge to
$\chi^2_{m-p-q}$ distribution for fixed $m$ and for ARMA adequacy test, proposed a portmanteau test
of natural Whittle type $T_{WLR}$ and showed that $T_{WLR}$ is always asymptotically chi-square
distributed. This paper compares $T_{WLR}$ with another famous portmanteau test statistics
$T_{BP}$, $T_{LB}$ and $T_{MN}$ and we observe that $T_{WLR}$ behaves well numerically.

This paper is organized as follows. Section 2 describes the construction of $T_{WLR}$ and
its asymptotics. In Section 3, we compare the means and variances of $T_{WLR}$ with those of
other portmanteau tests $T_{BP}$, $T_{LB}$ and $T_{MN}$ by simulation. Then the empirical significance
levels and the empirical powers under contiguous alternatives are compared numerically.
2. Asymptotics of $T_{WLR}$

A stationary process $\{X_i\}$ is assumed to satisfy

$$
\sum_{j=0}^p \alpha_j X_{t-j} = \sum_{j=0}^q \beta_j u_{t-j}, \quad (\alpha_0 = \beta_0 = 1, \ \alpha_p \neq 0, \ \beta_q \neq 0),
$$

(2.1)

where $\{u_t\}$ is an $m$-dimensional sequence with autocovariance $\{\theta_{2,j}\}$ ($\theta_{2,0} = 1, \ \theta_{2,-j} = \theta_{2,j}$) and the innovation process of $\{u_t\}$ is identically distributed with mean 0, variance $\sigma_u^2$ and fourth-order cumulant $\kappa_4$. Let $\alpha(z) = \sum_{j=0}^p \alpha_j z^j$ and $\beta(z) = \sum_{j=0}^q \beta_j z^j$, and they are assumed to satisfy $\alpha(z) \neq 0$ and $\beta(z) \neq 0$ on $D = \{z \in \mathbb{C} : |z| \leq 1\}$ and the equations $\alpha(z) = 0$ and $\beta(z) = 0$ have no common roots. We define $\theta_1 = (\theta_{1,1}, \ldots, \theta_{1,p+q})' \equiv (\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q)'$, $\theta_2 = (\theta_{2,1}, \ldots, \theta_{2,m})'$ and $\theta = (\theta_1', \theta_2')'$, then the spectral density of $\{X_t\}$ is

$$
f_0(\lambda) \equiv f(\theta_1, \theta_2)(\lambda) = \frac{|\sum_{j=0}^q \beta_j e^{ij\lambda}|^2}{|\sum_{j=0}^p \alpha_j e^{ij\lambda}|^2} \cdot \frac{\sigma_u^2}{2\pi} \left\{ \sum_{j=-m}^m \theta_{2,j} e^{-ij\lambda} \right\}
$$

For the construction of a portmanteau test, let $\hat{X}_n = (X_1, \ldots, X_n)'$ be an observed stretch from (1), and write the periodogram as

$$
I_n(\lambda) = \frac{1}{2\pi n} \left| \sum_{t=1}^n X_t e^{it\lambda} \right|^2, \quad \lambda \in [-\pi, \pi].
$$

(2.2)

By use of Whittle likelihood

$$
D(f_0, I_n) = -\frac{1}{4\pi} \int_{-\pi}^\pi \log f_0(\lambda) + \frac{I_n(\lambda)}{f_0(\lambda)} d\lambda
$$

(2.3)

estimators for $(\theta_1', \theta_2')$ are given by

$$
\hat{\theta}_1 = \arg \max_{\theta_1} D(f(\theta_1, 0), I_n), \quad (\hat{\theta}_1, \hat{\theta}_2) = \arg \max_{(\theta_1, \theta_2)} D(f(\theta_1, \theta_2), I_n),
$$

(2.4)

where 0 in (4) is the $m$-dimensional zero vector. As an adequacy test for ARMA(p,q) model, a portmanteau test of natural Whittle likelihood type

$$
T_{WLR} \equiv 2n[D(f(\hat{\theta}_1, \hat{\theta}_2), I_n) - D(f(\theta_1, 0), I_n)]
$$

(2.5)

was proposed in Taniguchi and Amano (2008).

The following lemmas are due to Taniguchi and Amano (2008).

**Lemma 2.1.** Write $F \equiv \frac{1}{4\pi} \int_{-\pi}^\pi \frac{\partial}{\partial \sigma^2} \log f_0(\lambda) \frac{\partial}{\partial \sigma^2} \log f_0(\lambda) d\lambda = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$. Suppose that $F$ is positive definite. If ARMA(p,q) model is adequate, then for any fixed $m = \dim \theta_2$, it holds that

$$
T_{WLR} \to \chi_m^2, \quad \text{in distribution as } n \to \infty.
$$

(2.6)

**Lemma 2.2.** Under $A_G^{(m)} : \theta_2 = \frac{1}{\sqrt{n}} h$, where $h$ is a fixed $m$-dimensional vector, the following holds

$$
T_{WLR} \to \chi_m^2(h'F_{22,1}h) \quad \text{in distribution as } n \to \infty
$$

(2.7)

where $F_{22,1} = F_{22} - F_{21}F_{11}^{-1}F_{12}$, and $\chi_m^2(h'F_{22,1}h)$ is a noncentral chi-square random variable with $m$ degrees of freedom and noncentrality parameter $h'F_{22,1}h$. 382
3. **Numerical study**

In this section, we give a comparison of the test statistic $T_{WLR}$ with another portmanteau tests

$$T_{LB} = n(n + 2) \sum_{k=1}^{m} \frac{\hat{r}_k^2}{n-k},$$

(3.1)

$$T_{LM} = \frac{m(m+1)}{2n} + n \sum_{k=1}^{m} \hat{r}_k^2$$

(3.2)

and

$$T_{MN} = n(n + 2) \sum_{k=1}^{m} \frac{\hat{\pi}_k^2}{n-k},$$

(3.3)

by simulation. Here, $\hat{r}_k$ and $\hat{\pi}_k$ are the $k$th sample autocorrelations and sample partial autocorrelations of the estimated residual process of ARMA(p,q) model, respectively. Under the null hypothesis that ARMA(p,q) is adequate, these portmanteau tests $T_{LB}$, $T_{LM}$ and $T_{MN}$ are supposed to be approximated by $\chi^2_{m-p-q}$-distribution.

In Example 3.1, the empirical means and variances of $T_{WLR}$ for $m = 1$ are compared with those of $T_{LB}$, $T_{LM}$ and $T_{MN}$ for $m = 2$ under null hypothesis. In Example 3.2, we compare the significance levels of $T_{WLR}$ for $m = 1$ with those of $T_{LB}$, $T_{LM}$ and $T_{MN}$ for $m = 2$, 20 under null hypothesis. Then we can observe that the test statistic $T_{WLR}$ is more accurate than $T_{LB}$, $T_{LM}$ and $T_{MN}$. In Example 3.3, local powers of the test $T_{WLR}$ for $m = 1$ are compared with those of $T_{LB}$, $T_{LM}$ and $T_{MN}$ for $m = 10$, 20 under local alternative and we can see that our test $T_{WLR}$ is more powerful than $T_{LB}$, $T_{LM}$ and $T_{MN}$.

**Example 3.1.** Let $\{X_t\}$ be the AR(1) process

$$X_t + \alpha X_{t-1} = u_t,$$

(3.4)

where $u_t$’s are independent and identically distributed as $N(0,1)$. For (4), we compare the empirical means and variances of $T_{WLR}$ for $m = 1$ with those of $T_{LB}$, $T_{LM}$ and $T_{MN}$ for $m = 2$, respectively. The parameter values are chosen as $0.85 \leq \alpha \leq 0.99$. The empirical means and variances are calculated based on length of observations $n = 200$ and 1000 times simulation.

In Figure 1, the empirical means of $T_{WLR}$ for $m = 1$ and $T_{LB}$, $T_{LM}$ and $T_{MN}$ for $m = 2$ ($0.85 \leq \alpha \leq 0.99$) are plotted.

In Figure 2, the empirical variances of $T_{WLR}$ for $m = 1$ and $T_{LB}$, $T_{LM}$ and $T_{MN}$ for $m = 2$ ($0.85 \leq \alpha \leq 0.99$) are plotted.

From Figure 1, the empirical means of $T_{WLR}$ for $m = 1$ are closer to 1 than those of $T_{LB}$, $T_{LM}$ and $T_{MN}$ for $m = 2$. From Figure 2, the empirical variances of $T_{WLR}$ for $m = 1$ are closer to 2 than those of $T_{LB}$, $T_{LM}$ and $T_{MN}$ for $m = 2$. Due to Lemma 2.1, $T_{WLR}$ for $m = 1$ is approximated by $\chi^2_1$-distribution and $T_{LB}$, $T_{LM}$ and $T_{MN}$ for $m = 2$ is supposed to be approximated by $\chi^2_1$-distribution. Hence Figures 1 and 2 imply $T_{WLR}$ is more accurate than another portmanteau tests $T_{LB}$, $T_{LM}$ and $T_{MN}$.
Figure 1: The means of $T_{WLR}$, $T_{LB}$, $T_{LM}$ and $T_{MN}$ in Example 3.1 ($0.85 \leq \alpha \leq 0.99$)

Figure 2: The variances of $T_{WLR}$, $T_{LB}$, $T_{LM}$ and $T_{MN}$ in Example 3.1 ($0.85 \leq \alpha \leq 0.99$)
Example 3.2. For (4), we compare the empirical significance levels of \( T_{WLR} \) for \( m = 1 \) with those of \( T_{LB}, T_{LM} \) and \( T_{MN} \) for \( m = 2, 20 \), respectively. The parameter values are chosen as \( 0.85 \leq \alpha \leq 0.99 \). The empirical significance levels are calculated based on length of observations \( n = 200 \) and 1000 times simulations.

In Figure 3, the fractions of times that \( T_{WLR} \) for \( m = 1 \) and \( T_{LB}, T_{LM} \) and \( T_{MN} \) for \( m = 2 \) \((0.85 \leq \alpha \leq 0.99)\) exceed the critical values of \( \chi^2_2 \)-distribution for nominal level 5% are plotted.

In Figure 4, the fractions of times that \( T_{WLR} \) for \( m = 1 \) and \( T_{LB}, T_{LM} \) and \( T_{MN} \) for \( m = 20 \) \((0.85 \leq \alpha \leq 0.99)\) exceed the critical values of \( \chi^2_1 \) and \( \chi^2_{19} \)-distribution for nominal level 5% are plotted.

Figure 3: The significance levels with nominal size 5% of \( T_{WLR}, T_{LB} \) \((m = 2)\), \( T_{LM} \) \((m = 2)\) and \( T_{MN} \) \((m = 2)\) in Example 3.2 \((0.85 \leq \alpha \leq 0.99)\)

Due to Lemma 2.1, \( T_{WLR} \) for \( m = 1 \) is approximated by \( \chi^2_1 \)-distribution and \( T_{LB}, T_{LM} \) and \( T_{MN} \) for \( m \) are supposed to be approximated by \( \chi^2_{m-1} \)-distribution. From Figures 3 and 4, it is seen that \( T_{WLR} \) is closer to its asymptotic distribution than \( T_{LB}, T_{ML} \) and \( T_{MN} \).

Example 3.3. Let \( \{X_t\} \) be the AR(1) process

\[
X_t + \alpha X_{t-1} = u_t
\]  
(3.5)

where \( \{u_t\} \) is the MR(1) with the mean 0, the variance 1 and the autocovariance function \( \{\frac{H}{\sqrt{n}}\} \) where \( H = \frac{3}{\sqrt{p^2 + 1}} = \frac{3}{\alpha} \). If \( T_{WLR} \) for \( m = 1 \) exceeds the 95% point of \( \chi^2_1 \), we reject the null hypothesis. \( T_{WLR} \) for \( m = 1 \) is calculated with length of observations \( n = 200 \). By use of 1000 times simulation, we give the frequency that the test rejects the hypothesis. If the \( T_{LB} \) for \( m \) exceeds the 95% point of \( \chi^2_{m-1} \), we reject the null hypothesis. \( T_{LB} \) for \( m \) is calculated with length of observations \( n = 200 \). By use of 1000 times simulations, we give
Figure 4: The significance levels with nominal size 5% of $T_{WLR}$, $T_{LB}$ ($m = 20$), $T_{LM}$ ($m = 20$), $T_{MN}$ ($m = 20$) in Example 3.2 ($0.85 \leq \alpha \leq 0.99$)

the frequency that the test rejects the hypothesis. Also, we give empirical powers of $T_{LM}$ and $T_{MN}$ for $m$ similar.

In Figure 5, the empirical powers for a 5%-level test of $T_{WLR}$ for $m = 1$ and those of $T_{LB}$, $T_{LM}$ and $T_{MN}$ for $m = 10$ ($0.45 \leq \alpha \leq 0.99$) are plotted.

In Figure 6, the empirical powers for a 5%-level test of $T_{WLR}$ for $m = 1$ and those of $T_{LB}$, $T_{LM}$ and $T_{MN}$ for $m = 20$ ($0.45 \leq \alpha \leq 0.99$) are plotted.

From Figures 5 and 6, our test statistic $T_{WLR}$ is more powerful than $T_{LB}$, $T_{LM}$ and $T_{MN}$. Simulation results imply that $T_{WLR}$ is closer to theoretic $\chi^2$-distribution than another famous portmanteau tests $T_{LB}$, $T_{LM}$ and $T_{MN}$ under null hypothesis that ARMA(p,q) model is adequate. It is implied that under contiguous alternative hypothesis, the ability of $T_{WLR}$ to detect model misspecification is higher than that of another famous portmanteau tests $T_{LB}$, $T_{LM}$ and $T_{MN}$ by simulation.
Figure 5: The empirical powers with level test 5% of $T_{WLR}$, $T_{LB}$ ($m = 10$), $T_{LM}$ ($m = 10$), $T_{MN}$ ($m = 10$) in Example 3.3 ($0.45 \leq \alpha \leq 0.99$)

Figure 6: The empirical powers with level test 5% of $T_{WLR}$, $T_{LB}$ ($m = 20$), $T_{LM}$ ($m = 20$), $T_{MN}$ ($m = 20$) in Example 3.3 ($0.45 \leq \alpha \leq 0.99$)
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