MEAN ERGODIC THEOREMS FOR A SEQUENCE OF NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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Abstract. Let $C$ be a closed convex subset of a Hilbert space and $\{T_n\}$ a sequence of nonexpansive self-mappings of $C$. Then we consider the following iterative sequence $\{z_n\}$: $x_1 = x \in C$, $x_{n+1} = T_n x_n$, and $z_n = \frac{1}{n} \sum_{k=1}^{n} x_k$ for $n \in \mathbb{N}$. In this paper, we obtain a weak convergence theorem for such a sequence $\{z_n\}$. Using our result, we get a nonlinear ergodic theorem which is a generalization of Baillon [2]. Further we apply our result to the problem of finding a common fixed point of a countable family of nonexpansive mappings.

1. Introduction

Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Then a mapping $T: C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of $T$. In 1975, Baillon [2] proved the first nonlinear ergodic theorem: Define

$$z_n = \frac{1}{n} \sum_{k=1}^{n} T^{k-1} x$$

for every $n \in \mathbb{N}$ and $x \in C$ and suppose that $F(T)$ is nonempty. Then the sequence $\{z_n\}$ converges weakly to some element of $F(T)$. It is known that many results concerning the mean ergodic theorem for a nonlinear mapping have been obtained, for example, [2], [11], [12], [5], [7], [8], [19]; see also [6], [3], [18], [1], [10], [9], and the references therein. Reich [13] also proved the following weak convergence theorem; see [16] for a simple proof.

**Theorem 1.1** (Reich [13]). Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $T$ a nonexpansive self-mapping of $C$. Suppose that $F(T)$ is nonempty. Let $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n$$

for $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,1)$ satisfies $\sum_{n=1}^{\infty} \alpha_n (1 - \alpha_n) = \infty$. Then $\{x_n\}$ converges weakly to $z \in F(T)$.

Reich [13] really proved such a theorem in a uniformly convex Banach space whose norm is Fréchet differentiable. Motivated by Baillon [2] and Reich [13], we consider the following
iterative sequence \(\{z_n\}\): \(x_1 = x \in C\) and

\[
\begin{align*}
x_{n+1} &= T_n x_n, \\
z_n &= \frac{1}{n} \sum_{k=1}^{n} x_k
\end{align*}
\]

for \(n \in \mathbb{N}\), where \(\{T_n\}\) is a sequence of nonexpansive self-mappings of \(C\).

In this paper, we establish a weak convergence theorem for such a sequence \(\{z_n\}\) generated by (1.1). Using our result, we obtain a nonlinear ergodic theorem for a nonexpansive mapping which is a generalization of Baillon [2]. Further we apply our theorem to the problem of finding a common fixed point of a countable family of nonexpansive mappings in a Hilbert space.

2. Preliminaries

Throughout this paper, \(H\) denotes a real Hilbert space with inner product \((\cdot, \cdot)\) and norm \(\|\cdot\|\). Let \(\{x_n\}\) be a sequence in \(H\) and \(x \in H\). Weak convergence of \(\{x_n\}\) to \(x\) is denoted by \(x_n \rightharpoonup x\) and strong convergence by \(x_n \to x\).

Let \(C\) be a nonempty closed convex subset of \(H\) and \(T\) a mapping of \(C\) into \(H\). A mapping \(T\) is said to be nonexpansive if \(\|Tx - Ty\| \leq \|x - y\|\) for all \(x, y \in C\). The set of fixed points of \(T\) is denoted by \(F(T)\). It is known that \(F(T)\) is closed and convex if \(T\) is nonexpansive. For each \(x \in H\), there exists a unique point \(z \in C\) such that

\[
\|x - z\| = \min \{\|x - y\| : y \in C\}.
\]

Such a point \(z\) is denoted by \(Px\) and \(P\) is called the metric projection of \(H\) onto \(C\). It is known that

\[
\langle x - Px, Px - y \rangle \geq 0
\]

for all \(x \in H\) and \(y \in C\); see [15] for more details.

To prove our results, we need the following lemmas.

Lemma 2.1 (Takahashi-Toyoda [17]). Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H\), \(P\) the metric projection of \(H\) onto \(C\), and \(\{x_n\}\) a sequence in \(H\). If \(\|x_{n+1} - u\| \leq \|x_n - u\|\) for all \(u \in C\) and \(n \in \mathbb{N}\), then \(\{Px_n\}\) converges strongly.

Lemma 2.2 (Bruck [4]). Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(E\). Let \(\{S_k\}\) be a sequence of nonexpansive mappings of \(C\) into \(H\) and \(\{\beta_k\}\) a sequence of positive real numbers such that \(\sum_{k=1}^{\infty} \beta_k = 1\). If \(\bigcap_{k=1}^{\infty} F(S_k)\) is nonempty, then the mapping \(T = \sum_{k=1}^{\infty} \beta_k S_k\) is well-defined and \(F(T) = \bigcap_{k=1}^{\infty} F(S_k)\).

Bruck [4] showed this assertion for a strictly convex Banach space.

3. Mean ergodic theorems

Using the technique in [15, p.59], we obtain the following:

Lemma 3.1. Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H\). Let \(\{x_n\}\) be a sequence in \(H\), \(\{z_n\}\) a sequence in \(H\) defined by

\[
z_n = \frac{1}{n} \sum_{k=1}^{n} x_k
\]

for \(n \in \mathbb{N}\), \(\{\alpha_n\}\) a sequence of real numbers such that \(\alpha_n \to 0\), and \(T\) a mapping of \(C\) into \(H\). Suppose that there exists \(z \in C\) such that

\[
\alpha_n \leq \|x_n - z\|^2 - \|x_{n+1} - Tz\|^2
\]
for every $n \in \mathbb{N}$ and a subsequence $\{z_n\}$ of $\{z_n\}$ converges weakly to $z$. Then $z$ is a fixed point of $T$.

**Proof.** For all $k \in \mathbb{N}$ we have

$$
\alpha_k \leq \|x_k - z\|^2 - \|x_{k+1} - Tz\|^2
$$

$$
= \|x_k - Tz + Tz - z\|^2 - \|x_{k+1} - Tz\|^2
$$

$$
= \|x_k - Tz\|^2 - \|x_{k+1} - Tz\|^2 + 2 \langle x_k - Tz, Tz - z \rangle + \|Tz - z\|^2.
$$

Summing these inequalities from $k = 1$ to $n$ and dividing by $n$, we get

$$
\frac{1}{n} \sum_{k=1}^{n} \alpha_k \leq \frac{1}{n} (\|x_1 - Tz\|^2 - \|x_{n+1} - Tz\|^2) + 2 \langle z_n - Tz, Tz - z \rangle + \|Tz - z\|^2
$$

$$
\leq \frac{1}{n} \|x_1 - Tz\|^2 + 2 \langle z_n - Tz, Tz - z \rangle + \|Tz - z\|^2.
$$

Further, replacing $n$ by $n_i$, we obtain

$$
\frac{1}{n_i} \sum_{k=1}^{n_i} \alpha_k \leq \frac{1}{n_i} \|x_1 - Tz\|^2 + 2 \langle z_n - Tz, Tz - z \rangle + \|Tz - z\|^2.
$$

Since $z_{n_i} \rightharpoonup z$ and $1/n_i \sum_{k=1}^{n_i} \alpha_k \to 0$, we obtain

$$
0 \leq 2 \langle z - Tz, Tz - z \rangle + \|Tz - z\|^2 = -\|Tz - z\|^2
$$

and hence $Tz = z$. \(\square\)

We prove the main result of this paper.

**Theorem 3.2.** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\{T_n\}$ be a sequence of nonexpansive self-mappings of $C$. Let $\{x_n\}$ and $\{z_n\}$ be two sequences in $C$ defined by $x_1 = x \in C$ and

$$
\begin{cases}
   x_{n+1} = T_n x_n, \\
   z_n = \frac{1}{n} \sum_{k=1}^{n} x_k
\end{cases}
$$

for $n \in \mathbb{N}$. Suppose that $\{T_n\}$ is pointwise convergent and $T$ denotes the pointwise limit of $\{T_n\}$, that is, $Ty = \lim_{n \to \infty} T_n y$ for $y \in C$. Then the following hold:

(i) The mapping $T$ is nonexpansive and $\bigcap_{n=1}^{\infty} F(T_n) \subset F(T)$.

(ii) If $\{x_n\}$ is bounded, then $F(T)$ is nonempty.

(iii) If $F(T) = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, then $\{z_n\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n \to \infty} P x_n$ and $P$ is the metric projection of $H$ onto $F(T)$.

**Proof.** We first prove (i). Let $x, y \in C$ be fixed. Since each $T_n$ is nonexpansive, we have

$$
\|Tx - Ty\| \leq \|Tx - T_n x\| + \|T_n x - T_n y\| + \|T_n y - Ty\|
$$

$$
\leq \|Tx - T_n x\| + \|x - y\| + \|T_n y - Ty\|.
$$

Since $\|T_n y - Ty\| \to 0$ for all $y \in C$, we conclude that $\|Tx - Ty\| \leq \|x - y\|$. Suppose $u \in \bigcap_{n=1}^{\infty} F(T_n)$. It is easy to obtain that

$$
\|u - Tu\| \leq \|u - T_n u\| + \|T_n u - Tu\| = \|T_n u - Tu\| \to 0.
$$

Therefore $u \in F(T)$.
Let us show (ii). Assume that \( \{x_n\} \) is bounded. Then \( \{z_n\} \) is also bounded. Thus there exists a subsequence \( \{z_{n_i}\} \) of \( \{z_n\} \) such that \( z_{n_i} \rightharpoonup z \). Note that \( z \in C \). Since \( T_n \) is nonexpansive, it is clear that

\[
\|x_{n+1} - T_n z\| = \|T_n x_n - T_n z\| \leq \|x_n - z\|
\]

for every \( n \in \mathbb{N} \). This yields

\[
\begin{align*}
\|x_{n+1} - Tz\|^2 &= \|x_{n+1} - T_n z + T_n z - Tz\|^2 \\
&= \|x_{n+1} - T_n z\|^2 + \|T_n z - Tz\|^2 + 2 \langle x_{n+1} - T_n z, T_n z - Tz \rangle \\
&\leq \|x_n - z\|^2 + \|T_n z - Tz\|^2 (\|T_n z - Tz\| + 2 \|x_{n+1} - T_n z\|).
\end{align*}
\]

Hence we conclude that

\[
\alpha_n \leq \|x_n - z\|^2 - \|x_{n+1} - Tz\|^2
\]

for every \( n \in \mathbb{N} \), where \( \alpha_n = - \|T_n z - Tz\| (\|T_n z - Tz\| + 2 \|x_{n+1} - T_n z\|) \). Since \( \{T_n\} \) is pointwise convergent and both \( \{x_n\} \) and \( \{T_n z\} \) are bounded, it follows that \( \alpha_n \to 0 \). Thus Lemma 3.1 implies that \( z \in F(T) \). This means that (ii) holds.

Let us prove (iii). Let \( u \in \bigcap_{n=1}^{\infty} F(T_n) \). It is obvious that

\[
\|x_{n+1} - u\| = \|T_n x_n - T_n u\| \leq \|x_n - u\|
\]

for every \( n \in \mathbb{N} \). Thus \( \{x_n\} \) is bounded. Then \( \{z_n\} \) is also bounded. Let \( \{z_{n_i}\} \) be a subsequence of \( \{z_n\} \) such that \( z_{n_i} \rightharpoonup z \). As in the proof of (ii), we obtain \( z \in F(T) \). On the other hand, Lemma 2.1 and (3.1) imply that \( \lim_{n \to \infty} Px_n = w \in \bigcap_{n=1}^{\infty} F(T_n) \). To complete the proof, it is enough to prove \( z = w \). From \( z \in F(T) \) and (2.1), it holds that

\[
\langle z - w, x_k - Px_k \rangle = \langle z - Px_k, x_k - Px_k \rangle + \langle Px_k - w, x_k - Px_k \rangle \\
\leq \langle Px_k - w, x_k - Px_k \rangle \\
\leq \|Px_k - w\| \|x_k - Px_k\| \\
\leq \|Px_k - w\| M
\]

for every \( k \in \mathbb{N} \), where \( M = \sup\{\|x_k - Px_k\| : k \in \mathbb{N}\} \). Summing these inequalities from \( k = 1 \) to \( n_i \) and dividing by \( n_i \), we have

\[
\left\langle z - w, z_{n_i} - \frac{1}{n_i} \sum_{k=1}^{n_i} Px_k \right\rangle \leq \frac{1}{n_i} \sum_{k=1}^{n_i} \|Px_k - w\| M.
\]

Since \( z_{n_i} \rightharpoonup z \) as \( i \to \infty \) and \( Px_n \rightharpoonup w \) as \( n \to \infty \), we obtain \( \langle z - w, z - w \rangle \leq 0 \). This means \( z = w \). This completes the proof. \( \square \)

Let \( T : C \to C \) be a nonexpansive mapping. In Theorem 3.2, putting \( T_n = T \) for \( n \in \mathbb{N} \), we see that \( x_{n+1} = T^n x \) and \( z_n = 1/n \sum_{k=1}^{n} T^{k-1} x \) for every \( n \in \mathbb{N} \), and moreover, it is also clear that \( T_n y - Ty = 0 \) for all \( y \in C \) and \( F(T) = \bigcap_{n=1}^{\infty} F(T_n) \). Therefore Theorem 3.2 (ii) yields a fixed point theorem for a nonexpansive mapping in a Hilbert space.

**Theorem 3.3** ([15, Theorem 3.1.6]). Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \) and \( T \) a nonexpansive self-mapping of \( C \). Then \( F(T) \neq \emptyset \) if and only if \( \{T^n x\} \) is bounded for some \( x \in C \).

We also obtain a nonlinear ergodic theorem which was proved by Baillon [2]; see also [15, Theorem 3.2.1].
Theorem 3.4 (Baillon [2]). Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $T$ a nonexpansive self-mapping of $C$. Suppose that $F(T)$ is nonempty. Let $x \in C$ and let $\{z_n\}$ be a sequence in $C$ defined by
\[
 z_n = \frac{1}{n} \sum_{k=1}^{n} T^{k-1} x
\]
for $n \in \mathbb{N}$. Then $\{z_n\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n \to \infty} P x_n$ and $P$ is the metric projection of $H$ onto $F(T)$.

Further, we obtain the following theorem:

Theorem 3.5. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$ and $T$ a nonexpansive self-mapping of $C$. Suppose that $F(T)$ is nonempty. Let $x \in C$ and let $\{x_n\}$ and $\{z_n\}$ be two sequences in $C$ defined by $x_1 = x \in C$ and
\[
 \begin{cases}
 x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \\
 z_n = \frac{1}{n} \sum_{k=1}^{n} x_k
 \end{cases}
\]
for $n \in \mathbb{N}$, where $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n \to \infty} \alpha_n = 0$. Then $\{z_n\}$ converges weakly to $z \in F(T)$, where $z = \lim_{n \to \infty} P x_n$ and $P$ is the metric projection of $H$ onto $F(T)$.

Proof. Put $T_n = \alpha_n I + (1 - \alpha_n) T$ for $n \in \mathbb{N}$, where $I$ is the identity mapping on $C$. Then $T_n$ is nonexpansive and $F(T_n) = F(T)$ for every $n \in \mathbb{N}$. Therefore $\bigcap_{n=1}^{\infty} F(T_n) = F(T) \neq \emptyset$ and $\|T_n y - T y\| = \alpha_n \|y - T y\| \to 0$ for all $y \in C$. So, from Theorem 3.2 (iii), we have the desired result. $lacksquare$

Problem 3.6. Can we establish a theorem which unifies Theorem 1.1 and Theorem 3.5?

For the remainder of this paper we discuss the problem of approximating a common fixed point of a given countable family of nonexpansive mappings.

Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Let $\{S_n\}$ be a sequence of nonexpansive self-mappings of $C$ and $\{\beta_n\}$ a sequence of $(0, 1)$ such that $\sum_{n=1}^{\infty} \beta_n = 1$. We define a sequence $\{T_n\}$ of self-mappings of $C$ as follows:
\[
 T_1 = \beta_1 S_1 + (1 - \beta_1) S_2, \\
 T_2 = \beta_1 S_1 + \beta_2 S_2 + (1 - \beta_1 - \beta_2) S_3, \\
 \vdots \\
 T_n = \sum_{k=1}^{n} \beta_k S_k + (1 - \sum_{k=1}^{n} \beta_k) S_{n+1},
\]
for $n \in \mathbb{N}$. It is easy to verify that $F(T_n) = \bigcap_{k=1}^{n+1} F(S_k)$, so that we obtain
\[
 (3.2) \quad \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{k=1}^{\infty} F(S_k).
\]
From Lemma 2.2 we may define a nonexpansive self-mapping $T$ of $C$ by
\[
 T = \sum_{k=1}^{\infty} \beta_k S_k.
\]
It also follows from Lemma 2.2 and (3.2) that
\[ F(T) = \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{k=1}^{\infty} F(S_k). \]

Let \( u \in \bigcap_{k=1}^{\infty} F(S_k) \) be fixed. Since each \( S_k \) is nonexpansive, we see that
\[ \|S_ky\| \leq \|S_ky - S_ku\| + \|S_ku\| \leq \|y - u\| + \|u\| \]
for all \( y \in C \) and \( k \in \mathbb{N} \). Then we obtain
\[
\|Ty - T_ny\| = \left\| \sum_{k=n+1}^{\infty} \beta_k S_k y - \left( \sum_{k=1}^{n} \beta_k S_k y + (1 - \sum_{k=1}^{n} \beta_k) S_{n+1} y \right) \right\|
\]
\[
= \left\| \sum_{k=n+1}^{\infty} \beta_k S_k y - (1 - \sum_{k=1}^{n} \beta_k) S_{n+1} y \right\|
\]
\[
\leq \sum_{k=n+1}^{\infty} \beta_k \|S_ky\| + (1 - \sum_{k=1}^{n} \beta_k) \|S_{n+1}y\|
\]
\[
\leq M \sum_{k=n+1}^{\infty} \beta_k + M(1 - \sum_{k=1}^{n} \beta_k)
\]
for all \( y \in C \) and \( n \in \mathbb{N} \), where \( M = \|y - u\| + \|u\| \). From the assumption that \( \sum_{k=1}^{\infty} \beta_k = 1 \), we conclude that
\[ \lim_{n \to 0} \|Ty - T_ny\| = 0 \]
for all \( y \in C \). So, we obtain the following theorem:

**Theorem 3.7.** Let \( C \) be a nonempty closed convex subset of a real Hilbert space \( H \). Let \( \{S_k\} \) be a sequence of nonexpansive self-mappings of \( C \) such that \( \bigcap_{k=1}^{\infty} F(S_k) \) is nonempty and \( \{\beta_k\} \) a sequence in \((0,1)\) such that \( \sum_{k=1}^{\infty} \beta_k = 1 \). Let \( \{x_n\} \) and \( \{z_n\} \) be two sequences defined by \( x_1 = x \in C \) and
\[
\begin{cases}
x_{n+1} = \sum_{k=1}^{n} \beta_k S_k x_n + (1 - \sum_{k=1}^{n} \beta_k) S_{n+1} x_n, \\
z_n = \frac{1}{n} \sum_{k=1}^{n} x_k
\end{cases}
\]
for \( n \in \mathbb{N} \). Then \( \{z_n\} \) converges weakly to \( z \in \bigcap_{k=1}^{\infty} F(S_k) \), where \( z = \lim_{n \to \infty} P x_n \) and \( P \) is the metric projection of \( H \) onto \( \bigcap_{k=1}^{\infty} F(S_k) \).

**References**

Mean ergodic theorems for a sequence of nonexpansive mappings in Hilbert spaces


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