ON OPERATION-PREOPEN SETS IN TOPOLOGICAL SPACES *

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Abstract. In this paper, we present concepts of pre $\gamma_p$-open sets and pre $\gamma_p$-closures of a subset in a topological space, where $\gamma_p$ is an operation on the family of all preopen sets of the topological space, and study some topological properties on them. As its application, we introduce the concept of pre $\gamma_p$-$T_i$ spaces ($i = 0, 1/2, 1, 2$) and study some properties of these spaces.

1 Introduction Throughout this paper, $(X, \tau)$ represents a nonempty topological space on which no separation axioms are assumed, unless otherwise mentioned. The closure and interior of $A \subset X$ are denoted by $Cl(A)$ and $Int(A)$ respectively. The power set of $X$ will be denoted by $P(X)$. An operation $\gamma$ on $\tau$ is a function from $\tau$ into $P(X)$ such that $U \subset U^{\gamma}$ for every set $U \in \tau$, where $U^{\gamma}$ denotes the value $\gamma(U)$ of $\gamma$ at $U$. In 1979, Kasahara [11] firstly defined and investigated the concept of operations on $\tau$. He used the following symbol "$\alpha$" as the operation on $\tau$, i.e., a function $\alpha : \tau \rightarrow P(X)$ is called an operation on $\tau$ if $U \subset \alpha(U)$ holds for any $U \in \tau$. He generalized the notion of compactness with help of operation. After the work of Kasahara, Janković [8] defined the concept of operation-closures (cf. Definition 2.4 below) and investigated some properties of functions with operation-closed graphs. In 1991, Ogata [20] defined and investigated the concept of operation-open sets, i.e., $\gamma$-open sets, and used it to investigate some new separation axioms. He used the symbol $\gamma : \tau \rightarrow P(X)$ as an operation on $\tau$. Thus, he avoided a confusion between the concept of $\alpha$-open sets [18] and one of operation "$\alpha$"-open sets (where the later symbol "$\alpha$" is operation in the sense of Kasahara [11]).

Let $\gamma : \tau \rightarrow P(X)$ be an operation on $\tau$. A nonempty subset $A$ is said to be $\gamma$-open (in the sense of Ogata) [20] if for each point $x \in A$, there exists an open set $U$ containing $x$ such that $U^{\gamma} \subset A$. An arbitrary union of $\gamma$-open sets is also $\gamma$-open [20, Proposition 2.3]. Using the concepts of operation-open sets and operation-closures, some operator-approaches to topological properties were studied [20]. Recently, Krishnan et al. [12] investigated operations on the family of all semi-open sets [13].

In the present paper, we shall introduce an alternative operation-open sets, i.e., pre $\gamma_p$-open sets (cf. Definition 2.3), and investigate more operator-approaches to properties of topological spaces. Let $\gamma_p : PO(X, \tau) \rightarrow P(X)$ be an operation from the family $PO(X, \tau)$ of all preopen sets of $(X, \tau)$ into $P(X)$ (cf. Definition 2.1). The concept of preopen sets was introduced and investigated by Mashhour et al. [16]. Next section contains fundamental definitions of $\gamma_p$-open sets and $\gamma_p$-closures. In Section 3, the notions of pre $\gamma_p$-open sets


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and four kinds of operation-closures, $\tau_\gamma$-Cl$(A)$, $PO(X)\gamma$-$\text{Cl}(A)$, pCl$_\gamma(A)$, $\text{Cl}_\gamma(A)$, are introduced and studied (cf. Definitions 3.9, 3.10, Theorem 3.16). In Section 4, pre $\gamma_p$-generalized closed sets and pre $\gamma_p$-$T_1$ separation axioms are introduced and investigated, where $i=0, 1/2, 1$ or 2. The concept of $\gamma_p$-$T_{1/2}$ (resp. pre $\gamma_p$-$T_{1/2}$) spaces is characterized by using $\gamma_p$-open singletons and $\gamma_p$-closed singletons (resp. pre $\gamma_p$-open singletons and pre $\gamma_p$-closed singletons) (cf. Theorem 4.6(ii) (resp. (i))). Especially, assume $\gamma_p$ is the “identity operation” (cf. Example 3.2(i)), then the concept of “$id$”-$T_{1/2}$ spaces coincides with the concept of $T_{1/2}$-spaces due to Levine [14] (cf. [4, Theorem 2.5]). The digital line $(Z, \kappa)$ is a typical example of $T_{1/2}$-spaces (e.g., [6, p.31 and the list of the references]).

We have other examples of operations (cf. [20] [8]; Example 3.2 and Remark 3.4 below). For some undefined or related concepts, we refer the reader to [17] and [7].

2 Preliminaries A subset $A$ of topological space $(X, \tau)$ is said to be preopen [16] if $A \subset \text{Int}(\text{Cl}(A))$ holds. We denote by $PO(X, \tau)$ (sometimes, $PO(X)$) the set of all preopen sets in $(X, \tau)$ [16]. The complement of a preopen set is called preclosed. The intersection of all preclosed sets of $(X, \tau)$ containing a subset $A$ is called the preclosure of $A$ and is denoted by $p\text{Cl}(A)$ [5]. The union of all preopen sets contained in a subset $A$ is called the preinterior of $A$ and is denoted by $p\text{Int}(A)$. The set $p\text{Cl}(A)$ is preclosed and $p\text{Int}(A)$ is preopen in $(X, \tau)$ for any subset $A$ of $(X, \tau)$, because an arbitrary union of preopen sets of $(X, \tau)$ is preopen [1]. It is well known that [2, Theorem 1.5 (e)(f)] $p\text{Cl}(A) = A \cup \text{Cl}(\text{Int}(A))$ and $p\text{Int}(A) = A \cap \text{Int}(\text{Cl}(A))$ hold for any subset $A$ of $(X, \tau)$. We note that $\tau \subset PO(X, \tau)$ for any topological space $(X, \tau)$ and $PO(X, \tau)$ is not a topology on $X$ in general.

Definition 2.1 Let $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ be a mapping from $PO(X, \tau)$ into $\mathcal{P}(X)$ satisfying the following property: $V \subset \gamma_p(V)$ for any $V \in PO(X, \tau)$. We call the mapping $\gamma_p$ an operation on $PO(X, \tau)$. We denote $V^{\gamma_p} := \gamma_p(V)$ for any $V \in PO(X, \tau)$.

Remark 2.2 For an operation $\gamma_p : PO(X) \to \mathcal{P}(X)$, the restriction of $\gamma_p$ onto $\tau$ (say $\gamma_p|\tau : \tau \to \mathcal{P}(X)$) is well defined. Indeed, $\tau \subset PO(X, \tau)$ holds and so $(\gamma_p|\tau)(V) = V^{\gamma_p}$ is well defined for any set $V \in \tau$. This restriction $\gamma_p|\tau : \tau \to \mathcal{P}(X)$ is the operation on $\tau$ in the sense of Ogata [20, Definition 2.1] (cf. Section 1 above). By [20, Definition 2.2] (cf. Section 1 above), a nonempty set $A$ is called a $\gamma_p|\tau$-open set of $(X, \tau)$ if for each point $x \in A$, there exists an open set $U$ containing $x$ such that $U^{\gamma_p}|\tau \subset A$. Moreover, a subset $A$ is said to be $\gamma_p|\tau$-closed in $(X, \tau)$, if $X \setminus A$ is $\gamma_p|\tau$-open in $(X, \tau)$. We suppose that the empty set is $\gamma_p|\tau$-open and we denote the set of all $\gamma_p|\tau$-open sets of $(X, \tau)$ by $\tau_{\gamma_p}|\tau$. We note that:

* $(*) U^{\gamma_p} = U^{\gamma_p}|\tau$ holds for any set $U \subset \tau$.

Definition 2.3 (cf. [20]) Let $(X, \tau)$ be a topological space and $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ an operation on $PO(X, \tau)$. A nonempty subset $A$ of $(X, \tau)$ is called a $\gamma_p$-open set of $(X, \tau)$ if for each point $x \in A$, there exists an open set $U$ such that $x \in U$ and $U^{\gamma_p} \subset A$. We suppose that the empty set $\emptyset$ is also $\gamma_p$-open for any operation $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$. The complement of a $\gamma_p$-open set is called $\gamma_p$-closed in $(X, \tau)$. We denote the set of all $\gamma_p$-open sets in $(X, \tau)$ by $\tau_{\gamma_p}$.

Definition 2.4 (cf. [8, Definition 2.2]) Let $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ be an operation and $A$ a subset of a topological space $(X, \tau)$.

(i) The point $x \in X$ is in the $\gamma_p$-closure of a set $A$ if $U^{\gamma_p} \cap A \neq \emptyset$ for each open set $U$ containing $x$. The $\gamma_p$-closure of a set $A$ is denoted by $\text{Cl}_{\gamma_p}(A)$. 
Proposition 2.6 \( (\text{cf. Remark 2.2(ii)}) \). It is obvious that 
\( U^{\varphi}_{A} \).

Remark 2.5 \( \) holds. We note that \( Cl_{\gamma}(A) := \{ x \in X | U^{\varphi}_{A} \cap A \neq \emptyset \} \) for each open set \( U \) containing \( x \).

Remark 2.5. \( \) is defined and related properties are investigated, where \( \gamma \) is a \( \gamma \)-open set of \( A \) in \( (X, \tau) \).

\( \gamma \)-open set of \( A \) is \( \gamma \)-open in \( (X, \tau) \) if and only if \( A \) is \( \gamma \)-open in \( (X, \tau) \) (in the sense of Ogata \[20, Definition 2.2\]; cf. Section 1 above). Namely, \( \tau_{\gamma} = \tau_{\gamma} \) holds.

(ii) \( A \) is \( \gamma \)-closed (in the sense of Janković), i.e., \( A = Cl_{\gamma}(A) \) in \( (X, \tau) \) (cf. [20, Definition 2.2]).

(iii)\[20, Theorem 3.7\] The following properties are equivalent:

(1) \( A \) is \( \gamma \)-open in \( (X, \tau) \);
(2) \( \tau_{\gamma} = Cl(X \setminus A) = Cl(X) \setminus A \) holds;
(3) \( \tau_{\gamma} = Cl(X \setminus A) = Cl(X) \setminus A \) holds;
(4) \( X \setminus A \) is \( \gamma \)-closed in \( (X, \tau) \) (cf. [20, Definition 2.2]).

(iv) \( A \) is \( \gamma \)-closed in \( (X, \tau) \), i.e., \( A = Cl_{\gamma}(A) \) if and only if \( X \setminus A \) is \( \gamma \)-open.

(v) \( \) (cf. [20, p.176]) Every \( \varphi \)-open set is \( \varphi \)-open, i.e., \( \gamma_{\tau} \subseteq \tau_{\gamma} \).

(vi) \( \) An arbitrary union of \( \gamma \)-open sets is \( \gamma \)-open.

Proof. (i) (Necessity) Let \( A \subseteq X \). There exists an open set \( U \) such that \( x \in U \) and \( U^{\varphi}_{A} \subseteq A \). By \( \ast \) in Remark 2.2, \( U^{\varphi}_{A} \subseteq A \) and so \( A \) is \( \gamma \)-open. (Sufficiency) It is easy to prove by using \( \ast \) in Remark 2.2. (ii) This follows from Definition 2.4 and Remark 2.5. (iii) By (i), (ii) and [20, Theorem 3.7], (iii) is proved. (iv) This is shown by (i), (ii) and (iii). (v) Let \( A \in \tau_{\gamma} \). Then, for each point \( x \in A \) there exists an open set \( U(x) \) containing \( x \) such that \( A = \bigcup \{ U(x) | x \in A \} \). Thus, we have that \( A \in \tau_{\gamma} \). (vi) By [20, Proposition 2.3] (cf. Section 1 above), an arbitrary union of \( \gamma \)-open sets is \( \gamma \)-open. Thus, using (i), (iv) is obtained. \( \square \)

3 Pre \( \gamma \)-open sets and operation-closures

In this section the notion of pre-\( \gamma \)-open sets is defined and related properties are investigated, where \( \gamma_{\tau} : PO(X, \tau) \to P(X) \) is an operation on \( PO(X, \tau) \).

Definition 3.1 Let \( (X, \tau) \) be a topological space and \( \gamma_{\tau} : PO(X, \tau) \to P(X) \) an operation on \( PO(X, \tau) \). A nonempty subset \( A \subseteq (X, \tau) \) is called a \( \gamma \)-open set of \( (X, \tau) \) if for each point \( x \in A \), there exists a pre-open set \( U \) such that \( x \in U \) and \( U^{\varphi}_{A} \subseteq A \). We suppose that the emptyset \( \emptyset \) is also pre-\( \gamma \)-open for any operation \( \gamma_{\tau} : PO(X, \tau) \to P(X) \). We denote the set of all pre-\( \gamma \)-open sets \( (X, \tau) \) by \( PO(X, \tau)_{\gamma_{\tau}} \) (or simply, \( PO(X, \gamma)_{\gamma_{\tau}} \)).

Example 3.2 (i) A subset \( A \subseteq (X, \tau) \) is a pre-\( \ast \)-open set of \( (X, \tau) \) if and only if \( A \) is pre-open in \( (X, \tau) \). The operation \( \ast : PO(X, \tau) \to P(X) \) is defined by \( V^{\ast} = V \) for any set \( V \in PO(X, \tau) \); this operation is called the identity operation on \( PO(X, \tau) \) (cf. [8]). A subset \( A \) is an \( \ast \)-open set of \( (X, \tau) \) if and only if \( A \) is open in \( (X, \tau) \). Therefore, we have that \( PO(X, \tau)_{\ast} = PO(X, \tau) \) and \( \tau_{\ast} = \tau \).
(ii) (ii-1) We characterize pre "$C^{l}$"-open sets, where "$C^{l}$": $PO(X, \tau) \rightarrow \mathcal{P}(X)$ is the operation defined by $V^{c^{l}} := Cl(V)$ for any subset $V \in PO(X, \tau)$. A nonempty subset $A$ is pre "$C^{l}$"-open in $(X, \tau)$ if and only if, by definition, for each point $x \in A$ there exists a subset $U \in PO(X, \tau)$ such that $x \in U$ and $U^{c^{l}} \subseteq A$; if and only if for each point $x \notin X \setminus A$, there exists a subset $V \in PO(X, \tau)$ such that $x \in V$ and $V^{c^{l}} \cap (X \setminus A) = \emptyset$; if and only if $pCl_{c^{l}}(X \setminus A) \subseteq X \setminus A$, where $pCl_{c^{l}}(B) := \{z \in X|Cl(W) \cap B \neq \emptyset \}$ for any subset $W \in PO(X, \tau)$ such that $z \in W$ for a subset $B$ of $(X, \tau)$ (cf. Definition 3.10 below). Thus, a nonempty set $A$ is pre "$C^{l}$"-open in $(X, \tau)$ if and only if $pCl_{c^{l}}(X \setminus A) = X \setminus A$ holds. The following property holds: a nonempty set $A$ is pre "$C^{l}$"-open in $(X, \tau)$ if and only if $A$ is $\theta$-open in $(X, \tau)$.

Proof. Let $A$ be pre "$C^{l}$"-open. For any point $x$ of a, there exists $U \in PO(X, \tau)$ such that $x \in U$ and $U^{c^{l}} \subseteq A$; if and only if, by definition, for each point $x \notin X \setminus A$, there exists a subset $V \in PO(X, \tau)$ such that $x \in V$ and $V^{c^{l}} \cap (X \setminus A) = \emptyset$; if and only if $pCl_{c^{l}}(X \setminus A) \subseteq X \setminus A$. It is obvious obtained that $A$ is $\theta$-open. The converse is obvious.

(ii-2) The operation "$pC^{l}$" : $PO(X, \tau) \rightarrow \mathcal{P}(X)$ is defined by $V^{pC^{l}} := Cl(V)$ for any set $V \in PO(X, \tau)$. We note that "$c^{l} \neq \text{"} C^{l}"": PO(X, \tau) \rightarrow \mathcal{P}(X)$ in general and

$\ast$ a subset $A$ is a pre "$pC^{l}$"-open set if and only if $A$ is pre $\theta$-open [22] in $(X, \tau)$.

A closure $pCl_{l}(B)$ of a subset $B$ is defined by $pCl_{l}(B) := \{y \in X|pCl(V) \cap B \neq \emptyset \}$ for every preopen set $V$ containing $y$ [22]. A subset $B$ is said to be pre $\theta$-closed in $(X, \tau)$ if $B = pCl_{l}(B)$ holds; a subset $A$ is said to be pre $\theta$-open in $(X, \tau)$ if $X \setminus A = pCl_{l}(X \setminus A)$ holds. It is obviously obtained that $A \subseteq pCl_{l}(A)$ for any subset $A$ of $(X, \tau)$.

The proof of $\ast$: A subset $A$ is pre "$pC^{l}$"-open if and only if, for each point $x \in A$, there exists a subset $U \in PO(X, \tau)$ such that $x \in U$ and $U^{pC^{l}} \subseteq A$; if and only if, for each point $x \notin X \setminus A$, there exists a subset $U \in PO(X, \tau)$ such that $x \in U$ and $U^{pC^{l}} \cap (X \setminus A) = \emptyset$; if and only if $pCl_{l}(X \setminus A) \subseteq X \setminus A$ and so $A$ is pre $\theta$-open in $(X, \tau)$.

(ii-3) The following example shows that the operations "$pC^{l}$" and "$C^{l}$" are distinct operations on $PO(X, \tau)$ in general. Indeed, let $X := \{a, b, c\}$ and $\tau := \{\emptyset, \{a, b\}, X\}$. In a topological space $(X, \tau), PO(X, \tau) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$ holds and so $Cl(\{a\}) = X, pCl(\{a\}) = \{a\}$. Thus, we have that, for a preopen set $\{a\},$ "$pC^{l}$"(\{a\} $\neq$ "$C^{l}$"(\{a\}.

(iii) The operation "$Int \circ C^{l}$" : $PO(X, \tau) \rightarrow \mathcal{P}(X)$ is well defined by $V^{Intc^{l}} := Int(Cl(V))$ for any subset $V \in PO(X, \tau)$. Indeed, $V \subseteq V^{Intc^{l}} = Int(Cl(V))$ holds for any $V \in PO(X, \tau)$ by definition of the preopen sets. This is called the interior-closure operation on $PO(X, \tau)$ (cf. [8]). For this operation we note that:

$\ast \ast$ a subset $A$ is pre "$Int \circ C^{l}$"-open in $(X, \tau)$ if and only if $A$ is "$Int \circ C^{l}$"-open in $(X, \tau)$, i.e., $A$ is $\delta$-open in $(X, \tau)$ [26].

The proof of $\ast \ast$: Suppose that $A$ is pre "$Int \circ C^{l}$"-open in $(X, \tau)$. For a point $x \in A$, there exists a subset $U \in PO(X, \tau)$ such that $x \in U$ and $Int(Cl(U)) \subseteq A$ if and only if there exists a subset $G \in \tau$ such that $x \in G$ and $Int(Cl(G)) \subseteq A$ (i.e., by definition, $A$ is "$Int \circ C^{l}$"-open in $(X, \tau)$). For the proof of necessity of the last equivalence, we can take $G = Int(Cl(U))$. By definitions, $A$ is "$Int \circ C^{l}$"-open in $(X, \tau)$ if and only if $A$ is $\delta$-open in $(X, \tau)$. Recall that $\tau_{3}$ denotes the collection of all $\delta$-open sets in $(X, \tau)$. It is well known that $\tau_{3}$ is a topology of $X$. By means of $\ast \ast$, we conclude that $\tau_{3} = \tau_{Intc^{l}} = PO(X, \tau)^{\ast Intc^{l}}$ holds and so $PO(X, \tau)^{\ast Intc^{l}}$ is a topology of $X$ (cf. Theorem 3.8(iv) below).

(iv) For more examples, operations from $PO(X, \tau)$ into $\mathcal{P}(X)$ are well defined as follows: The operations "$Cl_{a}^{l}$" , "$C^{l}$" , "$pCl_{l}$" , "$\alpha C^{l}$" , "$sC^{l}$" , "$\theta C^{l}$" : $PO(X, \tau) \rightarrow \mathcal{P}(X)$
are well defined, respectively, by $V^{"Cl_o"} := Cl_o(V)$ [26], $V^{"Cl_i"} := Cl_i(V)$ [26], $V^{"pCl_o"} := pCl_o(V)$ [16], $V^{"pCl_i"} := pCl_i(V)$ [18], $V^{"Cl"} := Cl(V)$ [13], $V^{"sCl"} := sCl(V)$ [13], where $a := \theta$-$sCl$ [10] for every set $V \in PO(X, \tau)$. We recall some definitions as follows: For a subset $B$ of $(X, \tau)$, $\delta$-closure $Cl_\delta(B)$ [26] (resp. $\theta$-closure $Cl_\theta(B)$ [26]) of $B$ is defined by $Cl_\delta(B) := \{y \in X| \text{Int}(Cl(U)) \cap B \neq \emptyset \text{ for every open set } U \text{ containing } y\}$ (resp. $Cl_\theta(B) := \{y \in X| Cl(U) \cap B \neq \emptyset \text{ for every open set } U \text{ containing } y\}$). For a subset $B$ of $(X, \tau)$, the $\alpha$-closure $\alpha Cl(B)$ [18] (resp. semi-closure $sCl(B)$ [13]) of the set $B$ is the intersection of all $\alpha$-closed sets (resp. semi-closed sets) containing $B$; $\alpha Cl(B)$ (resp. $sCl(B)$) is $\alpha$-closed (resp. semi-closed) in $(X, \tau)$. For these operations above, the following results are probably unexpected:

$\text{"Cl"} = \text{"Cl}_\theta = \text{"Cl}_i = \text{"pCl}" : PO(X, \tau) \to P(X)$, $\text{"pCl"} = \text{"pCl}_\theta : PO(X, \tau) \to P(X)$ hold.

Indeed, it is shown that $Cl(V) = Cl_o(V) = Cl_i(V) = \alpha Cl(V)$ hold for any set $V \in PO(X, \tau)$ ([9, Corollary 2.5 (c)], e.g., [19, Lemma 2.1]), $pCl(V) = pCl_o(V)$ holds for any set $V \in PO(X, \tau)$ ([3, Proposition 4.2]) and $sCl(V) = \theta$-$sCl(V)$ holds for any set $V \in PO(X, \tau)$ ([23, Lemma 1]).

(v) Let $\text{"Cl"}|_\tau, \text{"pCl"}|_\tau, \text{"Cl}_\theta|_\tau : \tau \to P(X)$ be the restrictions to $\tau$ of operations $\text{"Cl"}, \text{"pCl"}, \text{"Cl}_\theta : PO(X, \tau) \to P(X)$, respectively. Then, $\text{"Cl"}|_\tau = \text{"pCl"}|_\tau = \text{"Cl}_\theta|_\tau : \tau \to P(X)$ holds over $\tau$, because $Cl(V) = pCl(V) = \alpha Cl(V)$ for any open set $V$ of $(X, \tau)$.

(vi) Suppose that $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then, it is shown that $PO(X, \tau) = \{\emptyset, X, \{a\}, \{a, b\}\} \text{ such that } \gamma_p(A) := A \text{ if } b \in A, \gamma_p(A) := pCl(A) \text{ if } b \notin A$. We have that $PO(X, \tau)_{\gamma_p} = \{\emptyset, X, \{a\}\}.$

**Theorem 3.3** Let $\gamma_p : PO(X, \tau) \to P(X)$ be any operation on $PO(X, \tau)$.

(i) Every pre $\gamma_p$-open set of $(X, \tau)$ is preopen in $(X, \tau)$, i.e., $PO(X, \tau)_{\gamma_p} \subset PO(X, \tau)$.

(ii) Every $\gamma_p$-open set of $(X, \tau)$ is pre $\gamma_p$-open, i.e., $\tau_{\gamma_p} \subset PO(X, \tau)_{\gamma_p}$.

(iii) If $\{A_i| i \in J\}$ is a collection of pre $\gamma_p$-open sets in $(X, \tau)$, then $\bigcup\{A_i| i \in J\}$ is pre $\gamma_p$-open in $(X, \tau)$, where $J$ is any index set.

**Proof.** (i) Suppose that $A \in PO(X)_{\gamma_p}$. Let $x \in A$. Then, there exists a preopen set $U$ such that $x \in U \subset U^{\gamma_p} \subset A$. Because $U$ is a preopen set, this implies $x \in U \subset \text{Int}(Cl(U)) \subset \text{Int}(Cl(A))$. Thus we show that $A \subset \text{Int}(Cl(A))$ and hence $A \in PO(X)$. Thus we have that $PO(X, \tau)_{\gamma_p} \subset PO(X, \tau)$.

An alternative proof: Suppose that $A \in PO(X)_{\gamma_p}$. Let $x \in A$. There exists a preopen set $U(x)$ containing $x$ such that $U(x)^{\gamma_p} \subset A$. Then, $\bigcup\{U(x)| x \in A\} \subset \bigcup\{U(x)^{\gamma_p}| x \in A\} \subset A$ and so $A = \bigcup\{U(x)| x \in A\} \in PO(X, \tau)$ holds. (ii) Let $A$ be a $\gamma_p$-open set in $(X, \tau)$ and $x \in A$. There exists an open set $U$ such that $x \in U \subset U^{\gamma_p} \subset A$. Since every open set is a preopen set, this implies that $A$ is a pre $\gamma_p$-open set. Hence, it follows from definitions that $\tau_{\gamma_p} \subset PO(X, \tau)_{\gamma_p}$ holds. (iii) Let $x \in \bigcup\{A_i| i \in J\}$, then $x \in A_i$ for some $i \in J$. Since $A_i$ is a $\gamma_p$-open set, there exists a preopen set $U$ containing $x$ such that $U^{\gamma_p} \subset A_i \subset \bigcup\{A_i| i \in J\}$. Hence $\bigcup\{A_i| i \in J\}$ is a pre $\gamma_p$-open set. □

**Remark 3.4** (i) The converses of Theorem 3.3 (i) and (ii) above need not be true. Let $X := \{a, b, c, d\}$ and $\tau := \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then, for a topological space $(X, \tau)$, we have $PO(X, \tau) = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Define an operation $\gamma_p : PO(X, \tau) \to P(X)$ by putting $\gamma_p(A) := A$ if $a \in A$, $\gamma_p(A) := pCl(A)$ if $a \notin A$. Then it is clearly to see that $\{b\} \in PO(X, \tau)$ but $\{b\}$ is not pre $\gamma_p$-open; $\{a, b, d\}$ is a pre $\gamma_p$-open set but not a $\gamma_p$-open set.
(ii) In general, the intersection of two pre $\gamma_p$-open sets need not be a pre $\gamma_p$-open set. Let $X := \{a, b, c\}$, $\tau := \{\emptyset, X, \{a\}, \{a, b\}\}$. For a topological space $(X, \tau)$, $PO(X, \tau) = \{\emptyset, X, \{a, b\}, \{a, c\}\}$. Define an operation $\gamma_p : PO(X, \tau) \to P(X)$ by putting $\gamma_p(A) := A$ if $A \neq \{a\}$, $\gamma_p(A) := \{a, b\}$ if $A = \{a\}$. Then $A = \{a, b\}$ and $B = \{a, c\}$ are pre $\gamma_p$-open sets but $A \cap B = \{a\}$ is not a pre $\gamma_p$-open set.

**Definition 3.5** Let $(X, \tau)$ be a topological space and $\gamma_p : PO(X, \tau) \to P(X)$ an operation. Then, $(X, \tau)$ is said to be pre $\gamma_p$-regular (resp. $\gamma_p$-regular) if for each point $x \in X$ and for every preopen (resp. open) set $V$ containing $x$, there exists a preopen (resp. an open) set $U$ containing $x$ such that $U^{\gamma_p} \subseteq V$.

**Theorem 3.6** Let $(X, \tau)$ be a topological space and $\gamma_p : PO(X, \tau) \to P(X)$ an operation on $PO(X, \tau)$.

(i) The following properties are equivalent:

1. $PO(X, \tau) = PO(X, \tau)\gamma_p$;
2. $(X, \tau)$ is a pre $\gamma_p$-regular space;
3. For every $x \in X$ and for every preopen set $U$ of $(X, \tau)$ containing $x$, there exists a pre $\gamma_p$-open set $W$ of $(X, \tau)$ such that $x \in W$ and $W \subseteq U$.

(ii) A space $(X, \tau)$ is $\gamma_p$-regular if and only if $(X, \tau)$ is $\gamma_p|\tau$-regular (in the sense of Kasahara)[11], e.g., [20].

(iii) The following properties are equivalent:

1. $\tau = \tau_{\gamma_p}$ holds;
2. $(X, \tau)$ is a $\gamma_p$-regular space;
3. For every $x \in X$ and for every open set $U$ of $(X, \tau)$ containing $x$, there exists a $\gamma_p$-open set $W$ of $(X, \tau)$ such that $x \in W$ and $W \subseteq U$.

**Proof.** (i) $(1) \Rightarrow (2)$ Let $x \in X$ and $V$ a preopen set containing $x$. It follows from assumption that $V$ is a pre $\gamma_p$-open set. This implies that there exists a preopen set $U$ containing $x$ such that $U^{\gamma_p} \subseteq V$. Hence, $(X, \tau)$ is a pre $\gamma_p$-regular space. $(2) \Rightarrow (3)$ Let $x \in X$ and $U$ be a preopen set containing $x$. Then, by $(2)$ there is a preopen set $W$ containing $x$ and $W \subseteq W^{\gamma_p} \subseteq U$. By using $(2)$ for the set $W$, it is shown that $W$ is pre $\gamma_p$-open. Hence, $W$ is a pre $\gamma_p$-open set containing $x$ such that $W \subseteq U$.

$(3) \Rightarrow (1)$ By $(3)$ and Theorem 3.3(iii), it follows that every preopen set is pre $\gamma_p$-open, i.e., $PO(X, \tau) \subseteq PO(X, \tau)\gamma_p$. It follows from Theorem 3.3(i) that the converse inclusion $PO(X, \tau)\gamma_p \subseteq PO(X, \tau)$ holds. (ii) By definition, $U^{\gamma_p} = U^{\gamma_p|\tau}$ holds for every $U \in \tau$. Thus the proof is obtained. (iii) $(1) \Rightarrow (2)$ By $(1)$ and Proposition 2.6(i), $\tau = \tau_{\gamma_p|\tau} = \tau_{\gamma_p}$. Using [20, Proposition 2.4], we have that $(X, \tau)$ is $\gamma_p|\tau$-regular and so, by (ii), $(X, \tau)$ is $\gamma_p$-regular. $(2) \Rightarrow (3)$ Let $x \in X$ and $U$ be an open set containing $x$. By (2), there exists a subset $W \subseteq \tau$ such that $x \in W$ and $W^{\gamma_p} \subseteq U$. Using (2) for the set $W$ and any point $y \in W$, it is shown that $W$ is $\gamma_p$-open. Then, $U$ is $\gamma_p$-open and so $(X, \tau)$ is a $\gamma_p$-regular space. $(3) \Rightarrow (1)$ It is enough to prove $\tau \subseteq \tau_{\gamma_p}$, because $\tau_{\gamma_p} \subseteq \tau$ (cf. Proposition 2.6(v)). Let $U \in \tau$. By using (3) for the set $U$ and each point $x \in U$, there exists a subset $W(x) \subseteq \tau_{\gamma_p}$ such that $W(x) \subseteq U$. Thus we have that $U = \bigcup\{W(x)\} \subseteq U^{\gamma_p}$ (cf. Proposition 2.6(vi)). Therefore, we have that $\tau \subseteq \tau_{\gamma_p}$. □

**Definition 3.7** An operation $\gamma_p : PO(X, \tau) \to P(X)$ is called to be prerregular (resp. regular, cf.[11, p.98], e.g., [20, Definition 2.5]) if for each point $x \in X$ and for every pair of preopen (resp. open) sets $U$ and $V$ containing $x \in X$, there exists a preopen (resp. an open) set $W$ such that $x \in W$ and $W^{\gamma_p} \subseteq U^{\gamma_p} \cap V^{\gamma_p}$. 
Theorem 3.8 (i) Let $\gamma_p : PO(X, \tau) \rightarrow \mathcal{P}(X)$ be a preregular operation on $PO(X, \tau)$. If $A$ and $B$ are pre $\gamma_p$-open in $(X, \tau)$, then $A \cap B$ is also pre $\gamma_p$-open in $(X, \tau)$.

(ii) An operation $\gamma_p : PO(X, \tau) \rightarrow \mathcal{P}(X)$ is regular if and only if $\gamma_p|\tau : \tau \rightarrow \mathcal{P}(X)$ is regular (in the sense of [11, p.98], e.g., [20, Definition 2.5]).

(iii) Let $\gamma_p : PO(X, \tau) \rightarrow \mathcal{P}(X)$ be a regular operation on $PO(X, \tau)$. If $A$ and $B$ are $\gamma_p$-open in $(X, \tau)$, then $A \cap B$ is also $\gamma_p$-open in $(X, \tau)$.

(iv) If $\gamma_p : PO(X, \tau) \rightarrow \mathcal{P}(X)$ is a preregular (resp. regular) operation, then $PO(X, \tau)_{\gamma_p}$ (resp. $\tau_{\gamma_p}$) is a topology of $X$.

Proof. (i) Let $x \in A \cap B$. Since $A$ and $B$ are pre $\gamma_p$-open sets, there exists preopen sets $U, V$ such that $x \in U, x \in V$ and $U_{\gamma_p} \subset A$ and $V_{\gamma_p} \subset B$. By preregularity of $\gamma_p$, there exists a preopen set $W$ containing $x$ such that $W_{\gamma_p} \subset U_{\gamma_p} \cap V_{\gamma_p} \subset A \cap B$. Therefore, $A \cap B$ is a pre $\gamma_p$-open set. (ii) Since $U_{\gamma_p} = U_{\gamma_p}|\tau$ for any open subset $A$ of $(X, \tau)$, we have the equivalence. (iii) It is proved by (ii) above, Proposition 2.6(i) and [20, Proposition 2.9]. (iv) It is proved by (i) above and Theorem 3.3(iii) (resp. (iii) above and Proposition 2.6(vi)). $\square$

Definition 3.9 Let $\gamma_p : PO(X, \tau) \rightarrow \mathcal{P}(X)$ be an operation and $A$ a subset of a topological space $(X, \tau)$.

(i) A subset $A$ is said to be $\gamma_p$-closed in $(X, \tau)$ if $X \setminus A$ is a $\gamma_p$-open set of $(X, \tau)$ (cf. Proposition 2.6(iv)).

(ii) A subset $A$ is said to be pre $\gamma_p$-closed in $(X, \tau)$ if $X \setminus A$ is pre $\gamma_p$-open in $(X, \tau)$.

(iii) The following subsets are well defined as follows:

$\tau_{\gamma_p}Cl(A) := \bigcap\{F | F$ is a $\gamma_p$-closed set of $(X, \tau)$ such that $A \subset F\}$ (cf. (i) above, Proposition 2.6(ii)(iii));

$PO(X)_{\gamma_p}Cl(A) := \bigcap\{F | F$ is pre-$\gamma_p$-closed in $(X, \tau)$ such that $A \subset F\}$.

Definition 3.10 Let $\gamma_p : PO(X, \tau) \rightarrow \mathcal{P}(X)$ be an operation and $A$ a subset of a topological space $(X, \tau)$. A point $x \in X$ is in the pre $\gamma_p$-closure of a set $A$ if $U_{\gamma_p} \cap A \neq \emptyset$ for each preopen set $U$ containing $x$. The pre $\gamma_p$-closure of $A$ is denoted by $pCl_{\gamma_p}(A)$. Namely, $pCl_{\gamma_p}(A) := \{x \in X | U_{\gamma_p} \cap A \neq \emptyset$ for any preopen set $U$ containing $x\}$.

Theorem 3.11 Let $A$ be a subset of a topological space $(X, \tau)$. Then we have the following properties on $PO(X, \tau)_{\gamma_p}$-closures and $\tau_{\gamma_p}$-closures.

(i) $PO(X)_{\gamma_p}Cl(A) = \{y \in X | V \cap A \neq \emptyset$ for every set $V \in PO(X, \tau)_{\gamma_p}$ such that $y \in V\}$.

(ii) $\tau_{\gamma_p}Cl(A) = \{y \in X | V \cap A \neq \emptyset$ for every set $V \in \tau_{\gamma_p}$ such that $y \in V\}$.

Proof. (i) Denote $E := \{y \in X | V \cap A \neq \emptyset$ for every set $V \in PO(X, \tau)_{\gamma_p}$ such that $y \in V\}$. We shall prove that $PO(X)_{\gamma_p}Cl(A) = E$. Let $x \notin E$. Then there exists a pre $\gamma_p$-open set $V$ containing $x$ such that $V \cap A = \emptyset$. This implies that $X \setminus V$ is pre $\gamma_p$-closed and $A \subset X \setminus V$. Hence $PO(X)_{\gamma_p}Cl(A) \subset X \setminus V$. It follows that $x \notin PO(X)_{\gamma_p}Cl(A)$. Thus, we have that $PO(X)_{\gamma_p}Cl(A) \subset E$. Conversely, let $x \notin PO(X)_{\gamma_p}Cl(A)$. Then there exists a pre $\gamma_p$-closed set $F$ such that $A \subset F$ and $x \notin F$. Then we have that $x \in X \setminus F, X \setminus F \in PO(X, \tau)_{\gamma_p}$ and $(X \setminus F) \cap A = \emptyset$. This implies that $x \notin E$. Hence $E \subset PO(X)_{\gamma_p}Cl(A)$. Therefore, we have that $PO(X)_{\gamma_p}Cl(A) = E$. (ii) By using Definition 3.9(iii), Proposition 2.6(i) and [20, (3.2), Proposition 3.3], it is obtained that $\tau_{\gamma_p}Cl(A) = \bigcap\{F | A \subset F, X \setminus F \in \tau_{\gamma_p}\} = \bigcap\{F | A \subset F, X \setminus F \in \tau_{\gamma_p}\} = \tau_{\gamma_p}Cl(A) = \{y \in X | V \cap A \neq \emptyset$ for any set $V \in \tau_{\gamma_p}$ such that $y \in V\}$ for any $V \in \tau_{\gamma_p}$ such that $y \in V$. $\square$
For $pCl_{\gamma_p}(A)$ (cf. Definition 3.10) and $PO(X)_{\gamma_p}\Cl(A)$ (cf. Definition 3.9(iii)), where $A$ is a subset of a topological space $(X, \tau)$, we have the following properties Theorem 3.12 and Theorem 3.13.

**Theorem 3.12** Let $\gamma_p : PO(X, \tau) \rightarrow P(X)$ be an operation on $PO(X, \tau)$ and $A$ and $B$ subsets of a topological space $(X, \tau)$. Then, we have the following properties on $pCl_{\gamma_p}(A)$ and $pCl_{\gamma_p}(B)$.

1. The set $pCl_{\gamma_p}(A)$ is a preclosed set of $(X, \tau)$ and $A \subseteq pCl_{\gamma_p}(A)$.
2. $pCl_{\gamma_p}(\emptyset) = \emptyset$ and $pCl_{\gamma_p}(X) = X$.
3. $A$ is pre $\gamma_p$-closed (i.e., $X \setminus A$ is pre $\gamma_p$-open) in $(X, \tau)$ if and only if $pCl_{\gamma_p}(A) = A$ holds.
4. If $A \subseteq B$, then $pCl_{\gamma_p}(A) \subseteq pCl_{\gamma_p}(B)$.
5. $pCl_{\gamma_p}(A) \cup pCl_{\gamma_p}(B) \subseteq pCl_{\gamma_p}(A \cup B)$ holds.
6. If $\gamma_p : PO(X, \tau) \rightarrow P(X)$ is preregular, then $pCl_{\gamma_p}(A) \cup pCl_{\gamma_p}(B) = pCl_{\gamma_p}(A \cup B)$ holds.
7. $pCl_{\gamma_p}(A \cap B) \subseteq pCl_{\gamma_p}(A) \cap pCl_{\gamma_p}(B)$ holds.

**Proof.** (i) For each point $x \in X \setminus pCl_{\gamma_p}(A)$, by Definition 3.10, there exists a preopen set $U(x)$ containing $x$ such that $U(x) \cap A = \emptyset$. We set $V := \bigcup \{U(x) \mid x \in X \setminus pCl_{\gamma_p}(A)\}$. Then, it is shown that $V = X \setminus pCl_{\gamma_p}(A)$ holds. Indeed, for a point $y \in V$, there exists a subset $U(y) \subseteq PO(X, \tau)$ such that $y \in U(y)$ and $U(y) \cap A = \emptyset$. This shows that $y \notin pCl_{\gamma_p}(A)$ and so $V \subseteq X \setminus pCl_{\gamma_p}(A)$. Conversely, let $y \in X \setminus pCl_{\gamma_p}(A)$. There exists a subset $U(y) \subseteq PO(X, \tau)$ such that $U(y) \cap A = \emptyset$ and so $y \in U(y) \subseteq V$. Thus, we conclude that $X \setminus pCl_{\gamma_p}(A) \subseteq V$; we have that $V = X \setminus pCl_{\gamma_p}(A)$. Therefore, $pCl_{\gamma_p}(A)$ is preclosed in $(X, \tau)$, because $V \subseteq PO(X, \tau)$. Obviously, by Definition 3.10, we have that $A \subseteq pCl_{\gamma_p}(A)$. (ii) They are obtained from Definition 3.10. (iii) Suppose that $X \setminus A$ is pre $\gamma_p$-open in $(X, \tau)$. We claim that $pCl_{\gamma_p}(A) \subseteq A$. Let $x \notin A$. There exists a preopen set $U$ containing $x$ such that $U \cap A = \emptyset$, i.e., $U \cap A = \emptyset$. Hence, using Definition 3.10, we have that $x \notin pCl_{\gamma_p}(A)$ and so $pCl_{\gamma_p}(A) \subseteq A$. By (i), it is proved that $A = pCl_{\gamma_p}(A)$. (Sufficiency) Suppose that $A = pCl_{\gamma_p}(A)$. Let $x \in X \setminus A$. Since $x \notin pCl_{\gamma_p}(A)$, there exists a preopen set $U$ containing $x$ such that $U \cap A = \emptyset$, i.e., $U \cap A = \emptyset$. Namely, $X \setminus A$ is pre $\gamma_p$-open in $(X, \tau)$ and so $A$ is pre $\gamma_p$-closed. (v) They are obtained from (iv). (vi) Let $x \notin pCl_{\gamma_p}(A) \cup pCl_{\gamma_p}(B)$, then there exist two preopen sets $U$ and $V$ containing $x$ such that $U \cap A = \emptyset$ and $V \cap B = \emptyset$. By Definition 3.7, there exists a preopen set $W$ containing $x$ such that $W \cap (A \cup B) = \emptyset$. Namely, we have that $x \notin pCl_{\gamma_p}(A \cup B)$ and so $pCl_{\gamma_p}(A \cup B) \subseteq pCl_{\gamma_p}(A) \cup pCl_{\gamma_p}(B)$. We can obtain (vi) using (v). □

**Theorem 3.13** Let $\gamma_p : PO(X, \tau) \rightarrow P(X)$ be an operation on $PO(X, \tau)$ and $A$ and $B$ subsets of a topological space $(X, \tau)$. Then, we have the following properties on $PO(X)_{\gamma_p}\Cl(A)$ and $PO(X)_{\gamma_p}\Cl(B)$.

1. The set $PO(X)_{\gamma_p}\Cl(A)$ is a pre $\gamma_p$-closed set of $(X, \tau)$ and $A \subseteq PO(X)_{\gamma_p}\Cl(A)$.
2. $PO(X)_{\gamma_p}\Cl(\emptyset) = \emptyset$ and $PO(X)_{\gamma_p}\Cl(X) = X$.
3. $A$ subset $A$ is pre $\gamma_p$-closed (i.e., $X \setminus A$ is pre $\gamma_p$-open) in $(X, \tau)$ if and only if $PO(X)_{\gamma_p}\Cl(A) = A$ holds.
4. If $A \subseteq B$, then $PO(X)_{\gamma_p}\Cl(A) \subseteq PO(X)_{\gamma_p}\Cl(B)$.
5. $(PO(X)_{\gamma_p}\Cl(A)) \cup (PO(X)_{\gamma_p}\Cl(B)) \subseteq PO(X)_{\gamma_p}\Cl(A \cup B)$ holds.
Thus, we have that $U(\gamma)(vi)$ Cl$(\gamma)(vi)$ They are proved by (iv).

They are proved by (iv). (vi) Let $x \notin (PO(X)_{\gamma_p}-\text{Cl}(A)) \cup (PO(X)_{\gamma_p}-\text{Cl}(B))$. Then, there exist two pre $\gamma_p$-open sets $U$ and $V$ containing $x$ such that $U \cap A = \emptyset$ and $V \cap B = \emptyset$. By Theorem 3.8, it is proved that $U \cap V$ is $\gamma_p$-open in $(X, \tau)$ such that $(U \cap V) \cap (A \cup B) = \emptyset$. Thus, we have that $x \notin PO(X)_{\gamma_p}-\text{Cl}(A \cup B)$ and hence $PO(X)_{\gamma_p}-\text{Cl}(A \cup B) \subset (PO(X)_{\gamma_p}-\text{Cl}(A)) \cup (PO(X)_{\gamma_p}-\text{Cl}(B))$. Using (v), we have the equality. (viii) The proof is obvious from (i) and (iii).

For $\text{Cl}_{\gamma_p}(A)$ (cf. Definition 2.4(i)) and $\tau_{\gamma_p}\text{-Cl}(A)$ (cf. Definition 3.9(iii)), where $A$ is a subset of $(X, \tau)$, we have the following properties Theorem 3.14 and Theorem 3.15:

**Theorem 3.14** Let $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ be an operation on $PO(X, \tau)$ and $A$ and $B$ subsets of a topological space $(X, \tau)$. Then, we have the following properties on $\text{Cl}_{\gamma_p}(A)$ and $\text{Cl}_{\gamma_p}(B)$ (cf. Definition 2.4(i)).

(i) The set $\text{Cl}_{\gamma_p}(A)$ is a closed set of $(X, \tau)$ and $A \subset \text{Cl}_{\gamma_p}(A)$.

(ii) $\text{Cl}_{\gamma_p}(\emptyset) = \emptyset$ and $\text{Cl}_{\gamma_p}(X) = X$.

(iii) (Proposition 2.6(iv), Definition 3.9(iii)) $A$ subset $A$ is $\gamma_p$-closed (i.e., $X \setminus A$ is $\gamma_p$-open) in $(X, \tau)$ if and only if $A$ is $\gamma_p$-closed in $(X, \tau)$ (in the sense of Janković) (i.e., $\text{Cl}_{\gamma_p}(A) = A$ holds).

(iv) If $A \subset B$, then $\text{Cl}_{\gamma_p}(A) \subset \text{Cl}_{\gamma_p}(B)$.

(v) $\text{Cl}_{\gamma_p}(A) \cup \text{Cl}_{\gamma_p}(B) \subset \text{Cl}_{\gamma_p}(A \cup B)$ holds.

(vi) $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ is regular, then $\text{Cl}_{\gamma_p}(A) \cup \text{Cl}_{\gamma_p}(B) = \text{Cl}_{\gamma_p}(A \cup B)$ holds.

(vii) $\text{Cl}_{\gamma_p}(A \cap B) \subset \text{Cl}_{\gamma_p}(A) \cap \text{Cl}_{\gamma_p}(B)$ holds.

(viii) $\text{Cl}_{\gamma_p}(\tau_{\gamma_p}\text{-Cl}(A)) = \tau_{\gamma_p}\text{-Cl}(A)$ holds.

**Proof.** (i) By Remark 2.5 and [20, Theorem 3.6(i)], respectively, it is known that $\text{Cl}_{\gamma_p}(A) = \text{Cl}_{\gamma_p|\tau}(A)$ and every $\text{Cl}_{\gamma_p|\tau}(A)$ is closed in $(X, \tau)$ for any subset $A$ of $(X, \tau)$ and any operation $\gamma_p|\tau : \tau \to \mathcal{P}(X)$. (ii) (iv) They are obtained from Definition 2.4. (v) (vii) They are obtained from (iv). (vi) This follows from Remark 2.5, Theorem 3.8(ii) and [20, Lemma 3.10].

**Theorem 3.15** Let $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ be an operation on $PO(X, \tau)$ and $A$ and $B$ subsets of a topological space $(X, \tau)$. Then, we have the following properties on $\tau_{\gamma_p}\text{-Cl}(A)$ and $\tau_{\gamma_p}\text{-Cl}(B)$.

(i) The set $\tau_{\gamma_p}\text{-Cl}(A)$ is a $\gamma_p$-closed set of $(X, \tau)$ and $A \subset \tau_{\gamma_p}\text{-Cl}(A)$.

(ii) $\tau_{\gamma_p}\text{-Cl}(\emptyset) = \emptyset$ and $\tau_{\gamma_p}\text{-Cl}(X) = X$.

(iii) $A$ is $\gamma_p$-closed (i.e., $X \setminus A$ is $\gamma_p$-open) in $(X, \tau)$ if and only if $\tau_{\gamma_p}\text{-Cl}(A) = A$ holds.

(iv) If $A \subset B$, then $\tau_{\gamma_p}\text{-Cl}(A) \subset \tau_{\gamma_p}\text{-Cl}(B)$.

(v) $(\tau_{\gamma_p}\text{-Cl}(A)) \cup (\tau_{\gamma_p}\text{-Cl}(B)) \subset \tau_{\gamma_p}\text{-Cl}(A \cup B)$ holds.

(vi) $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ is regular, then $(\tau_{\gamma_p}\text{-Cl}(A)) \cup (\tau_{\gamma_p}\text{-Cl}(B)) = \tau_{\gamma_p}\text{-Cl}(A \cup B)$ holds.

(vii) $\tau_{\gamma_p}\text{-Cl}(A \cap B) \subset (\tau_{\gamma_p}\text{-Cl}(A)) \cap (\tau_{\gamma_p}\text{-Cl}(B))$ holds.

(viii) $\tau_{\gamma_p}\text{-Cl}(\tau_{\gamma_p}\text{-Cl}(A)) = \tau_{\gamma_p}\text{-Cl}(A)$ holds.
Proof. (i) By Proposition 2.6(iv)(vi) and Definition 3.9(iii), it is obtained that $\tau_{\gamma_p}$-$\text{Cl}(A)$ is a $\gamma_p$-closed set. (ii)-(iv) They are obtained from Definition 3.9(iii). (v) (vii) They are proved by using (iv). (vi) Let $x \notin (\tau_{\gamma_p}\text{-Cl}(A)) \cup (\tau_{\gamma_p}\text{-Cl}(B))$. Then, there exist two $\gamma_p$-open sets $U$ and $V$ containing $x$ such that $U \cap A = \emptyset$ and $V \cap B = \emptyset$. By Theorem 3.8(iii), it is proved that $U \cap V$ is $\gamma_p$-open in $(X, \tau)$ such that $(U \cap V) \cap (A \cup B) = \emptyset$. Thus, we have that $x \notin \tau_{\gamma_p}\text{-Cl}(A \cup B)$ and hence $\tau_{\gamma_p}\text{-Cl}(A \cup B) \subseteq (\tau_{\gamma_p}\text{-Cl}(A)) \cup (\tau_{\gamma_p}\text{-Cl}(B))$. Using (v), we have the equality. (viii) The proof is obvious from (i) and (iii). □

**Theorem 3.16** For a subset $A$ of a topological space $(X, \tau)$ and any operation $\gamma_p : PO(X, \tau) \rightarrow \mathcal{P}(X)$ an operation on $PO(X, \tau)$, the following relations hold.

(i) $p\text{Cl}(A) \subseteq p\gamma_p\text{-Cl}(A) \subseteq \tau_{\gamma_p}\text{-Cl}(A)$.

(ii) $p\text{Cl}(A) \subseteq \gamma_p\text{-Cl}(A) \subseteq \tau_{\gamma_p}\text{-Cl}(A)$.

**Proof.** (i) The implication that $p\text{Cl}(A) \subseteq p\gamma_p\text{-Cl}(A)$ is proved by Definition 3.10 and Definition 2.1: $p\gamma_p\text{-Cl}(A) \subseteq PO(X)\gamma_p\text{-Cl}(A)$ is proved by using Theorem 3.11 (i), Definition 3.1 and Definition 3.10: $PO(X)\gamma_p\text{-Cl}(A) \subseteq \tau_{\gamma_p}\text{-Cl}(A)$ is obtained by Theorem 3.3(ii) and Definition 3.9(iii). (ii) The implication that $p\gamma_p\text{-Cl}(A) \subseteq \gamma_p\text{-Cl}(A)$ is proved by a fact that $\tau \subseteq PO(X, \tau); \text{Cl}(A) \subseteq \tau_{\gamma_p}\text{-Cl}(A)$ is obtained by Definition 2.4: $\gamma_p\text{-Cl}(A) \subseteq \tau_{\gamma_p}\text{-Cl}(A)$ is proved by using Theorem 3.11(ii), Definition 2.3 and Definition 2.4(i). □

**Corollary 3.17** Let $A$ be a subset of a topological space $(X, \tau)$ and $\gamma_p : PO(X, \tau) \rightarrow \mathcal{P}(X)$ an operation on $PO(X, \tau)$.

(i) The following properties are equivalent:

(1) A subset $A$ is pre $\gamma_p$-open in $(X, \tau)$ (cf. Definition 3.1);

(2) $p\gamma_p\text{-Cl}(X - A) = X - A$;

(3) $PO(X)\gamma_p\text{-Cl}(X - A) = X - A$;

(4) $X - A$ is pre $\gamma_p$-closed in $(X, \tau)$ (cf. Definition 3.9(ii)).

(ii) The following properties are equivalent:

(1) A subset $A$ is $\gamma_p$-open in $(X, \tau)$ (cf. Definition 2.3);

(2) $\gamma_p\text{-Cl}(X - A) = X - A$;

(3) $\gamma_p\text{-Cl}(X - A) = X - A$;

(4) $X - A$ is $\gamma_p$-closed in $(X, \tau)$ (cf. Definition 3.9(ii)).

(5) $X - A$ is $\gamma_p\text{-Cl}(X, \tau)$ (cf. [20, Definition 2.2]).

**Proof.** (i) $(1) \iff (2)$ (resp. $(3) \iff (4)$) It is obtained by Theorem 3.12(iii) (resp. Theorem 3.13(iii)). $(4) \iff (1)$ This follows from Definition 3.1 and Definition 3.9(ii). (ii) $(1) \iff (2)$ (resp. $(3) \iff (4)$) This follows from Theorem 3.14(iii) (resp. Theorem 3.15(iii)). $(4) \iff (1)$ (resp. $(5) \iff (1)$) This follows from Definition 2.3 and Definition 3.9(i) (resp. Proposition 2.6(ii)(iii)). □

**Corollary 3.18** Let $A$ be a subset of a topological space $(X, \tau)$ and $\gamma_p : PO(X, \tau) \rightarrow \mathcal{P}(X)$ an operation on $PO(X, \tau)$.

(i) If $(X, \tau)$ is a pre $\gamma_p$-regular space, then $p\text{Cl}(A) = p\gamma_p\text{-Cl}(A) = PO(X)\gamma_p\text{-Cl}(A)$.

(ii) If $(X, \tau)$ is a $\gamma_p$-regular space, then $\text{Cl}(A) = p\gamma_p\text{-Cl}(A) = \tau_{\gamma_p}\text{-Cl}(A)$.

**Proof.** (i) By Theorem 3.6(i), it is shown that $p\text{Cl}(A) = PO(X)\gamma_p\text{-Cl}(A)$. Using Theorem 3.16(i), we have that $p\text{Cl}(A) = p\gamma_p\text{-Cl}(A)$. (ii) By Theorem 3.6(iii), it is shown that $\text{Cl}(A) = \tau_{\gamma_p}\text{-Cl}(A)$. Using Theorem 3.16(ii), we have that $\text{Cl}(A) = \gamma_p\text{-Cl}(A)$. □
In order to investigate the relationship among $PO(X)_{\gamma_p}$-$Cl(A)$, $pCl_{\gamma_p}(A)$, $\tau_{\gamma_p}$-$Cl(A)$ and $Cl_{\gamma_p}(A)$ for any set $A \in \mathcal{P}(X)$, we introduce the following notions of preopen operations and open operations (cf. [20, Definition 2.6, Example 2.7]):

**Definition 3.19** An operation $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ is called to be preopen (resp. open, cf. [20]) if for each point $x \in X$ and for every preopen set (resp. open set) $U$ containing $x$, there exists a pre $\gamma_p$-open set (resp. a $\gamma_p$-open set) $V$ such that $x \in V$ and $V \subset U^{\gamma_p}$ (cf. Remark 3.21 below for examples etc).

**Theorem 3.20** Let $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ be an operation on $PO(X, \tau)$ and $A$ be a subset of a topological space $(X, \tau)$.

(i) If $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ is a preopen operation, then $pCl_{\gamma_p}(A) = PO(X)_{\gamma_p}$-$Cl(A)$ and $pCl_{\gamma_p}(pCl_{\gamma_p}(A)) = pCl_{\gamma_p}(A)$ hold and $pCl_{\gamma_p}(A)$ is pre-$\gamma_p$-closed in $(X, \tau)$.

(ii) $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ is open if and only if $\gamma_p|_{\tau} : \tau \to \mathcal{P}(X)$ is open (in the sense of [20, Definition 4.4]).

(iii) (cf. Remark 3.21(v)) If $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ is an open operation, then $Cl_{\gamma_p}(A) = \tau_{\gamma_p}$-$Cl(A)$ and $Cl_{\gamma_p}(Cl_{\gamma_p}(A)) = Cl_{\gamma_p}(A)$ hold and $Cl_{\gamma_p}(A)$ is $\gamma_p$-closed in $(X, \tau)$.

**Proof.** (i) By Theorem 3.16(i) we have $pCl_{\gamma_p}(A) \subset PO(X)_{\gamma_p}$-$Cl(A)$. Suppose that $x \notin pCl_{\gamma_p}(A)$. Then, there exists a preopen set $U$ containing $x$ such that $U^{\gamma_p} \cap A = \emptyset$. Since $\gamma_p$ is preopen, by Definition 3.19, there exists a pre $\gamma_p$-open set $V$ such that $x \in V \subset U^{\gamma_p}$ and so $V \cap A = \emptyset$. By Theorem 3.11(i), $x \notin PO(X)_{\gamma_p}$-$Cl(A)$. Hence we have that $pCl_{\gamma_p}(A) = PO(X)_{\gamma_p}$-$Cl(A)$. Furthermore, using the above result and Theorem 3.13, we have that $pCl_{\gamma_p}(pCl_{\gamma_p}(A)) = PO(X)\gamma_p$-$Cl(PO(X)_{\gamma_p}$-$Cl(A)) = PO(X)\gamma_p$-$Cl(A)$ and so $pCl_{\gamma_p}(A)$ is pre-$\gamma_p$-closed in $(X, \tau)$. (ii) It is proved by Definition 3.19, Proposition 2.6(i) and a fact that $U^{\gamma_p} = U^{\gamma_p}_{|_{\tau}}$ holds for any open set $U$ of $(X, \tau)$. (iii) By Theorem 3.16(iii), we have $Cl_{\gamma_p}(A) \subset \tau_{\gamma_p}$-$Cl(A)$. Suppose that $x \notin Cl_{\gamma_p}(A)$. Then there exists an open set $U$ containing $x$ such that $U^{\gamma_p} \cap A = \emptyset$. Since $\gamma_p$ is an open operation, by Definition 3.19, there exists a $\gamma_p$-open set $V$ such that $x \in V \subset U^{\gamma_p}$ and so $V \cap A = \emptyset$. By Theorem 3.11(ii), $x \notin \tau_{\gamma_p}$-$Cl(A)$. Hence we have that $Cl_{\gamma_p}(A) = \tau_{\gamma_p}$-$Cl(A)$. Furthermore, using the above result and Theorem 3.15, we have that $Cl_{\gamma_p}(Cl_{\gamma_p}(A)) = \tau_{\gamma_p}$-$Cl(Cl_{\gamma_p}(A)) = \tau_{\gamma_p}$-$Cl(A) = Cl_{\gamma_p}(A)$ and $Cl_{\gamma_p}(A)$ is $\gamma_p$-closed in $(X, \tau)$. $\square$

**Remark 3.21** (i) Let $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$ be the “Int $\circ Cl$”-operation (cf. Example 3.2(iii)), where $(X, \tau)$ is any topological space. Then, it is shown that the operation $\gamma_p = “\text{Int} \circ Cl”$ is preopen (resp. open) on $PO(X, \tau)$. Indeed, let $x \in X$ and $U_x$ be a preopen (resp. open) set containing $x$. Put $G = \text{Int} Cl(U_x)$. Then it is shown that the set $G$ is a pre $\gamma_p$-open (resp. $\gamma_p$-open) and $x \in G \subset (U_x)^{\gamma_p}$, because $x \in \text{Int} Cl(U_x) = G = G^{\gamma_p} = \text{Int} Cl(\text{Int} Cl(U_x)))$ hold in $(X, \tau)$.

(ii) The identity operation $\gamma_p = “id” : PO(X, \tau) \to \mathcal{P}(X)$ is preopen and open.

(iii) If $(X, \tau)$ is a pre $\gamma_p$-regular (resp. a $\gamma_p$-regular) space for an operation $\gamma_p : PO(X, \tau) \to \mathcal{P}(X)$, then $\gamma_p$ is preopen (resp. open). Indeed, by Theorem 3.6(i) (2) $\Rightarrow$ (3) (resp. (iii) (2) $\Rightarrow$ (3)), Definition 2.1 and Definition 3.19, it is obtained.

(iv) The following example shows that the converse of (iii) above needs not be true. Let $(X, \tau)$ be a topological space, where $X := \{a, b, c\}$ and $\tau := \{\emptyset, X, \{a\}\}$. Define an operation $\gamma_p : PO(X) \to \mathcal{P}(X)$ as follows: $\gamma_p(A) := A$ if $b \notin A$, $\gamma_p(A) := pCl(A)$ if $b \notin A$. Then, we have that $PO(X, \tau) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$, $PO(X, \tau)_{\gamma_p} = \{\emptyset, \{a, b\}, X\}$.
and \( \tau_{\gamma_p} = \{\emptyset, X\} \). Since \( PO(X, \tau) \neq PO(X, \tau)_{\gamma_p} \) (resp. \( \tau \neq \tau_{\gamma_p} \)) holds, \((X, \tau)\) is not pre-\(\gamma_p\)-regular (resp. \(\gamma_p\)-regular), cf. Theorem 3.6 (i) (resp. (iii)). But we can check that the operation \( \gamma_p \) is preopen and open.

(v) We have a non-open operation \( \gamma_p \) and a property that \( Cl_{\gamma_p}(Cl_{\gamma_p}(A)) \neq Cl_{\gamma_p}(A) \) for some subset \( A \) of a topological space \((X, \tau)\) (cf. Theorem 3.20(iii)). Let \((X, \tau)\) be a topological space, where \( X := \{a, b, c\} \) and \( \tau := \{\emptyset, \{a\}, \{b\},\{a, b\}, X\} \). Define \( \gamma_p: PO(X, \tau) \to P(X) \) by \( A^p := Cl(A) \) for any \( A \in PO(X, \tau) \). Then, \( \tau_{\gamma_p} = \{\emptyset, X\} = PO(X, \tau)_{\gamma_p} \) and \( \gamma_p \) is not open and it is also not preopen. For a subset \( \{a\} \) of \((X, \tau)\), \( Cl_{\gamma_p}(Cl_{\gamma_p}(\{a\})) = Cl_{\gamma_p}(\{a, c\}) = X \neq Cl_{\gamma_p}(\{a\}) = \{a, c\} \).

Corollary 3.22 Let \("Int \circ Cl": PO(X, \tau) \to P(X)\) be the Interior-closure operation and \("id": PO(X, \tau) \to P(X)\) the identity operation on \( PO(X, \tau) \). Then, we have the following properties:

(i) \( PO(X, \tau)^{\circ Int\circ Cl} = \tau^{\circ Int\circ Cl} = \tau_{\delta} \);
(ii) \( PCl^{\circ Int\circ Cl}(A) = PO(X)^{\circ Int\circ Cl} - Cl(A) = \delta_{\circ \delta}(A) = Cl_{\delta}(A) = \delta^{\circ Int\circ Cl}(A) \) hold for any subset \( A \) of \((X, \tau)\).

Proof. (i) It follows from Example 3.2(iii) that \( PO(X, \tau)^{\circ Int\circ Cl} = \tau^{\circ Int\circ Cl} = \tau_{\delta} \) and so \( \tau_{\circ \delta} - Cl(A) = PO(X)^{\circ Int\circ Cl} - Cl(A) = \tau_{\delta} - Cl(A) \) for a subset \( A \) of \((X, \tau)\). Since \("Int \circ Cl" \) and \("id" \) are preopen and also open (cf. Example 3.21(i)), we have that \( PO(X, \tau)^{\circ Int\circ Cl} - Cl(A) = PCl^{\circ Int\circ Cl}(A) \) (cf. Theorem 3.20(i)), \( \tau_{\circ \delta} - Cl(A) = Cl_{\delta}(A) \) (cf. Theorem 3.20(iii)) and, by definitions, \( Cl_{\delta}(A) = Cl_{\circ \delta}(A) \), where \( A \) is a subset of \((X, \tau)\). Therefore, we have the required equalities. (ii) We have that \( PO(X, \tau)^{\circ id} = PO(X, \tau) \) and \( \tau^{\circ id} = \tau \) (cf. Example 3.2(i)). Since \("id" \) is preopen and also open (cf. Remark 3.21(ii)), we have that \( PCl^{\circ id}(A) = PO(X)^{\circ id} - Cl(A) = \tau_{\delta} \) (resp. \( Cl_{\delta}(A) \)) for any subset \( A \) of \((X, \tau)\) (cf. Theorem 3.20(i)). \( \square \)

4 Pre-\(\gamma_p\)-generalized closed sets and pre-\(\gamma_p\)-T\(_i\) spaces, where \( i=0, 1/2, 1 \) or 2

Throughout this section, let \( \gamma_p: PO(X, \tau) \to P(X) \) be an operation on \( PO(X, \tau) \).

Definition 4.1 Let \( A \) be a subset of a topological space \((X, \tau)\).

(i) A subset \( A \) is said to be pre-\(\gamma_p\)-generalized closed (shortly, pre-\(\gamma_p\)-g.closed) in \((X, \tau)\) if \( PCl_{\gamma_p}(A) \subset U \) whenever \( A \subset U \) and \( U \) is a pre-\(\gamma_p\)-open set of \((X, \tau)\).

(ii) (cf. [20, Definition 4.4]) A subset \( A \) is said to be pre-\(\gamma_p\)-generalized closed (shortly, pre-\(\gamma_p\)-g.closed) in \((X, \tau)\) if \( Cl_{\gamma_p}(A) \subset U \) whenever \( A \subset U \) and \( U \) is a pre-\(\gamma_p\)-open in \((X, \tau)\).

(iii) A subset \( A \) of \((X, \tau)\) is said to be pre-\(\gamma_p\)-open (resp. pre-\(\gamma_p\)-g.open) in \((X, \tau)\) if the complement \( X \setminus A \) is pre-\(\gamma_p\)-open (resp. pre-\(\gamma_p\)-g.closed) in \((X, \tau)\).

Theorem 4.2 Let \( \gamma_p: PO(X; \tau) \to P(X) \) be an operation and \( A \) a subset of a topological space \((X, \tau)\).

(i) The following properties are equivalent:

1. A subset \( A \) is pre-\(\gamma_p\)-g.closed in \((X, \tau)\);
2. \( PO(X)^{\gamma_p} - Cl(A) = \emptyset \) for every \( x \in PCl_{\gamma_p}(A) \);
3. \( PCl_{\gamma_p}(A) \subset PO(X)^{\gamma_p} - Ker(E) \) holds, where \( PO(X)^{\gamma_p} - Ker(E) = \{V \mid E \subset V, V \in PO(X, \tau)_{\gamma_p}\} \) for any subset \( E \) of \((X, \tau)\).

(ii) (cf. [20, Proposition 4.6] [15, Proposition 4.5]) The following properties are equivalent:
(1) A subset $A$ is $\gamma_p$-g.closed in $(X,\tau)$;
(2) $(\tau_{\gamma_p}\text{-Cl}(\{x\})) \cap A \neq \emptyset$ for every $x \in \text{Cl}_{\gamma_p}(A)$;
(3) $\tau_{\gamma_p}(A) \subset \tau_{\gamma_p}\text{-Ker}(A)$ holds, where $\tau_{\gamma_p}\text{-Ker}(E) := \bigcap\{V | E \subset V, V \in \tau_{\gamma_p}\}$ for any subset $E$ of $(X,\tau)$.

(iii) A subset $A$ is $\gamma_p$-g.closed in $(X,\tau)$ if and only if $A$ is $\gamma_p|\tau$-g.closed in $(X,\tau)$, where $\gamma_p|\tau$ is the restriction of $\gamma_p$ onto $\tau$ (cf. Remark 2.2(iii)).

Proof. (i) (1)$\Rightarrow$(2) Let $A$ be a pre $\gamma_p$-g.closed set of $(X,\tau)$. Suppose that there exists a point $x \in \text{pCl}_{\gamma_p}(A)$ such that $(PO(X)\gamma_p\text{-Cl}(\{x\})) \cap A = \emptyset$. By Theorem 3.13(i), $PO(X)\gamma_p\text{-Cl}(\{x\})$ is a pre $\gamma_p$-closed. Put $U := X \setminus (PO(X)\gamma_p\text{-Cl}(\{x\}))$. Then, we have that $A \subset U$, $x \notin U$ and $U$ is a pre $\gamma_p$-open set of $(X,\tau)$. Since $A$ is a pre $\gamma_p$-g.closed set, $\text{pCl}_{\gamma_p}(A) \subset U$. Thus, we have that $x \notin \text{pCl}_{\gamma_p}(A)$. This is a contradiction. (2)$\Rightarrow$(3)

Let $x \in \text{pCl}_{\gamma_p}(A)$. By (2), there exists a point $z$ such that $z \in (PO(X)\gamma_p\text{-Cl}(\{x\}))$ and $z \in A$. Let $U \in PO(X)\gamma_p$ be a subset of $X$ such that $A \subset U$. Since $z \in U$ and $z \in (PO(X)\gamma_p\text{-Cl}(\{x\}))$, we have that $U \cap \{x\} \neq \emptyset$. Namely, we show that $x \in PO(X)\gamma_p\text{-Ker}(A)$. Therefore, we prove that $\text{pCl}_{\gamma_p}(A) \subset PO(X)\gamma_p\text{-Ker}(A)$. (3)$\Rightarrow$(1) Let $U$ be any pre $\gamma_p$-open set such that $A \subset U$. Let $x$ be a point such that $x \in \text{pCl}_{\gamma_p}(A)$. By (3), $x \in PO(X)\gamma_p\text{-Ker}(A)$. Namely, we have that $x \in U$, because $A \subset U$ and $U \subset PO(X)\gamma_p$. (ii) (1)$\Rightarrow$(2) Let $A$ be a pre $\gamma_p$-g.closed set of $(X,\tau)$. Suppose that there exists a point $x \in \text{Cl}_{\gamma_p}(A)$ such that $(\tau_{\gamma_p}\text{-Cl}(\{x\})) \cap A = \emptyset$. By Theorem 3.15(i), $\tau_{\gamma_p}\text{-Cl}(\{x\})$ is $\gamma_p$-closed. Put $U := X \setminus \tau_{\gamma_p}\text{-Cl}(\{x\})$. Then, we have that $A \subset U$, $x \notin U$ and $U$ is a $\gamma_p$-open set of $(X,\tau)$. Since $A$ is a pre $\gamma_p$-g.closed set, $\text{Cl}_{\gamma_p}(A) \subset U$. Thus, we have that $x \notin \text{Cl}_{\gamma_p}(A)$. This is a contradiction. (2)$\Rightarrow$(3) Let $x \in \text{Cl}_{\gamma_p}(A)$. By (2), there exists a point $z$ such that $z \in \tau_{\gamma_p}\text{-Cl}(\{x\})$ and $z \in A$. Let $U \in \tau_{\gamma_p}$ be a subset of $X$ such that $A \subset U$. Since $z \in U$ and $z \in \tau_{\gamma_p}\text{-Cl}(\{x\})$, we have that $U \cap \{x\} \neq \emptyset$. Namely, we show that $x \in \tau_{\gamma_p}\text{-Ker}(A)$ for any point $x \in \text{Cl}_{\gamma_p}(A)$ and so $\text{Cl}_{\gamma_p}(A) \subset \tau_{\gamma_p}\text{-Ker}(A)$.

(iii) It is obtained by Definition 4.1, Remark 2.5, Proposition 2.6(i) and [20, Definition 4.4].

**Theorem 4.3** Let $A$ be a subset of a topological space $(X,\tau)$.

(i) If $A$ is pre $\gamma_p$-g.closed in $(X,\tau)$, then $\text{pCl}_{\gamma_p}(A) \setminus A$ does not contain any non-empty pre $\gamma_p$-closed set.

(ii') If $\gamma_p : PO(X,\tau) \rightarrow P(X)$ is a preopen operation (cf. Definition 3.19), then the converse of (i) is true.

(ii) (cf. [20, Remark 4.8] [15, Proposition 4.6(ii)]) If $A$ is $\gamma_p$-g.closed in $(X,\tau)$, then $\text{Cl}_{\gamma_p}(A) \setminus A$ does not contain any non-empty $\gamma_p$-closed set.

(ii') If $\gamma_p : PO(X,\tau) \rightarrow P(X)$ is an open operation (cf. Definition 3.19), then the converse of (ii) is true.

**Proof.** (i) Suppose that there exists a pre $\gamma_p$-closed set $F$ such that $F \subset \text{pCl}_{\gamma_p}(A) \setminus A$. Then, we have that $A \subset X \setminus F$ and $X \setminus F$ is pre $\gamma_p$-open. It follows from assumption that $\text{pCl}_{\gamma_p}(A) \subset X \setminus F$ and so $F \subset (\text{pCl}_{\gamma_p}(A) \setminus A) \cap (X \setminus \text{pCl}_{\gamma_p}(A))$. Therefore, we have that $F = \emptyset$. (i') Let $U$ be a pre $\gamma_p$-open set such that $A \subset U$. Since $\gamma_p$ is a preopen operation, it follows from Theorem 3.20(i) that $\text{pCl}_{\gamma_p}(A)$ is pre $\gamma_p$-closed in $(X,\tau)$. Thus, using Theorem 3.3(iii), Definition 3.9(ii) we have that $\text{pCl}_{\gamma_p}(A) \setminus (X \setminus U)$, $F$, is a pre $\gamma_p$-closed in $(X,\tau)$. Since $X \setminus U \subset X \setminus A$, $F \subset \text{pCl}_{\gamma_p}(A) \setminus A$. Using the assumption of the converse of (i) above, $F = \emptyset$ and hence $\text{pCl}_{\gamma_p}(A) \subset U$. (ii) Suppose that there exists a $\gamma_p$-closed set $F$ such that $F \subset \text{Cl}_{\gamma_p}(A) \setminus A$. Then, we have that $A \subset X \setminus F$ and $X \setminus F$ is
\(\gamma_p\)-open. It follows from assumption that \(\text{Cl}_{\gamma_p}(A) \subset X \setminus F\) and so \(F \subset (\text{Cl}_{\gamma_p}(A) \setminus A) \cap (X \setminus \text{Cl}_{\gamma_p}(A))\). Therefore, we have \(F = \emptyset\). (ii) Let \(U\) be a \(\gamma_p\)-open set such that \(A \subset U\). Since \(\gamma_p\) is an open operation, it follows from Theorem 3.20(iii) that \(\text{Cl}_{\gamma_p}(A)\) is \(\gamma_p\)-closed in \((X, \tau)\). Thus, using Proposition 2.6(vi), Corollary 3.17(iii) and Definition 3.9(i), we have that \(\text{Cl}_{\gamma_p}(A) \cap (X \setminus U)\), say \(F\), is a \(\gamma_p\)-closed in \((X, \tau)\). Since \(X \setminus U \subset X \setminus A, F \subset \text{Cl}_{\gamma_p}(A) \setminus A\). Using the assumption of the converse of (ii) above, \(F = \emptyset\) and hence \(p\text{Cl}_{\gamma_p}(A) \subset A\). \(\square\)

We define the following new classes of topological spaces called as \(\gamma_p\)-\(T_{1/2}\) spaces and pre \(\gamma_p\)-\(T_{1/2}\) spaces. We recall that every \(\gamma_p\)-closed (resp. pre \(\gamma_p\)-closed) set is \(\gamma_p\)-g.closed (resp. pre \(\gamma_p\)-g.closed) (cf. Definition 4.1).

**Definition 4.4** (i) A topological space \((X, \tau)\) is said to be a pre \(\gamma_p\)-\(T_{1/2}\) space if every pre \(\gamma_p\)-g.closed set of \((X, \tau)\) is pre \(\gamma_p\)-closed.

(ii) (cf. [20, Definition 4.5]) A topological space \((X, \tau)\) is said to be a \(\gamma_p\)-\(T_{1/2}\) space if every \(\gamma_p\)-g.closed set of \((X, \tau)\) is \(\gamma_p\)-closed.

We prove a lemma needed later.

**Lemma 4.5** For any operation \(\gamma_p : PO(X, \tau) \to P(X)\), the following properties hold.

(i) For each point \(x \in X\), \(\{x\}\) is pre \(\gamma_p\)-closed or \(X \setminus \{x\}\) is pre \(\gamma_p\)-g.closed in a topological space \((X, \tau)\).

(ii) (cf. [20, Proposition 4.9]) For each point \(x \in X\), \(\{x\}\) is \(\gamma_p\)-closed or \(X \setminus \{x\}\) is \(\gamma_p\)-g.closed in a topological space \((X, \tau)\).

**Proof.** (i) Suppose that \(\{x\}\) is not a pre \(\gamma_p\)-closed set. Then, by Corollary 3.17 (or definitions), \(X \setminus \{x\}\) is not a pre \(\gamma_p\)-open set. Let \(U\) be any pre \(\gamma_p\)-open set such that \(X \setminus \{x\} \subset U\). Then, \(U = X\) and so we have that \(p\text{Cl}_{\gamma_p}(X \setminus \{x\}) \subset U\). Therefore, \(X \setminus \{x\}\) is a pre \(\gamma_p\)-g.closed set in \((X, \tau)\). (ii) Suppose that \(\{x\}\) is not \(\gamma_p\)-closed in \((X, \tau)\). By Definition 3.9(i), \(X \setminus \{x\}\) is not \(\gamma_p\)-open. Then, it is shown that \(X \setminus \{x\}\) is \(\gamma_p\)-g.closed. \(\square\)

**Theorem 4.6** (i) A topological space \((X, \tau)\) is pre \(\gamma_p\)-\(T_{1/2}\) if and only if, for each point \(x \in X\), \(\{x\}\) is pre \(\gamma_p\)-open or pre \(\gamma_p\)-closed in \((X, \tau)\).

(ii) (cf. [20, Proposition 4.10]) The following properties on a topological space \((X, \tau)\) are equivalent:

1. \((X, \tau)\) is \(\gamma_p\)-\(T_{1/2}\);
2. For each point \(x \in X\), \(\{x\}\) is \(\gamma_p\)-open or \(\gamma_p\)-closed in \((X, \tau)\);
3. \((X, \tau)\) is \(\gamma_p\)-\(T_{1/2}\) in the sense of Ogata [20, Definition 4.5].

**Proof.** (i) (Necessity) Suppose that \(\{x\}\) is not a pre \(\gamma_p\)-closed set, by Lemma 4.5 (i), \(X \setminus \{x\}\) is a pre \(\gamma_p\)-g.closed set. Since \((X, \tau)\) is a pre \(\gamma_p\)-\(T_{1/2}\) space, this implies that \(X \setminus \{x\}\) is pre \(\gamma_p\)-closed. Hence \(\{x\}\) is a pre \(\gamma_p\)-open set. (Sufficiency) Let \(F\) be a pre \(\gamma_p\)-g.closed set in \((X, \tau)\). We shall prove that \(p\text{Cl}_{\gamma_p}(F) = F\) (cf. Corollary 3.17(i)). It is sufficient to show that \(p\text{Cl}_{\gamma_p}(F) \subset F\). Assume that there exits a point \(x\) such that \(x \in p\text{Cl}_{\gamma_p}(F) \setminus F\). Then, by assumption, \(\{x\}\) is pre \(\gamma_p\)-closed or pre \(\gamma_p\)-open. **Case 1.** \(\{x\}\) is a pre \(\gamma_p\)-closed set: For this case, we have a pre \(\gamma_p\)-closed set \(\{x\}\) such that \(\{x\} \subset p\text{Cl}_{\gamma_p}(F) \setminus F\). This is a contradiction to Theorem 4.3(i). **Case 2.** \(\{x\}\) is a pre \(\gamma_p\)-open set: Using Theorem 3.16(i), we have \(x \in PO(X)\gamma_p - \text{Cl}(F)\). Since \(\{x\}\) is pre \(\gamma_p\)-open, it implies that \(\{x\} \cap F \neq \emptyset\) (cf. Theorem 3.11(i)). This is a contradiction. Thus, we have that...
pClγp(F) = F and so, by Corollary 3.17(i), F is pre γp-closed. (ii) (1)⇒(2) Suppose that \( \{ x \} \) is not a γp-closed set. By Lemma 4.5(ii), \( X \setminus \{ x \} \) is a γp-g.closed set. Thus, we have that \( X \setminus \{ x \} \) is γp-closed (i.e., \( \{ x \} \) is a γp-open set, cf. Definition 3.9(i)). (2)⇒(1) Let F be a γp-g.closed set in \( (X, \tau) \). We claim that CLγp(F) = F (cf. Corollary 3.17(ii)). Assume that there exists a point x such that \( \{ x \} \subset CL_{\gamma_p}(F) \setminus F \). Case 1. \( \{ x \} \) is a γp-closed set: For this case, we have a contradiction to Theorem 4.3(ii). Case 2. \( \{ x \} \) is a γp-open set: Using Theorem 3.16(ii), we have \( x \in \tau_{\gamma_p}-\text{Cl}(F) \). This shows that \( \{ x \} \cap F \neq \emptyset \) (cf. Theorem 3.11(ii)). This is a contradiction. Thus, we have CLγp(F) = F. Hence, using Corollary 3.17(ii) every γp-g.closed set is γp-closed. (1)⇔(3) This follows from Theorem 4.2(iii), Corollary 3.17(ii), Definition 4.4(ii) and [20, Definition 4.5]. □

**Remark 4.7** By Theorem 4.6(ii) and Proposition 2.6(i), it is obtained that a topological space \( (X, \tau) \) is γp|\( \tau_1 \tau_2 \) (in the sense of Ogata)[20] if and only if for each point \( x \in X, \{ x \} \) is γ|\( \tau \)-open or γ|\( \tau \)-closed in \( (X, \tau) \). Therefore, this shows that the regularity on \( \gamma \) in [20, Proposition 4.10(ii)] can be omitted (cf. [25, Corollary 4.12 (ii)⇔(iii), Remark 4.13], [15, Remark 5.14 (ii) (5.15)]).

In the end of this section, we introduce further new operation-separation axioms on topological spaces called as pre-γp-\( T_i \), where \( i \in \{ 0, 1, 2 \} \).

**Definition 4.8** A topological space \( (X, \tau) \) is said to be

(a) a pre γp-T0 space if for any two distinct points \( x, y \in X \), there exists a pre open set \( U \) such that either \( x \in U \) and \( y \notin U^{\gamma_p} \) or \( y \in U \) and \( x \notin U^{\gamma_p} \); (a)' a pre γp-T0′ space if for any two distinct points \( x, y \in X \), there exists a pre γp-open set \( U \) such that either \( x \in U \) and \( y \notin U^{\gamma_p} \) or \( y \in U \) and \( x \notin U^{\gamma_p} \); (b) a pre γp-T1 space if for any two distinct points \( x, y \in X \), there exists two pre open sets \( U \) and \( V \) containing \( x \) and \( y \), respectively, such that \( y \notin U^{\gamma_p} \) and \( x \notin V^{\gamma_p} \); (b)' a pre γp-T1′ space if for any two distinct points \( x, y \in X \), there exists two pre γp-open sets \( U \) and \( V \) containing \( x \) and \( y \), respectively, such that \( y \notin U^{\gamma_p} \) and \( x \notin V^{\gamma_p} \); (c) a pre γp-T2 space if for any two distinct points \( x, y \in X \), there exists two pre open sets \( U \) and \( V \) such that \( x \in U \), \( y \in V \) and \( U^{\gamma_p} \cap V^{\gamma_p} = \emptyset \); (c)' a pre γp-T2′ space if for any two distinct points \( x, y \in X \), there exists two pre γp-open sets \( U \) and \( V \) such that \( x \in U \), \( y \in V \) and \( U \cap V = \emptyset \).

**Theorem 4.9** (i) A topological space \( (X, \tau) \) is a pre γp-T0′ space if and only if, for every pair \( x, y \in X \) with \( x \neq y \), \( PO(X)_{\gamma_p}-\text{Cl}(\{x\}) \neq PO(X)_{\gamma_p}-\text{Cl}(\{y\}) \) holds.

(ii) Suppose that \( \gamma_p : PO(X, \tau) \rightarrow \mathcal{P}(X) \) is preopen. A topological space \( (X, \tau) \) is a pre γp-T0′ space if and only if, for every pair \( x, y \in X \) with \( x \neq y \), \( p\text{Cl}_{\gamma_p}(\{x\}) \neq p\text{Cl}_{\gamma_p}(\{y\}) \) holds.

(iii) Suppose that \( \gamma_p : PO(X, \tau) \rightarrow \mathcal{P}(X) \) is preopen. A topological space \( (X, \tau) \) is pre γp-T0 if and only if \( (X, \tau) \) is pre γp-T0′.

**Proof.** (i) (Necessity) Let \( x \) and \( y \) be any two distinct points of a pre γp-T0′ space \( (X, \tau) \). Then, by definition, we assume that there exists a pre γp-open set \( U \) such that \( x \in U \) and \( y \notin U \). Hence \( y \in X \setminus U \). Because \( X \setminus U \) is a pre γ-closed set, we obtain that \( PO(X)_{\gamma_p}-\text{Cl}(\{y\}) \subset X \setminus U \) (cf. Theorem 3.13(iii)(iv)) and so \( PO(X)_{\gamma_p}-\text{Cl}(\{x\}) \neq PO(X)_{\gamma_p}-\text{Cl}(\{y\}) \). (Sufficiency) Suppose that for any \( x, y \in X, x \neq y \). Then we have \( PO(X)_{\gamma_p}-\text{Cl}(\{x\}) \neq PO(X)_{\gamma_p}-\text{Cl}(\{y\}) \). Thus we assume that there exists
z ∈ PO(X)γ_p-Cl(\{x\}) but z ∉ PO(X)γ_p-Cl(\{y\}). We shall prove that x ∉ PO(X)γ_p-Cl(\{y\}). Indeed, if x ∈ PO(X)γ_p-Cl(\{y\}), then we get PO(X)γ_p-Cl(\{x\}) ⊂ PO(X)γ_p-Cl(\{y\}) (cf. Theorem 3.13(iv)(viii)). This implies that z ∈ PO(X)γ_p-Cl(\{y\}). This contradiction shows that X \ PO(X)γ_p-Cl(\{y\}) is a pre γ_p-open set containing x but not y (cf. Theorem 3.13(i)). Hence (X, τ) is a pre γ_p-T_0 space. (ii) (Necessity) Let x and y be any two distinct points of a pre γ_p-T_0 space (X, τ). Then, by definition, we assume that there exists a preopen set U such that x ∈ U and y ∉ Uγ_p. It follows from assumption that there exists a pre γ_p-open set S such that x ∈ S and S ⊂ Uγ_p. Hence, y ∈ X \ Uγ_p ⊂ X \ S. Because X \ S is a pre γ_p-closed set, we obtain that pClγ_p(\{x\})⊂ X \ S and so pClγ_p(\{x\})≠pClγ_p(\{y\}). (Sufficiency) Suppose that x ≠ y for any x, y ∈ X. Then, we have that pClγ_p(\{x\})≠pClγ_p(\{y\}). Thus, we assume that there exists z ∈ pClγ_p(\{x\}) but z ∉ pClγ_p(\{y\}). If x ∈ pClγ_p(\{y\}), then we get pClγ_p(\{x\})⊂ pClγ_p(\{y\}) (cf. Theorem 3.20(i)). This implies that z ∈ pClγ_p(\{y\}). This contradiction shows that x ∉ pClγ_p(\{y\}), i.e., by Definition 3.10, there exists a preopen set W such that x ∈ W and Wγ_p ∩ \{y\} = ∅. Thus, we have that x ∈ W and y ∉ Wγ_p. Hence, (X, τ) is a pre γ_p-T_0. (iii) This follows from (i), (ii) above and a fact that, for any subset A of (X, τ), PO(X)γ_p-Cl(A) =pClγ_p(A) holds under the assumption that γ_p is preopen (cf. Theorem 3.20(i)). □

**Theorem 4.10** For a topological space (X, τ) and an operation γ_p : PO(X, τ) → P(X), the following properties hold.

(i) The following properties are equivalent:

1. \( (X, \tau) \) is pre γ_p-T_1;
2. For every point \( x \in X \), \( \{x\} \) is a pre γ_p-closed set;
3. \( (X, \tau) \) is pre γ_p-T_1.
4. Every pre γ_p-T_1 space is pre γ_p-T_i, where \( i \in \{1, 2\} \).
5. Every pre γ_p-T_1 space is pre γ_p-T_i, where \( i \in \{1\} \).
6. Every γ_p-T_1 space (in the sense of Ogata) is pre γ_p-T_i, where \( i \in \{1, 1/2\} \).
7. Every γ_p-T_1 space is pre γ_p-T_i, where \( i \in \{1\} \).

**Proof.** (i) (1)⇒(2) Let x ∈ X be a point. For each point \( y \in X \setminus \{x\} \), there exists a preopen set \( U_y \) such that \( y \in U_y \) and \( x ∉ \cup_{\{y\}} U_y \). Then, \( X \setminus \{x\} = \{ U_y \} \cup \{ y \in X \setminus \{x\} \} \). It is shown that \( X \setminus \{x\} \) is pre γ_p-open in (X, τ). (2)⇒(3) Let x and y be any distinct points of X. By (2), \( X \setminus \{x\} \) and \( X \setminus \{y\} \) are the required pre γ_p open sets such that \( y ∈ X \setminus \{x\}, x ∉ X \setminus \{y\} \) and \( x ∈ X \setminus \{x\}, x ∉ X \setminus \{y\} \). (3)⇒(1) It is shown that if \( x ∈ U \) where \( U ∈ PO(X, \tau) \), then there exists a preopen set \( S \) such that \( x ∈ S \setminus \gamma_p \subset U \). Using (3), we have that (X, τ) is pre γ_p-T_1. (ii) (iii) (vii) The proofs are obvious by Definition 4.8. (iv) This follows from (i) above and Theorem 4.6(i). (v) This follows from Theorem 4.6(i) and Definition 4.8(a)′. (vi) For an open set \( U \) of (X, τ), \( U^{\gamma_p} = U^{\gamma_p} \) and \( U ∈ PO(X, \tau) \) hold. Thus, the proofs of (vi) for \( i \in \{2, 1\} \) are obvious from definitions (cf. [20, Definitions 4.1, 4.2, 4.3], Definition 4.8). The proof for \( i = 1/2 \) is obtained by Remark 4.7, Proposition 2.6(i), Theorem 3.3(ii) and Theorem 4.6(i). □

**Remark 4.11** By Theorem 4.10, [20], [21] and [25, Proposition 5.8], we obtain the following diagram of implications. Moreover, the following Examples 4.12, 4.13, 4.14, 4.15, 4.16, 4.17 below and [21] [25, Section 5] show that some of these implications are not reversible.
Example 4.12 The converse of Theorem 4.10(iii) is not true in general. Let \((X, \tau)\) be the double origin topological space, where \(X := \mathbb{R}^2 \cup \{O^*\}\) and \(O^*\) denotes an additional point (eg., [24, p.92]). Let \(\gamma_p : PO(X, \tau) \rightarrow \mathcal{P}(X)\) be the closure operation, i.e., \(U^\gamma := \text{Cl}(U)\) for every preopen set \(U\) of \((X, \tau)\). We first prove that \((X, \tau)\) is not pre \(\gamma_p\)-\(T_2\). Let \(U\) be a preopen set containing \(O := (0,0) \in \mathbb{R}^2\) and \(V\) be a preopen set containing \(O^*\). Then, \(O \in U \subset \text{Int}(\text{Cl}(U))\) and \(O^* \in V \subset \text{Int}(\text{Cl}(V))\) hold in \((X, \tau)\). By the definition of \(\tau\), there exists an open neighbourhood of \(O\), say \(B_+^\tau(O) := \{O\} \cup \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < \varepsilon^2, y > 0\}\) for some positive real number \(\varepsilon\), such that \(B_+^\tau(O) \subset \text{Cl}(U)\). Similarly, for the point \(O^*\), there exists an open neighbourhood of \(O^*\), say \(B_-^\tau(O^*) := \{O^*\} \cup \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < \delta^2, y < 0\}\) for some positive real number \(\delta\), such that \(B_-^\tau(O^*) \subset \text{Cl}(V)\). Then, we have that \(\text{Cl}(B_+^\tau(O)) \cap \text{Cl}(B_-^\tau(O^*)) = \{O\} \cup \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < \min(\varepsilon, \delta)\} \neq \emptyset\) and \(\text{Cl}(B_+^\tau(O)) \cap \text{Cl}(B_-^\tau(O^*)) \subset U^\gamma \cap V^\gamma \neq \emptyset\). This shows that \((X, \tau)\) is not pre \(\gamma_p\)-\(T_2\) for the closure operation \(\gamma_p\). Finally, we have that \((X, \tau)\) is pre \(\gamma_p\)-\(T_1\). Indeed, \(\gamma_p : PO(X, \tau) \rightarrow \mathcal{P}(X)\) is the closure operation on \(PO(X, \tau)\) if and only if \(\gamma_p|\tau : \tau \rightarrow \mathcal{P}(X)\) is the closure operation on \(\tau\) (eg., [8], [20], [21]). Then, it follows from [21, (a) in Proof of Theorem 1] that the space \((X, \tau)\) is \(\gamma_p|\tau\)-\(T_2\) for the closure operation \(\gamma_p|\tau\) on \(\tau\). By Theorem 4.10(vi), it is obtained that \((X, \tau)\) is not pre \(\gamma_p\)-\(T_1\).

Example 4.13 The following example shows that \((X, \tau)\) is pre \(\gamma_p\)-\(T_0\); this is not pre \(\gamma_p\)-\(T_{1/2}\). Let \((X, \tau)\) be the double origin space of Example 4.12. We note that \(\text{Int}(\text{Cl}(\{O^*\})) = \emptyset\) holds in \((X, \tau)\). Let \(\gamma_p : PO(X, \tau) \rightarrow \mathcal{P}(X)\) be an operation defined by \(\gamma_p(A) := A \cup \{O^*\}\) for every set \(A \in PO(X, \tau)\) (cf. [25, Example 5.11]). First we show that \((X, \tau)\) is not pre \(\gamma_p\)-\(T_{1/2}\) for this operation \(\gamma_p\). The singleton \(\{O^*\}\) is neither pre \(\gamma_p\)-open nor pre \(\gamma_p\)-closed in \((X, \tau)\). Indeed, supposed that \(\{O^*\}\) is pre \(\gamma_p\)-open. There exists a preopen set \(U\) such that \(O^* \in U \subset U^\gamma \subset \{O^*\}\). Then, we have that \(U = \{O^*\} \subset \text{Int}(\text{Cl}(\{O^*\}))\). This shows a contradiction that \(\{O^*\} = \emptyset\). Suppose that \(\{O^*\}\) is pre \(\gamma_p\)-closed. For an origin \(O = (0,0) \in \mathbb{R}^2 \subset X \setminus \{O^*\}\), there exists a preopen set \(V\) such that \(O \in V\) and \(V^\gamma \subset X \setminus \{O^*\}\). We have also a contradiction that \(O^* \subset X \setminus \{O^*\}\). By Theorem 4.6 (i), it is shown that \((X, \tau)\) is not pre \(\gamma_p\)-\(T_{1/2}\) for this operation \(\gamma_p\).

Finally, we show that \((X, \tau)\) is pre \(\gamma_p\)-\(T_0\). Indeed, let \(x\) and \(y\) be distinct points of \(X\). In the below argument, let \(d(z, w)\) denote the distance of two point \(z\) and \(w\) of the Euclidean plane \(\mathbb{R}^2\).

**Case 1.** \(x, y \notin \{O, O^*\}\): Let \(\varepsilon\) be a positive real number such that \(\varepsilon < (1/2)d(x, y)\). Then, there exists a subset \(B_\varepsilon(x) \in PO(X, \tau)\) such that \(x \in B_\varepsilon(x)\) and \(y \notin B_\varepsilon(x)^\gamma\), where \(B_\varepsilon(x) := \{z \in \mathbb{R}^2 | d(x, z) < \varepsilon\}\). **Case 2.** \(x = O, y \neq O^*\): Let \(\varepsilon\) be a positive real number such that \(\varepsilon < (1/2)d(O, y)\). Then, there exists a subset \(B_\varepsilon^-(O)\) (cf. Example 4.12) such that \(x = O \in B_\varepsilon^-(O)^\gamma\) and \(y \notin B_\varepsilon^-(O)^\gamma\). **Case 3.** \(x = O^*, y \neq O\): Let \(\varepsilon\) be a positive real number such that \(\varepsilon < (1/2)d(O, y)\). Then, there exists a subset \(B_\varepsilon^+(O^*) \subset PO(X, \tau)\) (cf. Example 4.12) such that \(x \in B_\varepsilon^+(O^*)^\gamma\) and \(y \notin B_\varepsilon^-(O^*)^\gamma\). **Case 4.** \(x = O^*, y = O\): There exists a subset
By Definition 4.8(a) containing a γp X, τ and X \ {∅}.

Example 4.14 The converse of Theorem 4.10(iv) is not true in general. Let X := \{a, b, c\} and τ := \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}. For a topological space (X, τ), we have that PO(X, τ) = τ. Let γp : PO(X, τ) → P(X) be an operation defined by γp(A) := Int(Cl(A)) for every set A ∈ PO(X, τ). Then, the space (X, τ) is pre γp-T1/2 (= γp-T1/2). Indeed, singletons \{a\} and \{b\} are pre γp-open sets in (X, τ); a singleton \{c\} is pre γp-closed in (X, τ). By Theorem 4.6(i), (X, τ) is pre γp-T1/2 for the operation γp. However, (X, τ) is not pre γp-T1. Indeed, X \ \{a\} is not pre γp-open. For a point c ∈ X \ \{a\}, any preopen set containing c is only X and so X^p = X \ X \ \{a\}. Thus, \{a\} is not pre γp-closed; by Theorem 4.10(i), (X, τ) is not pre γp-T1.

Example 4.15 Some converses of Theorem 4.10(vi) are not true in general.

(i) For i = 0, the following example shows that (X, τ) is pre γp-T0; this is not γp|τ-T0 (in the sense of Ogata) [20], [21]. Let X := \{a, b, c\} and τ := \{\emptyset, \{a\}, \{a, b\}, X\}. For a topological space (X, τ), we have that PO(X, τ) = PO(X, τ) = τ. Let γp : PO(X, τ) → P(X) be defined by γp(A) := A for every set A such that A = \{a\}; γp(\{a\}) := \{a, b\}. Then, we have that PO(X, τ)γp = \{\emptyset, \{a, b\}, \{a, c\}, X\} and so (X, τ) is pre γp-T0. Hence, (X, τ) is pre γp-T0 (cf. Theorem 4.10(ii)). However, it is shown that (X, τ) is not γp|τ-T0.

(ii) For i = 1/2, the following example shows that (X, τ) is pre γp-T1/2; this is not γp|τ-T1/2 (in the sense of Ogata) [20], [21]. Let X := \{a, b, c\} and τ := \{\emptyset, \{a\}, \{b, c\}, X\}. For a topological space (X, τ), we have that PO(X, τ) = PO(X, τ) = τ. Let γp : PO(X, τ) → P(X) be defined by γp(A) := A for every set A ∈ \{\{a\}, \{b\}, \{a, c\}\}, γp(\{a\}) := \emptyset and γp(\{a\}) := X for every noneempty set B ∈ P(X) \ \{\{a\}, \{b\}, \{b, c\}\}. We have that PO(X, τ)γp = \{\emptyset, \{a\}, \{b\}, \{a, c\}, X\} and so singletons \{a\} and \{b\} are pre γp-open and \{c\} are pre γp-closed in (X, τ). By Theorem 4.6(i), (X, τ) is pre γp-T1/2. However, (X, τ) is not γp|τ-T1/2 (in the sense of Ogata). Indeed, τγp|τ = \{\emptyset, \{a, b\}, X\} and so \{c\} is neither γp|τ-open nor γp|τ-closed in (X, τ). By Remark 4.7, (X, τ) is not γp|τ-T1/2.

(iii) For i = 1, the same space (X, τ) of (ii) above shows that, for the identity operation γp, (X, τ) is pre γp-T1; this is not γp|τ-T1 (in the sense of Ogata) [20], [21].

Example 4.16 The converses of Theorem 4.10(vii) are not true in general. For i = 2, Example 4.12 shows that the space (X, τ) is not pre γp-T2 (cf. Theorem 4.10(ii) for i = 2); this is pre γp-T1′ (cf. Theorem 4.10(i)). For i = 1, Example 4.15(i) shows that the space (X, τ) is pre γp-T0′. The space (X, τ) is not pre γp-T1′, because \{a\} is not pre γp-closed in (X, τ).

Example 4.17 The converse of Theorem 4.10(ii) for i = 0 is not true in general. Let (X, τ) be the same topological space of Example 4.14, i.e., X := \{a, b, c\} and τ := \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}. Then, PO(X, τ) = τ holds. Let γp : PO(X, τ) → P(X) be an operation defined newly by γp(A) := \{a\}, \{b\}γp := \{a, b\}, \{a, b\}γp := \{a, b\}, \emptysetγp := \emptyset and Xγp := X. It is obtained that PO(X, τ)γp = \{\emptyset, \{a, b\}, X\} and γp is neither preregular nor preopen. Then, (X, τ) is not pre γp-T0′. Indeed, for every pre γp-open set V_a containing a, we have b ∈ V_b; for every pre γp-open set V_b containing b, we have a ∈ V_b. By Definition 4.8(a), the space (X, τ) is not pre γp-T0′. Moreover, the space (X, τ) is pre γp-T0.
Remark 4.18 Example 4.17 shows that, in Theorem 4.9(iii), the assumption of pre-openness on $\gamma_p$ cannot be removed.

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