A CHARACTERIZATION OF $\omega_1$-STRONGLY COUNTABLE-DIMENSIONAL METRIZABLE SPACES

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Abstract. In this paper, we characterize the class of $\omega_1$-strongly countable-dimensional metrizable spaces by a special metric. A characterization of locally finite-dimensional metrizable spaces is also obtained.

1 Introduction

If every finite open cover of a metrizable space $X$ has a finite open refinement of order $\leq n + 1$, then $X$ has covering dimension $\leq n$, $\dim X \leq n$. For $\varepsilon > 0$, we let $S_{\varepsilon}(x)$ denote the $\varepsilon$-ball $\{y \in X \mid \rho(x, y) < \varepsilon\}$ about $x$.

In [5], [6] and [7], J. Nagata gave a characterization of metrizable spaces of $\dim X \leq n$ by a special metric.

Theorem 1.1 (J. Nagata [5], [6], [7]) The following conditions are equivalent for a metrizable space $X$:

1. $\dim X \leq n$.
2. There is an admissible metric $\rho$ satisfying the following condition: for every $\varepsilon > 0$, every point $x$ of $X$ and every $n + 2$ many points $y_1, ..., y_{n+2}$ of $X$ with $\rho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ for each $i = 1, ..., n + 2$, there are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) < \varepsilon$.
3. There is an admissible metric $\rho$ satisfying the following condition: for every point $x$ of $X$ and every $n + 2$ many points $y_1, ..., y_{n+2}$ of $X$, there are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) \leq \rho(x, y_k)$.

For the case of the separable metrizable spaces, J. de Groot [2] gave the following characterization.

Theorem 1.2 (J. de Groot [2]) A separable metrizable space $X$ has $\dim X \leq n$ if and only if $X$ can introduce an admissible totally bounded metric satisfying the following condition:

For every point $x$ of $X$ and every $n + 2$ many points $y_1, ..., y_{n+2}$ of $X$, there are natural numbers $i, j$ and $k$ such that $i \neq j$ and $\rho(y_i, y_j) \leq \rho(x, y_k)$.

A metrizable space $X$ is strongly countable-dimensional if $X$ can be represented as a countable union of closed finite-dimensional subspaces. Let $\mathbb{N}$ denote the set of all natural numbers.

In [8], J. Nagata extended Theorems 1.1 and 1.2 to strongly countable-dimensional metrizable spaces.

Theorem 1.3 (J. Nagata [8]) The following conditions are equivalent for a metrizable space $X$:

1. $X$ is strongly countable-dimensional.

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(2) There is an admissible metric $\rho$ satisfying the following condition: for every point $x$ of $X$, there is an $n(x) \in \mathbb{N}$ such that for every $n(x) + 2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$, there are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) \leq \rho(x, y_j)$.

(3) There is an admissible metric $\rho$ satisfying the following condition: for every point $x$ of $X$, there is an $n(x) \in \mathbb{N}$ such that for every $n(x) + 2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$, there are natural numbers $i, j$ and $k$ such that $i \neq j$ and $\rho(y_i, y_j) \leq \rho(x, y_k)$.

In [3], Y. Hattori characterized the class of strongly countable-dimensional spaces by extending the condition (2) of Theorem 1.1.

**Theorem 1.4** (Y. Hattori [3]) A metrizable space $X$ is strongly countable-dimensional if and only if $X$ can introduce an admissible metric $\rho$ satisfying the following condition:

For every point $x$ of $X$, there is an $n(x) \in \mathbb{N}$ such that for every $\varepsilon > 0$, and every $n + 2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$ with $\rho(S_{\varepsilon/2}(x), y_i) < \varepsilon$ for each $i = 1, \ldots, n(x) + 2$, there are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) < \varepsilon$.

## 2 $\omega_1$-strongly countable-dimensional spaces

In this section, we characterize the class of $\omega_1$-strongly countable-dimensional metrizable spaces by a special metric. A characterization of locally finite-dimensional metrizable spaces is also obtained.

**Definition 2.1** A metrizable space $X$ is locally finite-dimensional if for every point $x \in X$ there exists an open subspace $U$ of $X$ such that $x \in U$ and $\dim U < \infty$.

The first infinite ordinal number is denoted by $\omega$ and $\omega_1$ is the first uncountable ordinal number.

**Definition 2.2** A metrizable space $X$ is called an $\omega_1$-strongly countable-dimensional space if $X = \bigcup \{P_\xi \mid 0 \leq \xi < \xi_0\}$, $\xi_0 < \omega_1$, where $P_\xi$ is an open subset of $X - \bigcup \{P_\eta \mid 0 \leq \eta < \xi\}$ and $\dim P_\xi < \infty$.

For a metrizable space $X$ and a non-negative integer $n$, we put

$$P_n(X) = \bigcup \{U \mid U \text{ is an open subspace of } X \text{ and } \dim U \leq n\}.$$  

We notice that for each ordinal number $\alpha$, we can put $\alpha = \lambda(\alpha) + n(\alpha)$, where $\lambda(\alpha)$ is a limit ordinal number or 0 and $n(\alpha)$ is a non-negative integer.

**Definition 2.3** Let $X$ be a metrizable space and $\alpha$ either an ordinal number $\geq 0$ or the integer $-1$. Then strong small transfinite dimension sind of $X$ is defined as follows:

(1) $\operatorname{sind} X = -1$ if and only if $X = \emptyset$.

(2) $\operatorname{sind} X \leq \alpha$ if $X$ is expressed in the form $X = \bigcup \{P_\xi \mid \xi < \alpha\}$, where $P_\xi = P_{n(\xi)}(X - \bigcup \{P_\eta \mid \eta < \lambda(\xi)\})$.

Furthermore, if $\operatorname{sind} X$ is defined, we say that $X$ has strong small transfinite dimension.

Clearly, a metrizable space $X$ is locally finite-dimensional if and only if $\operatorname{sind} X \leq \omega$ (R. Engelking [1]). And $X$ is $\omega_1$-strongly countable-dimensional if and only if there is a $\xi_0 < \omega_1$ such that $\operatorname{sind} X \leq \xi_0$.

Theorem 2.9 is a main theorem. Thus we characterize the class of $\omega_1$-strongly countable-dimensional metrizable spaces by a special metric. To prove this theorem, we need Theorem 2.4.
**Theorem 2.4** Let \( \alpha \) be an ordinal number with \( \alpha < \omega_1 \) and let \( n \) be a non-negative integer. The following conditions are equivalent for a metrizable space \( X \):

(a) \( \text{ind} X \leq \omega \alpha + n \).

(b) There are an admissible metric \( \rho \) for \( X \) and a family \( \{X_\beta \mid 0 \leq \beta \leq \alpha \} \) of closed sets of \( X \) satisfying the following conditions: (b-1) \( X_0 = X \), \( X_\beta \supseteq X_{\beta'} \) for \( \beta \leq \beta' \leq \alpha \), \( X_\beta = \bigcap \{X_{\beta'} \mid \beta' < \beta \} \) if \( \beta \) is a limit, and \( X_\alpha = \emptyset \) if \( n = 0 \). (b-2) For every point \( x \) of \( X \) there are an open neighborhood \( U(x) \) of \( x \) in \( X_{\beta(x)} \), where \( \beta(x) = \max \{ \beta \mid x \in X_\beta \} \), and an \( n(x) \in N_{\beta(x)} \) such that for every \( \varepsilon > 0 \), every point \( x' \) of \( U(x) \) and every \( n(x) + 2 \) many points \( y_1, ..., y_{n(x)+2} \) of \( X \) with \( \rho(S_{\varepsilon/2}(x'), y_i) < \varepsilon \) for each \( i = 1, ..., n(x) + 2 \), there are distinct natural numbers \( i \) and \( j \) such that \( \rho(y_i, y_j) < \varepsilon \), where

\[
N_{\beta(x)} = \left\{ \begin{array}{ll}
N, & \text{if } \beta(x) < \alpha, \\
(n-1), & \text{if } \beta(x) = \alpha.
\end{array} \right.
\]

(c) There are an admissible metric \( \rho \) for \( X \) and a family \( \{X_\beta \mid 0 \leq \beta \leq \alpha \} \) of closed sets of \( X \) satisfying the following conditions: (c-1) \( X_0 = X \), \( X_\beta \supseteq X_{\beta'} \) for \( \beta \leq \beta' \leq \alpha \), \( X_\beta = \bigcap \{X_{\beta'} \mid \beta' < \beta \} \) if \( \beta \) is a limit, and \( X_\alpha = \emptyset \) if \( n = 0 \). (c-2) For every point \( x \) of \( X \) there are an open neighborhood \( U(x) \) of \( x \) in \( X_{\beta(x)} \), where \( \beta(x) = \max \{ \beta \mid x \in X_\beta \} \), and an \( n(x) \in N_{\beta(x)} \) such that for every \( \varepsilon > 0 \), every point \( x' \) of \( U(x) \) and every \( n(x) + 2 \) many points \( y_1, ..., y_{n(x)+2} \) of \( X \), there are distinct natural numbers \( i \) and \( j \) such that \( \rho(y_i, y_j) < \rho(x', y_j) \), where

\[
N_{\beta(x)} = \left\{ \begin{array}{ll}
N, & \text{if } \beta(x) < \alpha, \\
(n-1), & \text{if } \beta(x) = \alpha.
\end{array} \right.
\]

**Remark 2.5** Let \( \{X_\beta \mid 0 \leq \beta \leq \alpha \} \) be a family of closed sets of \( X \) satisfying the condition (b-1). Then we shall show that for every point \( x \) of \( X \), there is a maximum element \( \beta(x) \) of \( \{ \beta \mid x \in X_\beta \} \). Indeed, if \( x \in X_{\lambda(x)} \), then \( \beta(x) = \max \{ \beta \mid x \in X_\beta, \lambda(x) \leq \beta \leq \alpha \} \). Now, we suppose that \( x \in X_{\lambda(x)} \), there is a minimum element \( \beta_0 \) of \( \{ \beta \mid x \notin X_\beta \} \). Assume that \( \beta_0 \) is limit. By the condition (b-1), \( x \in \bigcap \{X_\beta \mid \beta < \beta_0 \} = X_{\beta_0} \). This contradicts the definition of \( \beta_0 \). Therefore \( \beta_0 \) is not limit and hence \( \beta(x) = \beta_0 - 1 \).

To prove this theorem, we need the following lemmas. Essentially, the following lemma is the same as [3; Lemma 1.5]. By a minor modification in the proof of [3; Lemma 1.5], we obtain the following lemma.

**Lemma 2.6** ([3; Lemma 2.5], [8; Lemma 1]) Let \( n \) be a non-negative integer and let \( \{F_m \mid m = 0, 1, ...\} \) be a closed cover of a metrizable space \( X \) such that \( \dim F_m \leq (n-1) + m \), \( F_m \subseteq F_{m+1} \) for \( m = 0, 1, ... \). Then for every open cover \( U \) of \( X \), there are a sequence \( V_1, V_2, ... \) of discrete families of open sets of \( X \) and an open cover \( W \) of \( X \) which satisfy the following conditions:

1. \( \bigcup \{V_k \mid k \in \mathbb{N} \} \) is a cover of \( X \).
2. \( \bigcup \{V_k \mid k \in \mathbb{N} \} \) refines \( U \).
3. If \( W \in W \) satisfies \( W \cap F_m \neq \emptyset \), then \( W \) meets at most one member of \( V_k \) for \( k \leq (n+0)+(n+1)+...+(n+m) \).

Let \( Q^* \) denote the set of all rational numbers of the form \( 2^{-m_1} + ... + 2^{-m_t} \), where \( m_1, ..., m_t \) are natural numbers satisfying \( 1 \leq m_1 < ... < m_t \).

Essentially, the following lemma is the same as [3; Lemma 1.6]. By a minor modification in the proof of [3; Lemma 1.6], we obtain the following lemma.

**Lemma 2.7** ([3; Lemma 2.6], [8; Lemma 3]) Let \( n \) be a non-negative integer and let \( \{F_m \mid m = 0, 1, ...\} \) be a closed cover of a metrizable space \( X \) such that \( \dim F_m \leq (n-1) + m \),
$F_m \subset F_{m+1}$ for $m = 0, 1, \ldots$. Then for every $q \in Q^*$, there is an open cover $S(q)$ which satisfies the following conditions:

(1) $S(q) = \bigcup_{i=1}^{\infty} S^i(q)$, where each $S^i(q)$ is discrete in $X$.
(2) $\{St(x, S(q)) \mid q \in Q^*\}$ is a neighborhood base at $x \in X$.
(3) Let $p$, $q \in Q^*$ and $p < q$. Then $S(p)$ refines $S(q)$.
(4) Let $p$, $q \in Q^*$ and $p < q$. If $S_1 \in S^i(p)$ and $S_2 \in S^i(q)$, then $S_1 \cap S_2 = \emptyset$ or $S_1 \subset S_2$.
(5) Let $p$, $q \in Q^*$ and $p + q < 1$. Let $S_1 \in S(p)$, $S_2 \in S(q)$ and $S_1 \cap S_2 \neq \emptyset$. Then there is an $S_3 \in S(p + q)$ such that $S_1 \cup S_2 \subset S_3$.
(6) For every $q \in Q^*$ and every $S \in \bigcup\{S^i(q) \mid i > (n + 0) + (n + 1) + \ldots + (n + m)\}$, $S \cap F_m = \emptyset$.

For a cover $\mathcal{U}$ of a set $X$, we denote $\mathcal{U}^* = \{St(U, \mathcal{U}) \mid U \in \mathcal{U}\}$ and $\mathcal{U}^{**} = (\mathcal{U}^*)^*$, where $St(U, \mathcal{U}) = \bigcup\{V \in \mathcal{U} \mid U \cap V \neq \emptyset\}$.

Proof of Theorem 2.4. We prove the implication (a) $\Rightarrow$ (c). Let $\text{sind} X \leq \omega \alpha + n$. We put

$$Y_\gamma = X - \bigcup\{P_\xi \mid \xi < \gamma\} \quad \text{for} \quad \gamma \leq \omega \alpha + n$$

and

$$X_\beta = Y_{\omega \beta} \quad \text{for} \quad \beta \leq \alpha.$$ 

Clearly, the family of closed sets $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ satisfies the condition (c-1). Note that $X_\beta$ is a closed set of $X$ and $P_{\omega \beta + m}$ is an open set of $X_\beta$ such that $P_{\omega \beta + m} \subset P_{\omega \beta + (m+1)}$ for $m = 0, 1, \ldots$. Also $P_{\omega \alpha + (n-1)}$ is a closed set of $X$. Hence for each $\beta \leq \alpha$ there is a family $\{W_{\omega \beta + m} \mid m = 0, 1, \ldots\}$ of open sets of $X_\beta$ such that

(1) $W_{\omega \beta + m} \subset P_{\omega \beta + m}$,
(2) $W_{\omega \beta + m} \subset W_{\omega \beta + (m+1)}$,
(3) $\bigcup_{m=0}^{\infty} W_{\omega \beta + m} = \bigcup_{m=0}^{\infty} P_{\omega \beta + m}$.

Since $\{\beta \mid 0 \leq \beta < \alpha\}$ is countable, there is a mapping $f$ from $\mathbb{N}$ onto $\{\beta \mid 0 \leq \beta < \alpha\}$. For each $m = 0, 1, \ldots$, we put

$$V_0 = P_{\omega \alpha + (n-1)}$$

\begin{align*}
V_1 &= P_{\omega \alpha + (n-1)} \cup W_{\omega f(1) + (n-1)+1}, \\
V_2 &= P_{\omega \alpha + (n-1)} \cup W_{\omega f(1) + (n-1)+2} \cup W_{\omega f(2) + (n-1)+2}, \\
&\quad \ldots \\
V_m &= P_{\omega \alpha + (n-1)} \cup W_{\omega f(1) + (n-1)+m} \cup W_{\omega f(2) + (n-1)+m} \cup \ldots \cup W_{\omega f(m) + (n-1)+m}, \\
&\quad \ldots
\end{align*}

Then $V_0, V_1, \ldots$ are subsets of $X$ satisfying the following conditions:

(4) $V_m \subset V_{m+1}$,
(5) $\dim V_m \leq (n - 1) + m$.
(6) $X = \bigcup_{m=0}^{\infty} V_m$.

The latter half of the proof is similar to the proof of [3; Theorem 1.4]. By Lemma 2.7, for every $q \in Q^*$, there is an open cover $S(q)$ which satisfies the following conditions:

(7) $S(q) = \bigcup_{i=1}^{\infty} S^i(q)$, where each $S^i(q)$ is discrete in $X$.
(8) $\{St(x, S(q)) \mid q \in Q^*\}$ is a neighborhood base at $x \in X$. 

Let $p, q \in Q^*$ and $p < q$. Then $S(p)$ refines $S(q)$.

(10) Let $p, q \in Q^*$ and $p < q$. If $S_1 \in S(p)$ and $S_2 \in S(q)$, then $S_1 \cap S_2 = \emptyset$ or $S_1 \subset S_2$.

(11) Let $p, q \in Q^*$ and $p + q < 1$. Let $S_1 \in S(p), S_2 \in S(q)$ and $S_1 \cap S_2 \neq \emptyset$. Then there is an $S_3 \in S(p + q)$ such that $S_1 \cup S_2 \subset S_3$.

(12) For every $q \in Q^*$ and every $S \in \bigcup \{S(q) \mid i > (n + 0) + (n + 1) + \ldots + (n + m)\}$, $S \cap \mathcal{V}_m = \emptyset$.

We define a function $\rho : X \times X \to [0, 1]$ as follows: For $x, y \in X$,

$$\rho(x, y) = \begin{cases} 1, & \text{if } y \notin St(x, S(q)) \text{ for every } q \in Q^*, \\ \inf \{q \in Q^* \mid y \in St(x, S(q))\}, & \text{otherwise.} \end{cases}$$

It follows that $\rho$ is an admissible metric for $X$ by the proof of [9; Ch. 5.3, (D)]. To prove that the metric $\rho$ and the family of closed sets $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ satisfy the condition (c-2), let $x$ be a point of $X$. Put $n_0 = \min \{m \mid x \in V_m\}$ and $n(x) = (n + 0) + (n + 1) + \ldots + (n + n_0) - 1$. Clearly, if $n_0 = 0$ then $x \in V_0 = P_{\omega_\alpha + (n_1 - 1)} \subset X_\alpha$.

Now we shall show that if $n_0 > 0$, then $x \in W_{\omega \beta(x) + (n_1 + n_0)} \subset X_\beta(x)$. By the definition of $n_0$, $x \in V_{n_0} = P_{\omega_\alpha + (n_1 - 1)} \cup W_{\omega \beta(x) + (n_1 + n_0)} \cup \bigcup W_{\omega \beta(x) + (n_1 + n_0) + 1} \cup \ldots \cup W_{\omega \beta(x) + (n_1 + n_0) + n_0}$.

Since $x \notin P_{\omega_\alpha + (n_1 - 1)}$ by $n_0 > 0$, there is a natural number $i$ such that $x \in W_{\omega \beta(x) + (n_1 + n_0)}$.

Hence $x \in W_{\omega \beta(x) + (n_1 + n_0)} \subset P_{\omega \beta(x) + (n_1 + n_0)} \subset X_\beta(x) - X_\beta(x) + 1$. Also since $\beta(x) = \max \{\beta \mid x \in X_\beta\} < \alpha$, $x \in X_\beta(x) - X_\beta(x) + 1$. Hence $f(i) = \beta(x)$ and hence $x \in W_{\omega \beta(x) + (n_1 + n_0)} \subset X_\beta(x)$.

We put

$$U(x) = \begin{cases} P_{\omega_\alpha + (n_1 - 1)}, & \text{if } n_0 = 0, \\ W_{\omega \beta(x) + (n_1 + n_0)}, & \text{if } n_0 > 0. \end{cases}$$

Then $U(x)$ is an open neighborhood of $x$ in $X_\beta(x)$. Let $x'$ be a point of $U(x)$ and $y_1, \ldots, y_{n(x) + 2}$ be points of $X$. We can assume that $\rho(x', y_j) < 1$ for each $j = 1, \ldots, n(x) + 2$. Let $\varepsilon > 0$ be given. For each $j = 1, \ldots, n(x) + 2$, there is a $q(j) \in Q^*$ such that $\rho(x', y_j) \leq q(j) < \rho(x', y_j) + \varepsilon$. Then, by the definition of $\rho$, there is an $S_j \in S(q(j))$ such that $x', y_j \in S_j$.

Let $S_j \in S(q(j))$. Then, by (12), there are distinct natural numbers $j$ and $k$ such that $i(j) = i(k)$. Assume that $q(j) \leq q(k)$. By (10), we obtain $S_j \subset S_k$. Hence $y_j, y_k \in S_k$ and hence $\rho(y_j, y_k) \leq q(k) < \rho(x', y_k) + \varepsilon$. Therefore, there are distinct natural numbers $j$ and $k$ such that $\rho(y_j, y_k) < \rho(x', y_k) + 1/m$ for each $m \in \mathbb{N}$ and hence $\rho(y_j, y_k) \leq \rho(x', y_k)$. This completes the proof of the implication (a) $\Rightarrow$ (c).

Next, we prove the implication (a) $\Rightarrow$ (b). The proof is similar to the proof of [3; Theorem 1.1]. We use the above notations. By Lemma 2.6, there is a sequence $U_1, U_2, \ldots$ of open covers of $X$ which satisfies the following conditions.

(13) $U_{k+1}^*$ refines $U_k$ for each $k \in \mathbb{N}$.

(14) $\{St(x, U_k) \mid k \in \mathbb{N}\}$ is a neighborhood base at $x \in X$.

(15) For each $x \in V_m$ and each $k \in \mathbb{N}$, $St^2(x, U_{k+1})$ meets at most $(n + 0) + (n + 1) + \ldots + (n + m)$ members of $U_k$.

For each $q = 2^{-m_1} + \ldots + 2^{-m_t} \in Q^*$ and each $U \in U_{m_1}$, we put

$$S(U; m_1) = U,$$

$$S(U; m_1, \ldots, m_k) = St^2(S(U; m_1, \ldots, m_{k-1}), U_{m_k}) \text{ for } 2 \leq k \leq t,$$

$$S(U; q) = S(U; m_1, \ldots, m_t)$$

and

$$S(q) = \{S(U; q) \mid U \in U_{m_1}\}.$$
We define a function \( \rho: X \times X \to [0, 1] \) as follows: For \( x, y \in X \),
\[
\rho(x, y) = \begin{cases} 
1, & \text{if } y \notin \text{St}(x, S(q)) \text{ for every } q \in Q^*, \\
\inf\{q \in Q^* \mid y \in \text{St}(x, S(q))\}, & \text{otherwise}.
\end{cases}
\]

It follows that \( \rho \) is an admissible metric for \( X \) (see the proof of [9; Ch. 5.3,(D)]). To prove that the metric \( \rho \) and the family of closed sets \( \{X_\beta \mid 0 \leq \beta \leq \alpha\} \) satisfy the condition (b-2), let \( x \) be a point of \( X \). Put \( n_0 = \min\{m \mid x \in V_m\} \) and \( n(x) = (n+0)+(n+1)+...+(n+n_0)-1 \). We put
\[
U(x) = \begin{cases} 
P_{\omega n+(n-1)}, & \text{if } n_0 = 0, \\
W_{\omega \beta(x)+(n-1)+n_0}, & \text{if } n_0 > 0.
\end{cases}
\]

Then \( U(x) \) is an open neighborhood of \( x \) in \( X_{\beta(x)} \). Let \( x' \in U(x), \varepsilon > 0 \) and \( y_1, ..., y_{n(x)+2} \in X \) with \( \rho(S_{\varepsilon/2}(x'), y_i) < \varepsilon \) for each \( i = 1, ..., n(x)+2 \). For each \( i = 1, ..., n(x)+2 \), let \( x_i \) be a point of \( X \) such that \( \rho(x_i, x) < \varepsilon/2 \) and \( \rho(x_i, y_i) < \varepsilon \). Put
\[
\delta_i = \max\{2\rho(x', x_i), \rho(x_i, y_i)\}
\]
and \( \delta = \max\{\delta \mid i = 1, ..., n(x)+2\} \). Then there is a \( q = 2^{-m_1} + ... + 2^{-m_t} \in Q^* \) such that \( \delta < q < \varepsilon \). Since \( \rho(x', x_i) < 2^{-(m_1+1)} + ... + 2^{-(m_t+1)} \), there is a \( U_i \in U_{m_1+1} \) such that \( x_i, x \in S(U_i; m_1 + 1, ..., m_t + 1) \). Hence \( x_i \in \text{St}(x', U_{m_1+1}) \). On the other hand, there is a \( U_i' \in U_{m_1} \) such that \( x_i, y_i \in S(U_i'; m_1, ..., m_t) \). Therefore \( \text{St}(x_i, U_{m_1+1}) \cap U_i' \neq \emptyset \) and hence \( \text{St}(x, U_{m_1+1}) \cap U_i' \neq \emptyset \). By (15), there are distinct natural numbers \( i \) and \( j \) such that
\[
U_i' = U_j'.
\]
Then \( y_i, y_j \in S(U_i'; m_1, ..., m_t) = S(U_j'; m_1, ..., m_t) \). Therefore, \( \rho(y_i, y_j) \leq q < \varepsilon \).

This completes the proof of the implication (a) \( \Rightarrow \) (b).

Next, we prove the implication (c) \( \Rightarrow \) (a). Let \( \rho \) be an admissible metric for \( X \) and \( \{X_\beta \mid 0 \leq \beta \leq \alpha\} \) be a family of closed sets of \( X \) which satisfy the conditions (c-1) and (c-2).

We shall show that for every \( \beta \leq \alpha \)
\[
(16) \quad X - \bigcup\{P_\xi \mid \xi < \omega \beta\} \subset X_\beta.
\]

The validity of (16) is clear for \( \beta = 0 \). To prove (16) by transfinite induction we assume (16) for \( \gamma < \beta \).

Let \( x \notin X_\beta \). Then there are an open neighborhood \( U(x) \) of \( x \) in \( X_{\beta(x)} \) and an \( n(x) \in N_{\beta(x)} \) such that for every point \( x' \) of \( U(x) \) and every \( n(x)+2 \) many points \( y_1, ..., y_{n(x)+2} \) of \( X \), there are distinct natural numbers \( i \) and \( j \) such that \( \rho(y_i, y_j) \leq \rho(x', y_j) \). Since \( x \notin X_\beta \), \( \beta(x) < \beta \).

If \( x \in \bigcup\{P_\xi \mid \xi < \omega \beta(x)\} \), then \( x \in \bigcup\{P_\xi \mid \xi < \omega \beta\} \) by \( \beta(x) < \beta \).

We shall also show that if \( x \in X - \bigcup\{P_\xi \mid \xi < \omega \beta(x)\} \), then \( x \in \bigcup\{P_\xi \mid \xi < \omega \beta\} \). Since \( U(x) \) is an open neighborhood of \( x \) in \( X_{\beta(x)} \), by the induction hypothesis
\[
V(x) = U(x) \cap \left( X - \bigcup\{P_\xi \mid \xi < \omega \beta(x)\} \right)
\]
is an open neighborhood of \( x \) in \( X - \bigcup\{P_\xi \mid \xi < \omega \beta(x)\} \). Also \( \dim V(x) \leq \dim U(x) \leq n(x) \) by Theorem 1.1. Hence,
\[
x \in V(x) \subset P_{n(x)}(X - \bigcup\{P_\xi \mid \xi < \omega \beta(x)\}) \]
\[
= P_{\omega \beta(x)+n(x)} \subset \bigcup\{P_\xi \mid \xi < \omega (\beta(x)+1)\}.
\]
Therefore for every $\beta \leq \alpha$, (16) holds. In particular,

\begin{equation}
X - \bigcup \{ P_\xi \mid \xi < \omega \alpha \} \subset X_\alpha.
\end{equation}

We shall show that

\begin{equation}
X - \bigcup \{ P_\xi \mid \xi < \omega \alpha \} \subset \bigcup \{ P_\xi \mid \omega \alpha \leq \xi < \omega \alpha + n \}.
\end{equation}

If $n = 0$ then $X_\alpha = \emptyset$, and hence

\begin{equation*}
X - \bigcup \{ P_\xi \mid \xi < \omega \alpha \} = \emptyset
\end{equation*}

by (17).

Assume that $n > 0$. Let $x \in X - \bigcup \{ P_\xi \mid \xi < \omega \alpha \}$. Then there is an open neighborhood $U(x)$ of $x$ in $X_{\beta(x)}$ such that there is an $n(x) \in N_{\beta(x)}$ such that for every point $x'$ of $U(x)$ and every $n(x) + 2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$, there are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) \leq \rho(x', y_j)$. Since $x \in X_\alpha$ by (17), $\beta(x) = \alpha$. Hence $U(x)$ is an open neighborhood of $x$ in $X_\alpha$. By (17)

\begin{equation}
V(x) = U(x) \cap (X - \bigcup \{ P_\xi \mid \xi < \omega \alpha \})
\end{equation}

is an open neighborhood of $x$ in $X - \bigcup \{ P_\xi \mid \xi < \omega \alpha \}$. Also dim $V(x) \leq \dim U(x) \leq n(x)$ by Theorem 1.1. Furthermore $n(x) < n$ by $n(x) \in N_\alpha = \{ n-1 \}$. Hence,

\begin{align*}
x \in V(x) & \subset P_{n(x)}(X - \bigcup \{ P_\xi \mid \xi < \omega \alpha \}) \\
& = P_{\omega \alpha + n(x)} \subset \bigcup \{ P_\xi \mid \omega \alpha \leq \xi < \omega \alpha + n(x) \} \\
& \subset \bigcup \{ P_\xi \mid \omega \alpha \leq \xi < \omega \alpha + n \}.
\end{align*}

Therefore $X = \bigcup \{ P_\xi \mid 0 \leq \xi < \omega \alpha + n \}$ and hence $\text{sind}X \leq \omega \alpha + n$. This completes the proof of the implication (c) $\Rightarrow$ (a).

Finally, the proof of the implication (b) $\Rightarrow$ (a) is the same as the proof of the implication (c) $\Rightarrow$ (a). $\square$

By Theorems 1.2 and 2.4, we obtain the following theorem.

**Theorem 2.8** Let $\alpha$ be an ordinal number with $\alpha < \omega_1$ and let $n$ be a non-negative integer. The following conditions are equivalent for a compact metrizable space $X$:

(a) $\text{sind}X \leq \omega \alpha + n$.

(b) There are an admissible totally bounded metric $\rho$ for $X$ and a family $\{ X_\beta \mid 0 \leq \beta \leq \alpha \}$ of closed sets of $X$ satisfying the following conditions: (d-1) $X_0 = X$, $X_\beta \supseteq X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$, $X_\beta = \cap \{ X_{\beta'} \mid \beta' < \beta \}$ if $\beta$ is a limit, and $X_\alpha = \emptyset$ if $n = 0$. (d-2) For every point $x$ of $X$ there is an open neighborhood $U(x)$ of $x$ in $X_{\beta(x)}$, where $\beta(x) = \max \{ \beta \mid x \in X_\beta \}$, and an $n(x) \in N_{\beta(x)}$ such that for every point $x'$ of $U(x)$ and every $n(x) + 2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$, there are natural numbers $i$, $j$ and $k$ such that $i \neq j$ and $\rho(y_i, y_j) \leq \rho(x', y_k)$, where

\[ N_{\beta(x)} = \begin{cases} 
\mathbb{N}, & \text{if } \beta(x) < \alpha, \\
\{ n-1 \}, & \text{if } \beta(x) = \alpha.
\end{cases} \]
Theorem 2.9  The following conditions are equivalent for a metrizable space $X$:

(a) $X$ is an $\omega_1$-strongly countable-dimensional space.

(b) There are an admissible metric $\rho$ for $X$, an ordinal number $\alpha < \omega_1$ and a family $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ of closed sets of $X$ satisfying the following conditions: (b-1) $X_0 = X$, $X_\beta \supseteq X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$ and $X_{\beta} = \bigcap \{X_\beta \mid \beta' < \beta\}$ if $\beta$ is a limit. (b-2) For every point $x$ of $X$ there is an open neighborhood $U(x)$ of $x$ in $X_{\beta(x)}$, where $\beta(x) = \max\{\beta \mid x \in X_\beta\}$, and an $n(x) \in \mathbb{N}$ such that for every $\varepsilon > 0$, every point $x'$ of $U(x)$ and every $n(x) + 2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$ with $\rho(S_{x'2}(x'), y_i) < \varepsilon$ for each $i = 1, \ldots, n(x) + 2$, there are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) < \varepsilon$.

(c) There are an admissible metric $\rho$ for $X$, an ordinal number $\alpha < \omega_1$ and a family $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ of closed sets of $X$ satisfying the following conditions: (c-1) $X_0 = X$, $X_\beta \supseteq X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$ and $X_{\beta} = \bigcap \{X_\beta \mid \beta' < \beta\}$ if $\beta$ is a limit. (c-2) For every point $x$ of $X$ there is an open neighborhood $U(x)$ of $x$ in $X_{\beta(x)}$, where $\beta(x) = \max\{\beta \mid x \in X_\beta\}$, and an $n(x) \in \mathbb{N}$ such that for every point $x'$ of $U(x)$ and every $n(x) + 2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$, there are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) < \rho(x', y_k)$.

(d) There are an admissible metric $\rho$ for $X$, an ordinal number $\alpha < \omega_1$ and a family $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ of closed sets of $X$ satisfying the following conditions: (d-1) $X_0 = X$, $X_\beta \supseteq X_{\beta'}$ for $\beta \leq \beta' \leq \alpha$ and $X_{\beta} = \bigcap \{X_\beta \mid \beta' < \beta\}$ if $\beta$ is a limit. (d-2) For every point $x$ of $X$ there is an open neighborhood $U(x)$ of $x$ in $X_{\beta(x)}$, where $\beta(x) = \max\{\beta \mid x \in X_\beta\}$, and an $n(x) \in \mathbb{N}$ such that for every point $x'$ of $U(x)$ and every $n(x) + 2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$, there are natural numbers $i$, $j$ and $k$ such that $i \neq j$ and $\rho(y_i, y_j) < \rho(x', y_k)$.

Proof. The conditions (a), (b) and (c) are equivalent by Theorem 2.4. The implication (c) $\Rightarrow$ (d) is obvious.

We prove the implication (d) $\Rightarrow$ (a). Let $\rho$ be an admissible metric for $X$ and $\{X_\beta \mid 0 \leq \beta \leq \alpha\}$ be a family of closed sets of $X$ which satisfy the conditions (d-1) and (d-2). Let $x$ be a point of $X$. There are an open neighborhood $U(x)$ of $x$ in $X_{\beta(x)}$ and an $n(x) \in \mathbb{N}$ such that for every point $x'$ of $U(x)$ and every $n(x) + 2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$, there are natural numbers $i$, $j$ and $k$ such that $i \neq j$ and $\rho(y_i, y_j) < \rho(x', y_k)$. Then $\dim \mu \mu \dim U(x) \leq n(x)$, where $\mu \dim$ denotes the metric dimension, by [8; p. 500]. By M. Katětov's theorem [4] and J. Nagata [9; p. 93],

$$ \dim U(x) \leq 2\mu \dim U(x) \leq 2n(x). $$

Therefore $\dim X \leq \omega \alpha + \omega$ by the proof of the implication (c) $\Rightarrow$ (a) of Theorem 2.4, and hence $X$ is an $\omega_1$-strongly countable-dimensional space. □

Theorem 2.10  The following conditions are equivalent for a metrizable space $X$:
(a) $X$ is a locally finite-dimensional space.

(b) There is an admissible metric $\rho$ for $X$ satisfying the following conditions: For every point $x$ of $X$, there are an $n(x) \in \mathbb{N}$ and an open neighborhood $U(x)$ of $x$ in $X$ such that for every $\varepsilon > 0$, every point $x'$ of $U(x)$ and every $n(x)+2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$ with $\rho(S_{\varepsilon/2}(x'), y_i) < \varepsilon$ for each $i = 1, \ldots, n(x)+2$, there are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) < \varepsilon$.

(c) There is an admissible metric $\rho$ for $X$ satisfying the following conditions: For every point $x$ of $X$, there are an $n(x) \in \mathbb{N}$ and an open neighborhood $U(x)$ of $x$ in $X$ such that for every point $x'$ of $U(x)$ and every $n(x)+2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$, there are distinct natural numbers $i$ and $j$ such that $\rho(y_i, y_j) \leq \rho(x', y_j)$.

(d) There is an admissible metric $\rho$ for $X$ satisfying the following conditions: For every point $x$ of $X$, there are an $n(x) \in \mathbb{N}$ and an open neighborhood $U(x)$ of $x$ in $X$ such that for every point $x'$ of $U(x)$ and every $n(x)+2$ many points $y_1, \ldots, y_{n(x)+2}$ of $X$, there are natural numbers $i, j$ and $k$ such that $i \neq j$ and $\rho(y_i, y_j) \leq \rho(x', y_k)$.

References


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