DUAL BCK-ALGEBRA AND MV-ALGEBRA

KYUNG HO KIM AND YONG HO YON

Received March 23, 2007

Abstract. The aim of this paper is to study the properties of dual BCK-algebra and to prove that the MV-algebra is equivalent to the bounded commutative dual BCK-algebra.

1 Introduction Let $(L, \leq)$ be a poset. The greatest element of $L$ is called the top element of $L$ if it exists. Similarly, the least element of $L$ is called the bottom element of $L$ if it exists. We denote the top and the bottom element as 1 and 0, respectively. Let $S$ be a subset of $L$ and $u \in L$. $u$ is called an upper bound of $S$ if $s \leq u$ for all $s \in S$, and $u$ is called the join of $S$ if $u$ is the least upper bound of $S$. Dually, we define a lower bound of $S$ and the meet of $S$. A poset $L$ is an upper semilattice if $\sup \{x, y\}$ exists for all $x, y \in L$ and a poset $L$ is a lower semilattice if $\inf \{x, y\}$ exists for all $x, y \in L$ and a poset $L$ is a lattice if it is an upper and lower semilattice. A lattice $L$ is said to be bounded if there exists a bottom 0 and a top 1 in $L([5, 3])$.

BCK-algebras were introduced in 1966 by Iséki [4]. It is an algebraic formulation of the BCK-propositional calculus system of C. A. Meredith [7], and generalize the notion of implicative algebras. The notion of MV-algebra, originally introduced by C.C. Chang [2], is an attempt at developing a theory of algebraic systems that would correspond to the $\aleph_0$-valued propositional calculus; the axioms for this calculus are known as the Łukasiewicz axioms.

The purpose of this note is to study the relation between the MV-algebra and the dual concept of BCK-algebra. we will introduce some properties of dual BCK-algebras and MV-algebras, and prove that the MV-algebra is equivalent to the bounded commutative dual BCK-algebra.

2 Preliminaries In this section, we introduce the definitions and some properties of a BCK-algebra and a MV-algebra.

Definition 2.1. [6] An algebra $(X, *, 0)$ of type $(2, 0)$ is called a BCK-algebra if it satisfies:

(1) $(x * y) * (x * z) * (z * y) = 0$,
(2) $(x * (x * y)) * (z * y) = 0$,
(3) $x * x = 0$,
(4) $x * y = 0$ and $y * x = 0$ imply $x = y$,
(5) $0 * x = 0$,

for all $x, y, z \in X$.

2000 Mathematics Subject Classification. 06F35, 06D35, 03G25.

Key words and phrases. BCK-algebra, bounded commutative BCK-algebra, MV-algebra.
From the above definition, we have the definition of dual BCK-algebra as following definition.

**Definition 2.2.** A dual BCK-algebra is an algebra \((X, \circ, 1)\) of type \((2, 0)\) satisfying:

1. \((x \circ y) \circ ((y \circ z) \circ (x \circ z)) = 1.\)
2. \(x \circ ((x \circ y) \circ y) = 1.\)
3. \(x \circ x = 1.\)
4. \(x \circ y = 1\) and \(y \circ x = 1\) imply \(x = y,\)
5. \(x \circ 1 = 1.\)

for all \(x, y, z \in X.\)

Let \((X, \circ, 1)\) be a dual BCK-algebra. Then we can define a binary relation "\(\leq\)" on \(X\) as the following:

\[ x \leq y \text{ if and only if } x \circ y = 1 \]

for \(x, y \in X,\) and has the following Lemma form the definition of dual BCK-algebra.

**Lemma 2.3.** Let \((X, \circ, 1)\) be a dual BCK-algebra. Then for all \(x, y, z \in X,\)

1. \(x \circ y \leq (y \circ z) \circ (x \circ z),\)
2. \(x \leq (x \circ y) \circ y,\)
3. \(x \leq x,\)
4. \(x \leq y\) and \(y \leq x\) imply \(x = y,\)
5. \(x \leq 1.\)

**Theorem 2.4.** Let \((X, \circ, 1)\) be a dual BCK-algebra. Then for any \(x, y, z \in X\) the following hold

1. \(x \leq y\) implies \(y \circ z \leq x \circ z,\)
2. \(x \leq y\) and \(y \leq z\) imply \(x \leq z.\)

**Proof.** (1) Let \(x \leq y.\) Then \(1 = x \circ y \leq (y \circ z) \circ (x \circ z)\) by 2.3 (1). Hence \((y \circ z) \circ (x \circ z) = 1\) by 2.3 (5), and \(y \circ z \leq x \circ z.\)

(2) If \(x \leq y\) and \(y \leq z\), then \(1 = y \circ z \leq x \circ z\) by this lemma (1), hence \(x \circ z = 1\) and \(x \leq z.\) \(\square\)

A dual BCK-algebra \((X, \circ, 1)\) is a poset with the ordering \(\leq\) from 2.3 and 2.4, and the element 1 in \(X\) is the top element with respect to the order relation \(\leq.\)

**Theorem 2.5.** Let \((X, \circ, 1)\) is a dual \(BCK\)-algebra and \(x, y, z \in X.\) Then

1. \(x \circ (y \circ z) = y \circ (x \circ z),\)
2. \(x \leq y \circ z\) imply \(y \leq x \circ z,\)
3. \(x \circ y \leq (z \circ x) \circ (z \circ y),\)
4. \(x \leq y\) imply \(z \circ x \leq z \circ y,\)
(5) \( y \leq x \circ y \),
(6) \( 1 \circ x = x \).

Proof. (1) \((\leq)\) Since \( y \leq (y \circ z) \circ z \) by 2.3(2), \(( (y \circ z) \circ z ) \circ (x \circ z) \leq y \circ (x \circ z) \) by 2.4(1). And \( x \circ (y \circ z) \leq ((y \circ z) \circ z) \circ (x \circ z) \) by 2.3(1), hence \( x \circ (y \circ z) \leq y \circ (x \circ z) \). \((\geq)\) Interchanging the role of \( x \) and \( y \) in \((\leq)\), we get \( y \circ (x \circ z) \leq x \circ (y \circ z) \).

(2) Let \( x \leq y \circ z \). Then \( 1 = x \circ (y \circ z) = y \circ (x \circ z) \), hence \( y \leq x \circ z \).

(3) Since \( z \circ x \leq (x \circ y) \circ (z \circ y) \) by 2.3(1), \( x \circ y \leq (z \circ x) \circ (z \circ y) \) by the above (2).

(4) Let \( x \leq y \). Then \( 1 = x \circ y \leq (z \circ x) \circ (z \circ y) \) by the above (3), hence \((z \circ x) \circ (z \circ y) = 1 \).

It follows that \( z \circ x \leq z \circ y \).

(5) From the above (1), \( y \circ (x \circ y) = x \circ (y \circ y) = x \circ 1 = 1 \), hence \( y \leq x \circ y \).

(6) Since \( 1 \leq (1 \circ x) \circ y \) by 2.3(2), \((1 \circ x) \circ x = 1 \), hence \( 1 \circ x \leq x \). Conversely, \( x \circ (1 \circ x) = 1 \circ (x \circ x) = 1 \circ 1 = 1 \), hence \( x \leq 1 \circ x \). \(\square\)

Let \((X, \circ, 1)\) be a dual \(BCK\)-algebra. We define a binary operation "+" on \(X\) as the following : for any \(x, y \in X\)

\[ x + y = (x \circ y) \circ y. \]

Remark 2.6. 1) Let \((X, \circ, 1)\) be a dual \(BCK\)-algebra and \(x, y \in X\). Then \(x + y\) is an upper bound of \(x\) and \(y\), since \(x \leq (x \circ y) \circ y\) and \(y \leq (x \circ y) \circ y\) by 2.3(2) and 2.5(5).

2) Let \((X, \circ, 1)\) be a dual \(BCK\)-algebra and \(x \in X\). Then \(x + 1 = (x \circ 1) \circ 1 = 1 \circ 1 = 1\), \(1 + x = (1 \circ x) \circ x = x \circ x = 1\), and \(x + x = (x \circ x) \circ x = 1 \circ x = x\), by 2.2 and 2.5(6).

Lemma 2.7. If \((X, \circ, 1)\) is a dual \(BCK\)-algebra, then \((x + y) \circ y = x \circ y \) for all \(x, y \in X\).

Proof. \((\leq)\) Since \(x \leq x + y\), \((x + y) \circ y \leq x \circ y\) by 2.4(1). \((\geq)\) From 2.3(2), \(x \circ y \leq ((x \circ y) \circ y) \circ y = (x + y) \circ y\). \(\square\)

3 Bounded Commutative Dual \(BCK\)-algebras and \(MV\)-algebras

Definition 3.1. A dual \(BCK\)-algebra \((X, \circ, 1)\) is said to be bounded if there exists an element \(0 \in X\) such that \(0 \circ x = 1\) for all \(x \in X\).

The element \(0\) is the bottom element in \(X\), since \(0 \leq x\) for all \(x \in X\) from the definition of the order \(\leq\).

Definition 3.2. Let \((X, \circ, 1, 0)\) be a bounded dual \(BCK\)-algebra and \(x \in X\). \(x \circ 0\) is called a pseudocomplement of \(x\) and we write \(x^* = x \circ 0\) and \(x^{*\circ} = (x^*)^*\).

Theorem 3.3. Let \((X, \circ, 1, 0)\) be a bounded dual \(BCK\)-algebra and \(x, y \in X\). Then

1) \(1^* = 0\) and \(0^* = 1\),
2) \(x \leq x^{*\circ}\),
3) \(x \circ y \leq y^* \circ x^*\),
4) \(x \leq y\) implies \(y^* \leq x^*\),
5) \(x \circ y^* = y \circ x^*\),
6) \(x^{*\circ\circ} = x^*\).
Therefore,
\[ x \leq (x \circ 0) \circ 0 = x^{**} \]
by 2.3(2).

(3) \[ x \circ y \leq (y \circ 0) \circ (x \circ 0) = y^{*} \circ x^{*} \]
by 2.3(1).

(4) \[ x \leq y \]
implies \[ y^{*} = y \circ 0 \leq x \circ 0 = x^{*} \]
by 2.4(1).

(5) \[ x \circ y^{*} = x \circ (y \circ 0) = y \circ (x \circ 0) = y \circ x^{*} \]
by 2.5(1).

(6) \[ x^{***} = ((x \circ 0) \circ 0) \circ 0 = (x + 0) \circ 0 = x \circ 0 = x^{*} \]
by 2.7.

\[ \square \]

**Definition 3.4.** An element \( x \) in a bounded dual BCK-algebra \( X \) is said to be regular if \( x^{**} = x \).

**Definition 3.5.** A dual BCK-algebra \( (X, \circ, 1) \) is said to be commutative if \( x + y = y + x \)
for all \( x, y \in X \).

**Theorem 3.6.** Let \( (X, \circ, 1) \) be a dual BCK-algebra. Then the following conditions are equivalent:

1. \( x + y \leq y + x \) for all \( x, y \in X \),
2. \( X \) is commutative,
3. \( y \leq x \)
implies \( x = (x \circ y) \circ y \) for \( x, y \in X \).

**Proof.** (1) \( \Rightarrow \) (2). Interchanging the role of \( x \) and \( y \), it is trivial.

(2) \( \Rightarrow \) (3). Suppose \( X \) is commutative and \( y \leq x \) for \( x, y \in X \), then \( x = 1 \circ x = (y \circ x) \circ x = y + x = x + y = (x \circ y) \circ y \).

(3) \( \Rightarrow \) (1). Suppose that \( y \leq x \)
implies \( x = (x \circ y) \circ y \) for \( x, y \in X \). Since \( y \leq y + x \),
\[ y + x = ((y + x) \circ y) \circ y, \]
thus \( (x + y) \circ ((y + x) \circ y) = ((y + x) \circ y) \circ (x \circ y) \)
by 2.5(1) and 2.7, and since \( x \leq y + x \),
\[ 1 = x \circ (y + x) \leq ((y + x) \circ y) \circ (x \circ y) \]
by 2.3(1). Hence \( (x + y) \circ (y + x) = 1 \) and
\[ x + y \leq y + x. \]

\[ \square \]

**Theorem 3.7.** A commutative dual BCK-algebra is an upper semilattice.

**Proof.** Let \( (X, \circ, 1) \) be a commutative dual BCK-algebra and \( x, y \in X \). Then \( x + y \)
is an upper bound of \( x \) and \( y \) by 2.6(1). We shall show that the \( x + y \)
is the least upper bound of \( x \) and \( y \). Suppose \( z \) is an upper bound of \( x \) and \( y \). Then \( x \circ z = y \circ z = 1 \)
(i): \( z = 1 \circ z = (x \circ z) \circ z = (z \circ z) \circ z \)
by the commutativity, and (ii): \( z = 1 \circ z = (y \circ z) \circ z = (z \circ y) \circ y \).

(iii): \( z = (z \circ x) \circ x = ((z \circ y) \circ y) \circ x \)
from (i) and (ii).

Set \( u = (z \circ y) \circ y \), then \( z = (u \circ x) \circ x \)
from (iii). Since \( y \leq u \) by 2.5(5), \( u \circ x \leq y \circ x \)
by 2.4(1), and then \( (y \circ x) \circ x \leq (u \circ x) \circ x = z \).
Hence \( x + y = (x \circ y) \circ y = (y \circ x) \circ x \leq z \).
Therefore, \( x + y \) is the least upper bound of \( x \) and \( y \) and \( X \) is an upper semilattice
with the join, say \( x + y \), of any two elements \( x \) and \( y \).

We define a binary operation "\( . \)" on \( X \) by the following: \( x \cdot y = (x^{*} + y^{*})^{*} \)
for each \( x, y \in X \).

**Theorem 3.8.** Every element in a bounded commutative dual BCK-algebra \( X \) is a regular.

**Proof.** Since \( 0 \leq x = (x \circ 0) \circ 0 = x^{**} \) for all \( x \in X \) by 3.6(3).

**Theorem 3.9.** Let \( (X, \circ, 1, 0) \) be a bounded commutative dual BCK-algebra and \( x, y \in X \). Then

1. \( x^{*} \cdot y^{*} = (x + y)^{*} \),
Definition 3.12. [1] An MV-algebra is an algebra \((A, \oplus,')\) of type \((2, 1, 0)\) satisfying the following equations:

1. \(x \oplus (y \oplus z) = (x \oplus y) \oplus z\),
2. \(x \oplus y = y \oplus x\),
3. \(x \oplus 0 = x\),
4. \(x'' = x\),
5. \(x \oplus 0' = 0'\),
6. \((x' \oplus y)' \oplus y = (y' \oplus x)' \oplus x\).

On a MV-algebra \(A\), we define the constant 1 and the operations "\(\ominus\)" and "\(\odot\)" as follows: \(1 = 0'\), \(x \ominus y = (x' \ominus y)'\) and \(x \odot y = x \cdot y' = (x' \odot y)'.\)

Lemma 3.13. [1] For \(x, y \in A\), the following conditions are equivalent:

1. \(x' \odot y = 1\),
2. \(x \odot y' = 0\),
3. \(y = x \oplus (y \ominus x)\),
4. there is an element \(z \in A\) such that \(x \ominus z = y\).

We define the binary relation "\(\leq\)" on a MV-algebra \(A\) as follows: \(x \leq y\) if and only if \(x\) and \(y\) satisfy one of the equivalent axioms 1) - 4) in the above lemma. The relation \(\leq\) is a partial ordered relation on \(A\).
Theorem 3.14. [1] Let $A$ be a MV-algebra and $x, y, z \in A$. Then

1. $1' = 0$,
2. $x \oplus y = (x' \odot y')'$,
3. $x \oplus 1 = 1$,
4. $(x \oplus y) \ominus y = (y \ominus x) \oplus x$,
5. $x \ominus x' = 1$,
6. $x \ominus 0 = x, 0 \ominus x = 0, x \ominus x = 0, 1 \ominus x = x', x \ominus 1 = 0$,
7. $x \ominus x = x$ if $x \ominus x = x$,
8. $x \leq y$ if $y' \leq x'$,
9. if $x \leq y$, then $x \oplus z \leq y \oplus z$ and $x \ominus z \leq y \ominus z$,
10. if $x \leq y$, then $x \ominus z \leq y \ominus z$ and $z \ominus y \leq z \ominus x$,
11. $x \ominus y \leq x, x \ominus y \leq y'$,
12. $(x \ominus y) \ominus x \leq y$,
13. $x \ominus z \leq y$ if $z \leq x' \oplus y$,
14. $x \ominus y \ominus x \ominus y = x \ominus y$.

Theorem 3.15. A bounded commutative dual BCK-algebra $(X, \circ, 1, 0)$ is a MV-algebra $(X, \oplus, '\circ', 0)$ with the operations “$\ominus$” and “$''\ominus''$” defined as following:

\[ x \ominus y = x^* \odot y \] and \[ x' = x^* \]

for all $x, y \in X$.

Proof. For any $x, y, z \in X$, we have $x \ominus (y \ominus z) = x^* \odot (y^* \odot z) = x^* \odot (z^* \odot y) = z^* \odot (x^* \odot y) = (x^* \odot y)^* \odot z = (x \ominus y) \ominus z,$ $x \ominus y = x^* \odot y = y^* \odot x = y \ominus x,$ $x \ominus 0 = x^* \odot 0 = x^{**} = x,$ $x'' = x^{**} = x$ and $x \ominus 0' = x^* \odot 0' = x^* \odot 1 = 1 = 0^* = 0'$. Thus we get the properties (1), (2), (3), (4) and (5) of the definition of MV-algebra. Next we will prove the property (6) of the MV-algebra. $(x' \ominus y') \ominus y = (x \ominus y)^* \ominus y = (x \ominus y)^* \odot y = (y \odot x) \odot x = (y \circ x)^* \odot x = (X^* \odot y^*)^* \odot x = (y' \ominus x') \ominus x$. \qed

Theorem 3.16. A MV-algebra $(X, \ominus, '1', 0)$ is a bounded commutative dual BCK-algebra with the operation “$\odot$” and the top element 1 defined as following:

\[ x \odot y = x' \ominus y \] and \[ 1 = 0' \]

for all $x, y \in X$.

Proof. 1) $x \odot 1 = x' \ominus 1 = 1$. 2) $x \odot x = x' \ominus x = 1$. 3) $x \odot ((x \odot y) \odot y) = x' \ominus ((x' \ominus y') \ominus y) = (x' \ominus y') \ominus (x' \ominus y) = 1$. 4) If $x \odot y = 1$ and $y \odot x = 1$, then $x' \ominus y = 1$ and $y' \ominus x = 1$. From the definition of the order $\leq$ in MV-algebra, $x \leq y$ and $y \leq x$, hence $x = y$. \qed
5) Let $x, y, z \in X$. Then we have

\[
(x \circ y) \circ ((y \circ z) \circ (x \circ z)) = (x \circ y)' \oplus ((y \circ z)' \oplus (x \circ z))
\]

\[
= (x \circ y)' \oplus (((y' \oplus z)' \oplus (x' \oplus z)) \oplus x')
\]

\[
= (x \circ y)' \oplus (((z' \oplus y)' \oplus y) \oplus x') \quad \text{(by 3.12(6))}
\]

\[
= (x \circ y)' \oplus (((z' \oplus y)' \oplus (x \circ y))
\]

\[
= (x \circ y)' \oplus (z' \oplus (x \circ y))
\]

\[
= 1 \oplus (z' \oplus y)'
\]

\[
= 1.
\]

References


