BASIC BCI-ALGEBRAS AND ABELIAN GROUPS ARE EQUIVALENT

J. SHOHANI, R. A. BORZOOEI, M. AFSHAR JAHANSHAHI

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Abstract. In this paper we prove that Abelian groups and basic BCI-algebras are equivalent and by using it we have obtain more results.

1. Introduction

The notion of BCI-algebras was formulated in 1966 by K. Iséki[3] and since then a lot of work has been done on subalgebras, ideals and homomorphism, but it is not much known about relations of BCI-algebras to other algebra systems. T. Lei and C. Xi[8] displayed the relationship of p-semi simple BCI-algebras to Abelian groups. In any BCI-algebra, there exist two important subsets. One of them is BCK-part and another one is a set consisting of incomparable elements containing 0. It makes the basic BCI-algebra introduced by K. Iséki [4]. Also B-algebras and 0-commutative B-algebras were introduced for the first time by J. Neggers, H. S. Kim and H. G. Park[9, 7]. In this paper, we prove that above two concepts namely basic-BCI algebras and Abelian groups are equivalent and we obtain more results about them.

2. Preliminaries

Definition 2.1. [4] A basic BCI-algebra is an algebra \((X, *, 0)\) of type \((2,0)\) which satisfies the following axioms:
For all \(x, y, z \in X\).
(B-BC1) \((x * y) * (x * z) = z * y\),
(B-BC2) \(x * (x * y) = y\),
(B-BC3) \(x * y = 0 \) implies \(x = y\).

Definition 2.2. [7] A 0-commutative B-algebra is an algebra \((X, *, 0)\) of type \((2,0)\) satisfying the following axioms:
For all \(x, y, z \in X\). (B1) \(x * x = 0\),
(B2) \(x * 0 = x\),
(B3) \((x * y) * z = x * (z * (0 * y))\),
(B4) \(x * (0 * y) = y * (0 * x)\).

Theorem 2.3. [7] Let \((X, *, 0)\) be an algebra of type \((2,0)\). Then the following statements are equivalent.
(i) \((X, *, 0)\) is a 0-commutative B-algebra,
(ii) \((X, *, 0)\) is a p-semi simple BCI-algebras,
(iii) \((X, *, 0)\) is an Abelian group.

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3. Basic BCI-algebras and 0-Commutative B-algebras are equivalent

Proposition 3.1. [4] Let \((X, *, 0)\) be a basic BCI-algebra. Then, for all \(x, y, z \in X\),
(i) \(x * 0 = x\),
(ii) \(0 * (0 * x) = x\),
(iii) \(0 * x = 0 * y\) implies \(x = y\),
(iv) \((x * y) * z = (x * z) * y\),
(v) \(x * (0 * y) = y * (0 * x)\).

Lemma 3.2. [9, 7] Let \((X, *, 0)\) be a 0-commutative B-algebra. Then for all \(x, y \in X\),
(i) \(x * y = 0\) implies \(x = y\),
(ii) \(x * (x * y) = y\),
(iii) \((x * y) * (x * z) = z * y\).

Corollary 3.3. Any 0-commutative B-algebra is a basic BCI-algebra.

Proof. The proof comes from Lemma 3.2. \(\square\)

Proposition 3.4. Let \((X, *, 0)\) be a basic BCI-algebra. Then, for all \(x, y \in X\),
(i) \(0 * (x * y) = y * x\),
(ii) \(0 * (x * y) = (0 * x) * (0 * y)\),
(iii) \((0 * x) * (0 * y) = y * x\).

Proof. (i) Let \(x, y \in X\). Then,
\[
0 * (x * y) = (x * (x * 0)) * (x * y), \quad \text{(by B-BCI2)}
= (x * (x * y)) * (x * 0), \quad \text{(by Proposition 3.1(iv))}
= y * x. \quad \text{(by B-BCI2 and Proposition 3.1(i))}
\]

(ii) Let \(x, y \in X\). Then,
\[
0 * (x * y) = y * x, \quad \text{(by (i))}
= (0 * (0 * y)) * x, \quad \text{(by B-BCI2)}
= (0 * x) * (0 * y), \quad \text{(by Proposition 3.1(iv))}
\]

(iii) Let \(x, y \in X\). Then, by (i) and (ii) we have
\[
(0 * x) * (0 * y) = 0 * (x * y) = y * x
\]

\(\square\)

Theorem 3.5. \((X, *, 0)\) is a basic BCI-algebra if and only if \((X, *, 0)\) is a 0-commutative B-algebra.

Proof. By Corollary 3.3, any 0-commutative B-algebra is a basic BCI-algebra. Conversely, let \((X, *, 0)\) be a basic BCI-algebra. Then for all \(x \in X\),
\[
x * x = x * (0 * (0 * x)), \quad \text{(by B-BCI2)}
= (0 * x) * (0 * x), \quad \text{(by Proposition 3.1(v))}
= (0 * x) * ((0 * x) * 0), \quad \text{(by Proposition 3.1(i))}
= 0. \quad \text{(by B-BCI2)}
\]

Moreover, by Proposition 3.1, \(x * 0 = x\). Now, we should prove the axiom B3. For this, we will prove \(0 * ((x * y) * z) = 0 * (x * (z * (0 * y)))\). Let \(x, y, z \in X\). Then,
0 * ((x * y) * z) = 0 * ((x * (0 * (0 * y))) * z), (by B-BCI2)
= 0 * (((0 * y) * (0 * x)) * z), (by Proposition 3.1(v))
= 0 * (((0 * y) * z) * (0 * x)), (by Proposition 3.1(iv))
= (0 * ((0 * y) * z)) * (0 * (0 * x)), (by Proposition 3.4(ii))
= (z * (0 * y)) * x, (by Propositions 3.4(i) and 3.1(ii))
= 0 * (x * (z * (0 * y))), (by Proposition 3.4(i))

Now, by Proposition 3.1(iii), (x * y) * z = x * (z * (0 * y)) and so we have axiom B3. Finally, by Proposition 3.1(v), we have axiom B4. Therefore, (X, *, 0) is a 0-commutative B-algebra. □

**Corollary 3.6.** Let X be a non-empty set and “*” is a binary operation on X. Then the following statements are equivalent:
(i) (X, *, 0) is a basic BCI-algebra,
(ii) (X, *, 0) is a 0-commutative B-algebra,
(iii) (X, *, 0) is a p-semi simple BCI-algebra,
(iv) (X, *, 0) is an Abelian group.

**Proof.** The proof comes from by Theorems 3.5 and 2.3. □

**REFERENCES**


J. Shohani, M. Afshar Jahanshahi, Department of Mathematics, Sistan and Baluchistan University, Zahedan, Iran
E-mail address: shohani@math.usb.ac.ir

R. A. Borzooei, Department of Mathematics, Islamic Azad University, Zahedan, Iran
E-mail address: borzooei@hamoon.usb.ac.ir