THEORIES OF ORDERED COMMUTATIVE MONOIDS

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Abstract. In this paper, we study some theories of lexicographic products of ordered commutative monoids. In particular we show that the lexicographic product of the ordered commutative monoid of nonnegative integers and the ordered commutative monoid of nonnegative rational numbers admits elimination of quantifiers in some expansive language of the language of ordered monoids.

1. Introduction

Let $L_{\text{og}} := \{0, +, <\}$, $L_0 := L_{\text{og}} \cup \{n \mid n > 0, n \in \mathbb{N}\} \cup \{1\}$ and $\mathbb{Q}_{\geq 0} := \{a \in \mathbb{Q} : a \geq 0\}$. It is well-known that the ordered commutative monoid $\mathbb{N}$ and the ordered abelian group $\mathbb{Z}$ admit elimination of quantifiers in the language $L_0$, where $n \mid x$ means ‘$n$ divides $x$’; see for example [4] or [10]. In [3] and [11], Komori and Weispfenning independently showed that the lexicographically ordered abelian group $\mathcal{M} := \mathbb{Z} \times \mathbb{Q}$ admits elimination of quantifiers in the same language $L_0$; here $0^\mathcal{M} := \langle 0, 0 \rangle$ and $1^\mathcal{M} := \langle 1, 0 \rangle$. In [8], the author showed the converse of them.

Let $\mathcal{L}$ be an expansion of $L_{\text{og}}$. Suppose that $H$ is an $\mathcal{L}$-structure whose reduct to the language $L_{\text{og}}$ is an ordered abelian group and $K$ is an ordered divisible abelian group. Then, extending the result of Komori and Weispfenning, Suzuki [7] showed that if $H$ admits elimination of quantifiers in $\mathcal{L}$ and the set $\{0\} \times K$ is defined by some quantifier-free $\mathcal{L}$-formula in the lexicographic product $G := H \times K$, then $G$ admits elimination of quantifiers in $\mathcal{L}$. In [9], the author and Yokoyama showed the converse of it. However, the lexicographically ordered commutative monoid $\mathcal{N} := \mathbb{N} \times \mathbb{Q}_{\geq 0}$ does not admit elimination of quantifiers in $L_0$, where $0^\mathcal{N} := \langle 0, 0 \rangle$ and $1^\mathcal{N} := \langle 1, 0 \rangle$ (Lemma 3.1).

In section 2, we give some axioms for ordered commutative monoids.

In section 3, we show that the lexicographically ordered commutative monoid $\mathbb{N} \times \mathbb{Q}_{\geq 0}$ admits elimination of quantifiers in the language $L$, where the language $L$ is the union of $L_0$, a unary relation symbol $R(x)$ and binary relation symbols $E_1(x, y), E_2(x, y)$. By Definition 2.1 we notice that the language $L$ is a definable expansion of $L_0$. We also show the converse of it.

In [3] and [11], Komori and Weispfenning studied model completions of theories of ordered abelian groups. In section 4, we study model completions of theories of ordered commutative monoids.

In [2], Belegradek, Verbovskiy and Wagner showed that the algebraic closure in $\text{Th}_{L_0}(\mathbb{Z} \times \mathbb{Q})$ satisfies the Exchange Principle. However, in section 5, we show that the algebraic closure in $\text{Th}_{L}(\mathbb{N} \times \mathbb{Q}_{\geq 0})$ does not satisfy the Exchange Principle.

In [1], Belegradek, Peterzil and Wagner showed that the $L_0$-structure $\mathbb{Z} \times \mathbb{Q}$ is quasi-o-minimal, that is, in any structure elementarily equivalent to $\mathbb{Z} \times \mathbb{Q}$ the definable subsets

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are exactly the Boolean combinations of \(0\)-definable subsets and intervals. However, in section 6, we show that the \(L\)-structure \(\mathbb{N} \times \mathbb{Q}_{\geq 0}\) is not quasi-o-minimal.

The reader is assumed to be familiar with model theory; see for example [5] or [6].

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2. SOME AXIOMS FOR ORDERED COMMUTATIVE MONOIDS

Let \(\mathbb{N}\) be the ordered commutative monoid of nonnegative integers. Let \(\mathbb{Z}\) be the ordered abelian group of integers. Let \(\mathbb{Q}\) be the ordered abelian group of rational numbers and \(\mathbb{Q}_{\geq 0}\) the ordered commutative monoid of nonnegative rational numbers.

We denote \(L_0 := \{0, +, < \} \cup \{n \mid x : n > 0, n \in \mathbb{N}\} \cup \{1\}\), where \(n \mid x\) is a unary relation symbol for each positive integer \(n\). We denote \(L := L_0 \cup \{R(x), E_1(x, y), E_2(x, y)\}\), where \(R(x)\) is a unary relation symbol and \(E_1(x, y), E_2(x, y)\) are binary relation symbols.

The terms \(t + \cdots + t\) and \(1 + \cdots + 1\) \((t\ and\ 1\ repeated\ n\ times)\) are written as \(nt\) and \(n\), respectively. The formulas \(s < t \land t < u\) and \(s < t \lor s = t\) are written as \(s < t < u\) and \(s \leq t\), respectively.

We now give some axioms for ordered commutative monoids.

Definition 2.1.

1. The axioms for commutative monoids:
   \[
   \forall x \forall y \forall z ((x + y) + z = x + (y + z)); \\
   \forall x (x + 0 = x); \\
   \forall x \forall y (x + y = y + x).
   \]

2. The axioms for a linear ordering compatible with monoid structures:
   \[
   \forall x \forall y (x < y \lor x = y \lor y < x); \\
   \forall x (\neg(x < x)); \\
   \forall x \forall y \forall z (x < y \to x + z < y + z); \\
   0 < 1; \\
   \forall x (0 \leq x).
   \]

3. The axioms for \(n \mid x\):
   \[
   \forall x (n \mid x \iff \exists y \exists z (z < 1 \land x = ny + z)) \text{ for each integer } n > 0; \\
   \forall x \left( \bigvee_{0 \leq r < n} n \mid x + r \right) \text{ for each integer } n > 1.
   \]

4. The axioms for infinitesimals:
   \[
   \forall x (x < 1 \to nx < 1) \text{ for each integer } n > 1.
   \]

5. The axiom for \(R(x)\):
   \[
   \forall x (R(x) \iff \forall y \forall z (y < x \land z < 1 - y + z < x)).
   \]

6. The axioms for \(E_1(x, y)\) and \(E_2(x, y)\):
   \[
   \forall x \forall y (E_1(x, y) \iff \exists z (x + z = y \land R(z))); \\
   \forall x \forall y (E_2(x, y) \iff \exists z (x + z = y \land \neg R(z))).
   \]

7. \(\forall x \exists y \exists z (x = y + z \land R(y) \land z < 1)\).

8. The axioms for difference:
   \[
   \forall x \forall y (x < y \land y < 1 \to \exists z (x + z = y)); \\
   \forall x \forall y (x < y \land R(x) \land R(y) \to \exists z (x + z = y)).
   \]

9. The axioms for divisible infinitesimals:
   \[
   \forall x (x < 1 \to \exists y (x = ny)) \text{ for each integer } n > 1.
   \]

10. The axiom for discrete ordering:
    \[
    \forall x (\neg(0 < x < 1)).
    \]

11. The axiom for existence of infinitesimals:
    \[
    \exists x (0 < x < 1).
    \]
Note that the language $L$ is a definable expansion of the language $L_0$.

**Definition 2.2.** We denote $\text{NSS} := (1) \cup (2) \cup (3) \cup (4) \cup (5) \cup (6) \cup (7) \cup (8)$. We denote $\text{NDC} := \text{NSS} \cup (10)$ and $\text{NSC} := \text{NSS} \cup (9) \cup (11)$.

We consider the lexicographic order from left to right on the ordered commutative monoid $\mathbb{N} \times \mathbb{Q}_{\geq 0}$. Then $\mathbb{N} \times \mathbb{Q}_{\geq 0}$ is a model of the theory NSC, where 0 and 1 are interpreted as $(0, 0)$ and $(1, 0)$, respectively. It is well-known that $\text{Th}_{\mathbb{N}}(\mathbb{N})$ admits elimination of quantifiers and $\text{Th}_{\mathbb{N}}(\mathbb{N}) = (1) \cup (2) \cup (3) \cup (10) \cup \{\forall x \forall y(x < y \rightarrow \exists z(x + z = y)\}$. Thus, it follows that the theory NDC admits elimination of quantifiers in the language $L$ and $\text{NDC} = \text{Th}_L(\mathbb{N})$.

### 3. Quantifier eliminable ordered commutative monoids

In this section, we show that the theory NSC admits elimination of quantifiers in the language $L$ and $\text{NSC} = \text{Th}_L(\mathbb{N} \times \mathbb{Q}_{\geq 0})$. We also show that if $M$ is a model of the theory NSC and $\text{Th}_L(M)$ admits elimination of quantifiers, then $M$ is a model of either the theory NDC or the theory NSC.

**Lemma 3.1.** The theory $\text{Th}_{\mathbb{N}}(\mathbb{N} \times \mathbb{Q}_{\geq 0})$ does not admit elimination of quantifiers.

**Proof.** Let $\varphi(x) := \forall y \forall z(y < x \land z < 1 \rightarrow y + z < x)$. Then we have $\mathbb{N} \times \{0\} = \varphi(\mathbb{N} \times \mathbb{Q}_{\geq 0})$. Thus, any quantifier-free $L_0$-formula is not equivalent to the formula $\varphi(x)$ in the theory $\text{Th}_{\mathbb{N}}(\mathbb{N} \times \mathbb{Q}_{\geq 0})$. \hfill $\Box$

To show that the theory NSC admits elimination of quantifiers in the language $L$, we first prove some lemmas needed later.

**Lemma 3.2.** We have that $\text{NSS} \models \forall x \forall y(x < 1 \land y < 1 \rightarrow x + y < 1)$.

**Proof.** Suppose for a contradiction that there exists a model $M$ of NSS and $x, y \in M$ such that $x < 1, y < 1$ and $x + y \geq 1$. By Axiom (4), we have $2x < 1$ and $2y < 1$. Thus, we obtain $2(x + y) < 2$, a contradiction. \hfill $\Box$

**Lemma 3.3.** We have that $\text{NSS} \models \forall x \forall y(R(x) \land R(y) \leftrightarrow R(x + y))$.

**Proof.** Let $M$ be a model of NSS. Let $x, y \in M$.

Suppose that $R(x)$ and $R(y)$. By Axiom (7), there exist $z_1, z_2 \in M$ with $R(z_1)$ and $z_2 < 1$ such that $x + y = z_1 + z_2$. Since $R(x)$ and $R(y)$, we obtain $x \leq z_1$ and $y \leq z_1$. By Axiom (8), there exists $u \in M$ with $y + u = z_1$. Since $x + y = z_1 + z_2$, we have $x = u + z_2$. Because $R(x)$ and $z_2 < 1$, we get $x = u$. Thus, we have $z_2 = 0$. It follows $x + y = z_1$, as desired.

Suppose that $R(x + y)$. By Axiom (7), there exist $x_1, x_2 \in M$ with $R(x_1)$ and $x_2 < 1$ such that $x = x_1 + x_2$. Since $x + y = (x_1 + y) + x_2$ and $R(x + y)$, we have $x + y = x_1 + y$. Hence $x = x_1$, and therefore $R(x)$. Similarly, we get $R(y)$. \hfill $\Box$

By Axiom (6), the following lemma holds.

**Lemma 3.4.** Let $i \in \{1, 2\}$. Then, we have $\text{NSS} \models \forall x \forall y \forall z(E_i(x, y) \leftrightarrow E_i(x + z, y + z))$.

**Lemma 3.5.** Let $p$ be a positive integer and $i \in \{1, 2\}$. Then, we have that $\text{NSS} \models \forall x \forall y(E_i(x, y) \leftrightarrow E_i(px, py))$.

**Proof.** We only show that $\text{NSS} \models \forall x \forall y(E_1(x, y) \leftrightarrow E_1(px, py))$. The other case is similar. Let $M$ be a model of NSS and $x, y \in M$.

Suppose that $E_1(x, y)$. There exists $z \in M$ with $R(z)$ such that $x + z = y$. Then $px + pz = py$. By Lemma 3.3, we get $R(pz)$. Thus, it follows $E_1(px, py)$.

Suppose that $E_1(px, py)$. There exists $u \in M$ with $R(u)$ such that $px + u = py$. By Axiom (7), there exist $x_1, x_2, y_1, y_2 \in M$ with $R(x_1), x_2 < 1, R(y_1), y_2 < 1$ such that
Theorem 3.6. The theory NSC admits elimination of quantifiers in the language $L$.

Proof. Let $\exists x \varphi$ be a formula, where $\varphi$ is a quantifier-free formula in $L$. We may assume that $\varphi$ is of the form $\psi_1 \land \cdots \land \psi_m$, where each $\psi_i$ is an atomic formula or the negation of an atomic formula. In addition, each $\psi_i$ is of the form $t = s$, $\neg(t = s)$, $t < s$, $\neg(t < s)$, $n | t$, $\neg(n | t)$, $R(t)$, $\neg(R(t))$, $E_1(t, s)$, $\neg E_1(t, s)$, $E_2(t, s)$ or $\neg E_2(t, s)$, where $t$ and $s$ are terms and $n$ is a positive integer. Moreover $\neg(t = s)$, $\neg(t < s)$ and $\neg(n | t)$ are equivalent to $t < s \lor s < t$, $t = s \lor s < t$ and $n | t + 1 \lor \cdots \lor n | t + n - 1$, respectively.

Now, each term $t$ can be written in the form $px + s$ with $p \in \mathbb{N}$ and $s$ a term which does not contain $x$. Therefore $\exists x \varphi$ can be written as

$$\exists x \left( \bigwedge_{i \in A} p_i x + t_i = s_i \land \bigwedge_{i \in B} u'_i x < q_i x + u_i \land \bigwedge_{i \in B'} r_i x + v_i < v'_i \land \bigwedge_{i \in C} n_i | x + w_i \land \psi \right),$$

where each $p_i, q_i, r_i, m_i, n_i$ are positive integer, each $s_i, t_i, u_i, u'_i, v_i, v'_i, w_i$ are terms which do not contain $x$, the sets $A, B, B', C$ may be empty, and the formula $\psi$ is a finite conjunction of formulas of the forms $R, E_1, E_2$ or the negation of these. By Axiom (9), for each positive integer $p$ the formula $p | x$ is equivalent to $\exists y (x = py)$. Thus, by Lemmas 3.3 and 3.5, we may assume that the formula $\exists x \varphi$ is equivalent to

$$\exists x \left( \bigwedge_{i \in A} x + t_i = s_i \land \bigwedge_{i \in B} u'_i x < u_i \land \bigwedge_{i \in B'} x + v_i < v'_i \land \bigwedge_{i \in C} n_i | x + w_i \land \psi \right).$$

Let $\theta(x)$ be the formula

$$\bigwedge_{i \in I_1} E_1(x, \alpha_i) \land \bigwedge_{i \in I_2} \neg E_1(x, \alpha'_i) \land \bigwedge_{i \in I_3} E_1(\beta_i, x) \land \bigwedge_{i \in I_4} \neg E_1(\beta'_i, x)$$

$$\land \bigwedge_{i \in J_1} E_2(x, \gamma_i) \land \bigwedge_{i \in J_2} \neg E_2(x, \gamma'_i) \land \bigwedge_{i \in J_3} E_2(\delta_i, x) \land \bigwedge_{i \in J_4} \neg E_2(\delta'_i, x),$$

where each $\alpha_i, \alpha'_i, \beta_i, \beta'_i, \gamma_i, \gamma'_i, \delta_i, \delta'_i$ are terms which do not contain $x$ and each $I_i, J_i$ may be empty. By Lemmas 3.3 and 3.4, we may assume that the formula $\exists x \varphi$ is equivalent to

$$\exists x \left( \bigwedge_{i \in A} x + t = s_i \land \bigwedge_{i \in B} u_i < x + t \land \bigwedge_{i \in B'} x + t < v_i \land \bigwedge_{i \in C} n_i | x + t + w_i \land \bigwedge_{i \in D} R(x) \land \bigwedge_{i \in D'} \neg R(x) \land \theta(x + t) \right),$$

where $t$ is term which does not contain $x$ and the sets $D, D'$ may be empty. If $D$ is not empty and $D'$ is empty, then we may assume that the formula $\exists x \varphi$ is equivalent to

$$\exists y \left( \bigwedge_{i \in A} y = s_i \land \bigwedge_{i \in B} u_i < y \land \bigwedge_{i \in B'} y < v_i \land \bigwedge_{i \in C} n_i | y + w_i \land \theta(y) \land E_1(t, y) \right).$$
If $D'$ is not empty and $D$ is empty, then we may assume that the formula $\exists x \varphi$ is equivalent to
\[
\exists y \left( \bigwedge_{i \in A} y = s_i \land \bigwedge_{i \in B} u_i < y \land \bigwedge_{i \in B'} y < v_i \land \bigwedge_{i \in C} n_i \mid y + w_i \land \theta(y) \land E_2(t, y) \right).
\]
Hence, without loss of generality we may assume that the formula $\exists x \varphi$ is equivalent to
\[
\exists y \left( u < y < v \land \bigwedge_{i \in C} n_i \mid y + w_i \land \theta(y) \right),
\]
where $u, v$ are terms which do not contain $x$. By Axiom (3), this formula is equivalent to
\[
\bigvee_{0 \leq r < n} \left( \bigwedge_{i \in C} n_i \mid r + w_i \land \exists z \left( u < nz + r < v \land \theta(nz + r) \right) \right),
\]
where $n$ is the least common multiple of $n_i \ (i \in C)$. By Lemma 3.4, the formula $\exists z(u < nz + r < v \land \theta(nz + r))$ is equivalent to $\exists z(u + (n-r) < n(z+1) \land \theta(n(z+1)))$ for each integer $r$ with $0 \leq r < n$. Hence, we may assume that the formula $\exists x \varphi$ is equivalent to $\exists z(u < nz < v \land \theta(nz))$. Moreover, we may assume that $\exists x \varphi$ is equivalent to
\[
\exists z(u < nz \land E_1(nz, \alpha)) \land -E_1(nz, \alpha') \land E_1(\beta, nz) \land -E_1(\beta', nz)
\land E_2(nz, \gamma) \land -E_2(nz, \gamma') \land E_2(\delta, nz) \land -E_2(\delta', nz)),
\]
where $\alpha, \alpha', \beta, \beta', \gamma, \gamma', \delta, \delta'$ are terms which do not contain $x$. Then, this formula is equivalent to some quantifier-free formula. For example, the formula $\exists z(u < nz \land E_1(nz, \alpha))$ is equivalent to
\[
u < \alpha \land (n \mid \alpha \lor [u \mid u \rightarrow E_2(u, \alpha) \lor u + n < \alpha] \lor \bigvee_{1 \leq r < n} (n \mid u + r \rightarrow u + r < \alpha))\]
Hence, the formula $\exists x \varphi$ is equivalent to some quantifier-free formula. Therefore, the theory NSC admits elimination of quantifiers in the language $L$. 

**Fact 3.7 ([5, Proposition 1.1.8]).** Let $M$ be an $L$-structure and $N$ a substructure of $M$. Suppose that $\varphi$ is a quantifier-free $L$-sentence. Then, $M \models \varphi$ if and only if $N \models \varphi$.

**Theorem 3.8.** The theory NSC is complete. Namely, we have NSC = Th$_L(\mathbb{N} \times \mathbb{Q}_{\geq 0})$.

**Proof.** Let $M$ be a model of NSC. Suppose that $f : N \rightarrow M$ by $f(n) = n^M$. Then $f$ is an embedding. Thus, by Theorem 3.6 and Fact 3.7, the theory NSC is complete. 

**Lemma 3.9.** Let $\psi(x)$ be a quantifier-free $L$-formula with one free variable $x$. Suppose that $M \models \text{NSS}$. Then, either $M \models \psi(a)$ for each $a$ with $0 < a < 1$, or $M \models \neg \psi(a)$ for each $a$ with $0 < a < 1$.

**Proof.** Let $\psi(x)$ be a quantifier-free $L$-formula with one free variable $x$. The formula $\psi(x)$ is equivalent to a Boolean combination of formulas which is of the forms $px + q = 0$, $px = q$, $px + q < 0$, $px < 0$, $q < px$, $n \mid px + q$, $R(px + q)$, $E_1(px + q)$, $E_1(0, px + q)$, $E_1(q, px)$, $E_2(px + q, 0)$, $E_2(px, q)$, $E_2(0, px + q)$, $E_2(q, px)$, where $n, p \in \mathbb{N} \setminus \{0\}$ and $q \in \mathbb{N}$.

Let $M \models pa = q$ for some $a \in M$ with $0 < a < 1$. Then, by Axiom (4), we have $0 < pa < 1$, a contradiction. 

Let $M \models pa < q$ for some $a \in M$ with $0 < a < 1$. Then, by $0 < pa$, we have $1 < q$. Thus, $M \models px < q$ for each $x \in M$ with $0 < x < 1$.

Let $M \models q < pa$ for some $a \in M$ with $0 < a < 1$. Then, by $pa < 1$, we have $q = 0$. Thus, $M \models q < px$ for each $x \in M$ with $0 < x < 1$.

Let $M \models n \mid pa + q$ for some $a \in M$ with $0 < a < 1$. Then, by $0 < pa < 1$, there exists $m \in \mathbb{N}$ such that $q = mn$. Thus, we have $M \models n \mid px + q$ for each $x \in M$ with $0 < x < 1$.

Let $M \models E_1(0, pa + q)$ for some $a \in M$ with $0 < a < 1$. Then, by $0 < pa < 1$, we get $M \models -R(pa + q)$, a contradiction.
The other cases are similar. This completes the proof of the lemma. \hfill \Box

We show the converse of Theorem 3.6.

**Theorem 3.10.** Let $M$ be a model of NSS. Suppose that $\text{Th}_L(M)$ admits elimination of quantifiers. Then $M$ is a model of either NDC or NSC. Namely, we have either $M \equiv_L \mathbb{N}$ or $M \equiv_L \mathbb{N} \times \mathbb{Q}_{\geq 0}$.

**Proof.** First, suppose that Axiom (10) holds in $M$. Then, the structure $M$ is a model of NDC.

Secondly, suppose that Axiom (11) holds in $M$. Let $n$ be a positive integer with $n > 1$. Because $\text{Th}(M)$ admits elimination of quantifiers, there exists a quantifier-free $L$-formula $\psi_n(x)$ such that

$$\text{Th}(M) \models \forall x[(x < 1 \to \exists y(x = ny)) \leftrightarrow \psi_n(x)].$$

Let $a \in M$. Now $M \models \psi_n(a)$ if $a = 0$ or $1 \leq a$. Assume that $0 < a < 1$. Then $M \models \psi_n(na)$. By Lemma 3.9, we obtain $M \models \psi(a)$. It follows that $M \models \psi(a)$ for each $a \in M$. Thus, Axiom (9) holds in $M$. Therefore, the structure $M$ is a model of NSC. \hfill \Box

**Remark 3.11.** By Theorem 3.10, the lexicographically ordered commutative monoid $\mathbb{N} \times \mathbb{N}$ does not admit elimination of quantifiers in the language $L$, where 0 and 1 are interpreted as $(0, 0)$ and $(1, 0)$, respectively. However, in a similar way to Theorem 3.6 we show that the lexicographically ordered commutative monoid $\mathbb{N} \times \mathbb{N}$ admits elimination of quantifiers in the language $L \cup \{1^t\}$, where $1^t$ is interpreted as $(0, 1)$.

## 4. Model completion

In this section, we show that the theory NSC is a model completion of the theory NSS.

Recall the notion of the model companion and the model completion from [5].

**Definition 4.1.** Let $L$ be a language and $M$ an $L$-structure. Suppose that $\text{Diag}(M) := \{\varphi(m_1, \ldots, m_n) : \varphi(x_1, \ldots, x_n) \text{ is either an atomic } L\text{-formula or the negation of an atomic } L\text{-formula, } m_1, \ldots, m_n \in M \text{ and } M \models \varphi(m_1, \ldots, m_n)\}$. Suppose that $T$ and $T'$ are $L$-theories. We say that $T'$ is a model companion of $T$ if

(i) $T'$ is model-complete;

(ii) every model of $T$ has an extension that is a model of $T'$;

(iii) every model of $T'$ has an extension that is a model of $T$.

Moreover, if $T'$ is a model companion of $T$ and $T' \cup \text{Diag}(M)$ is a complete $L(M)$-theory for any $M \models T$, then $T'$ is called a model completion of $T$.

**Fact 4.2 ([3]).** $\text{Th}_{L_0}(\mathbb{Z} \times \mathbb{Q})$ is a model completion of the $L_0$-theory SS, where the $L_0$-theory SS is defined by [3].

**Lemma 4.3.** Any model of NSS can be embedded in a model of NSC.

**Proof.** Let $M$ be a model of NSS.

Suppose that Axiom (10) holds in $M$. We now consider the lexicographic order on $M \times \mathbb{Q}_{\geq 0}$. Then, the lexicographically ordered commutative monoid $M \times \mathbb{Q}_{\geq 0}$ is a model of NSC, where 0 and 1 are interpreted as $(0, 0)$ and $(1, 0)$, respectively. Moreover, $f : M \to M \times \mathbb{Q}_{\geq 0}$ by $f(a) = (a, 0)$ is an embedding.

On the other hand, suppose that Axiom (11) holds in $M$. Let $N := \{(a, n) : a \in M, n > 0, n \in \mathbb{N}, n \mid a\}$. We define an equivalence relation $\sim$ on $N$ by $(a, m) \sim (b, m)$ if $na = mb$. Let $[(a, m)]$ denote the $\sim$-class of $(a, m) \in N$ and $N' := \{[(a, m)] : (a, m) \in N\}$. We define $+$ on $N'$ by $[(a, m)] + [(b, n)] := [(na + mb, mn)]$. We also define an order on $N'$ by...
Theorem 4.5. Let \( acl(\langle a, b \rangle) \) be the algebraic closure of \( a \) in \( M \). Then, the ordered commutative monoid \( N' \) is a model of NSC. Moreover, \( g : M \to N' \) by \( g(a) = \langle [a, 1] \rangle \) is an embedding.

**Fact 4.4** ([6, Theorem 9.2.2]). Let \( L \) be a language and \( T \) an \( L \)-theory. Then, the following conditions are equivalent:

(i) \( T \) admits elimination of quantifiers in the \( L \);

(ii) For every model \( M \) of \( T \) and every substructure \( A \) of \( M \), we have that \( T \cup \text{Diag}(A) \) is a complete \( L(A) \)-theory.

By Theorem 3.6, Fact 4.4 and Lemma 4.3, we have the following.

**Theorem 4.5.** The theory NSS is a model completion of the theory NSC.

Recall that the \( L_0 \)-theory SC is the set of the following axioms ([3]).

- The axioms for ordered abelian groups.
- The axioms for a semi-discrete ordering:
  \( 0 < 1, \forall x (2x < 1 \lor 1 < 2x) \).
- The axioms for infinitesimals:
  \( \forall x (2x < 1 \rightarrow nx < 1) \) for each integer \( n > 2 \).
- The axioms for divisible infinitesimals:
  \( \forall x (\exists \exists 3z (0 < 2z + 1 \land 2z < 1 \land x = ny + z)) \) for each integer \( n > 0 \),
\( \forall x (\forall 0 \leq r < n \exists n | x + r) \) for each integer \( n > 0 \).
- The axioms for existence of infinitesimals:
  \( \exists x (0 < x < 1) \).

**Fact 4.6** ([3, Theorem 1.1]). The \( L_0 \)-theory SC admits elimination of quantifiers and is complete. In particular, we have \( SC = \text{Th}_{L_0}(\mathbb{Z} \times \mathbb{Q}) \), where we interpret 0 and 1 as \( \langle 0, 0 \rangle \) and \( \langle 1, 0 \rangle \), respectively.

**Theorem 4.7.** Any model of NSC can be embedded in a model of SC.

**Proof.** Let \( M \) be a model of NSC. We define an equivalence relation \( \sim \) on \( M \times M \) by \( \langle a, b \rangle \sim \langle a', b' \rangle \) if \( a + b' = a' + b \). Let \( [(a, b)] \) denote the \( \sim \)-class of \( \langle a, b \rangle \) in \( M \times M \) and \( N := \{ [(a, b)] : \langle a, b \rangle \in M \times M \} \). We define + on \( N \) by \( [(a, b)] + [(a', b')] := [(a + a', b + b')] \).

We also define an order on \( N \) by \( [(a, b)] < [(a', b')] \) if \( a + b' < a' + b \).

Then, the structure \( N \) is a model of SC. Moreover, \( f : M \to N \) by \( f(a) = [(a, 0)] \) is an embedding. This completes the proof of the theorem.

5. Exchange principle

In this section, we show that the algebraic closure in the theory NSS does not satisfy the Exchange Principle.

Let \( L \) be a language and \( M \) an \( L \)-structure. Let \( A \) be a subset of \( M \). We say that \( a \in M \) is algebraic over \( A \) if there exists an \( L \)-formula \( \varphi(x, y_1, \ldots, y_n) \) and \( b_1, \ldots, b_n \in A \) such that \( M \models \varphi(a, b_1, \ldots, b_n) \) and \( \{ c \in M : M \models \varphi(c, b_1, \ldots, b_n) \} \) is finite. The algebraic closure of \( A \) in \( M \), denoted \( acl(A) \), is given by \( \{ a \in M : a \text{ is algebraic over } A \} \).

**Definition 5.1.** Let \( L \) be a language and \( M \) an \( L \)-structure. We say that the algebraic closure in \( M \) satisfies the *Exchange Principle* if \( A \subseteq M \), \( a, b \in M \) and \( a \in acl(A \cup \{ b \}) \setminus acl(A) \), then \( b \in acl(A \cup \{ a \}) \). The algebraic closure in an \( L \)-theory \( T \) is said to satisfy the *Exchange Principle* if the algebraic closure in any model \( M \) of \( T \) satisfies the Exchange Principle.
**Fact 5.2** ([2, Corollary 50]). *The algebraic closure in the theory Th_{L_0}(\mathbb{Z} \times \mathbb{Q}) satisfies the Exchange Principle.*

**Theorem 5.3.** *The algebraic closure in the theory NSC does not satisfy the Exchange Principle.*

Proof. Let $\mathbb{Q}_{>0} := \{a \in \mathbb{Q} : a > 0\}$ and $\mathbb{R}_{\geq 0} := \{a \in \mathbb{R} : a \geq 0\}$. Let $M_1$ be the lexicographically ordered structure $\{0\} \times \mathbb{N} \times \mathbb{R}_{\geq 0}$ and $M_2$ the lexicographically ordered structure $\mathbb{Q}_{>0} \times \mathbb{Z} \times \mathbb{R}_{\geq 0}$. In $M := M_1 \sqcup M_2$ we define

- $+$ is defined coordinatewise;
- $a < b$ whenever $a \in M_1$ and $b \in M_2$;
- $0$ and $1$ are interpreted as $\langle 0, 0, 0 \rangle$ and $\langle 0, 1, 0 \rangle$, respectively.

Then $M$ is a model of the theory NSC.

Let $a = \langle 1, 3, 2 \rangle$, $b = \langle 2, 6, 4\sqrt{2} \rangle$ and $c = \langle 0, 0, 4\sqrt{2} \rangle$. Let $\varphi(x) := 2a < x < 2a + 1 \land E_1(c, x)$. Then $\{x \in M : M \models \varphi(x)\} = \{b\}$. Thus, we get $b \in acl(\{a, c\})$. There exists no $n \in \mathbb{N}$ with $\langle 2, 6, 0 \rangle = n\langle 0, 1, 0 \rangle$. Hence, by Theorem 3.6, we have $b \notin acl(\{c\})$. Now, there exists no $p \in \mathbb{Q}$ with $2 = 4p\sqrt{2}$. Thus, by Theorem 3.6, we obtain $a \notin acl(\{b, c\})$. Therefore, the algebraic closure in $M$ does not satisfy the Exchange Principle. This finishes the proof. \(\square\)

**Remark 5.4.** *The algebraic closure in the $L$-structure $\mathbb{N} \times \mathbb{Q}_{\geq 0}$ satisfies the Exchange Principle.*

### 6. Non-quasi-o-minimality

In this section, we show that the $L$-structure $\mathbb{N} \times \mathbb{Q}_{\geq 0}$ is not quasi-o-minimal.

Recall the notion of quasi-o-minimal structures from [1].

**Definition 6.1.** A structure $(M, <, \ldots)$, where $<$ is a linear ordering of $M$, is said to be *quasi-o-minimal* if in any structure elementarily equivalent to it the definable subsets are exactly the Boolean combinations of $\emptyset$-definable subsets and intervals.

The following fact is known.

**Fact 6.2** ([1]). *The $L_0$-structures $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Z} \times \mathbb{Q}$ are quasi-o-minimal.*

**Theorem 6.3.** *The $L$-structure $\mathbb{N} \times \mathbb{Q}_{\geq 0}$ is not quasi-o-minimal.*

Proof. Suppose for a contradiction that the $L$-structure $\mathbb{N} \times \mathbb{Q}_{\geq 0}$ is quasi-o-minimal. Let $\varphi(x) := E_1(\langle 2, 1, x \rangle)$ and $A := \varphi(\mathbb{N} \times \mathbb{Q}_{\geq 0})$. Then $A = \{\langle n, 1 \rangle : n \geq 2, n \in \mathbb{N}\}$, i.e., $A$ is infinite. By quasi-o-minimality of $\mathbb{N} \times \mathbb{Q}_{\geq 0}$, there exist $a, b \in \mathbb{N} \times \mathbb{Q}_{\geq 0}$ and infinite $\emptyset$-definable set $B$ such that $a < b$ and $A$ contains $(a, b) \cap B$. By Theorem 3.6, there exists some quantifier-free formula $\psi(x)$ without parameters such that $B = \psi(\mathbb{N} \times \mathbb{Q}_{\geq 0})$. Since $B$ is infinite, there exists $n \in \mathbb{N}$ and $d \in \mathbb{Q}_{\geq 0}$ such that $d \neq 1$ and $\langle n, d \rangle \in (a, b) \cap B$. It follows $(a, b) \cap B \not\subseteq A$, a contradiction. \(\square\)

**References**


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