EXPLICIT FORMULAS FOR THE REPRODUCING KERNELS OF THE SPACE OF HARMONIC POLYNOMIALS IN THE CASE OF CLASSICAL REAL RANK 1

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Abstract. We give the explicit formulas of the reproducing kernels of the space of harmonic polynomials of \( p \subset \mathfrak{g} \) in the case of classical real rank 1, which are generalizations of the well-known reproducing formulas of classical harmonic polynomials on the unit sphere or any other \( SO(p) \)-orbits in \( \mathbb{C}^p \). These formulas are expressed as integrals on a single orbit, simplifying our previous results that are expressed as double integrals on some family of nilpotent orbits.

Introduction.

For harmonic functions on \( \mathbb{R}^p \) there are many studies. Especially, the following reproducing formula on the unit sphere \( S^{p-1} \) is well-known:

\[
\delta_{n,m} f(s) = \dim H_{n,p} \int_{S^{p-1}} f(s_1) P_{n,p}(s \cdot s_1) ds_1 \quad (s \in S^{p-1}, f \in H_{m,p}),
\]

where \( H_{n,p} \) is the space of spherical harmonics of degree \( n \) in dimension \( p \), and \( P_{n,p}(t) = \frac{(p-3)\mu!(n^2 - \frac{p^2}{3})!}{(n+p-3)!} C_n^{\frac{p^2}{3}}(t) \) is the Legendre polynomial of degree \( n \) in dimension \( p \) and \( C_n^{\nu}(t) \) is the Gegenbauer function (cf. [1], [7], [8], [11], etc). We denote by \( H_n(\mathbb{C}^p) \) the space of polynomials \( f \) on \( \mathbb{C}^p \) of degree \( n \) which satisfy \( \sum_{j=1}^{p} \frac{\partial^2 f}{\partial z_j^2} = 0 \). Then homogeneous harmonic polynomials on \( \mathbb{R}^p \) of degree \( n \) are uniquely extended to the element of \( H_n(\mathbb{C}^p) \). The reproducing formulas of \( H_n(\mathbb{C}^p) \) on any non-trivial \( SO(p) \)-orbit in \( \mathbb{C}^p \) are also known in addition to the above case \( S^{p-1} \) (cf. [2], [9], [10], [17], [21]). For details on harmonic polynomials and harmonic functions on \( \mathbb{R}^p \), see also [15], [16].

In this paper, we further generalize these formulas from the Lie algebraic standpoint in a unified way. According to the formulation of [5], harmonic polynomials on \( \mathbb{R}^p \) can be canonically identified with harmonic polynomials on the vector space \( p \) which is the complexification of \( \mathfrak{p} \) appeared in a Cartan decomposition of the Lie algebra \( \mathfrak{g}_\mathbb{R} = \mathfrak{so}(p,1) \). In this situation, any \( SO(p) \)-orbit in \( \mathbb{C}^p \) corresponds to a \( K_R \)-orbit in \( \mathfrak{p} \), where \( K_R \) is a Lie subgroup of \( GL(p) \) generated by \( \exp \text{ad} X (X \in \mathfrak{f}_R) \). Thus, the integral formulas of harmonic polynomials on \( \mathbb{R}^p \) can be rewritten explicitly as integral representation formulas on each \( K_R \)-orbits (cf. Appendix of [18]).

In [20] we generalize these formulas to the case where the Lie algebra \( \mathfrak{g}_R \) is real rank 1: i.e. \( \mathfrak{g}_R = \mathfrak{so}(p,1) \) \( (p \geq 2) \), \( \mathfrak{su}(p,1) \), \( \mathfrak{sp}(p,1) \) \( (p \geq 1) \) or \( \mathfrak{f}_4(-20) \) by constructing the reproducing kernels for each case (cf. Theorems 1.2 and 1.3). We denote by \( \mathfrak{f}_R = \mathfrak{f}_R + \mathfrak{p}_R \) a Cartan decomposition of \( \mathfrak{g}_R \) and put \( K_R = \exp \mathfrak{f}_R \). In [20] we express these formulas as integrals of the reproducing kernels on a single \( K_R \)-orbit in a unified way, simplifying the

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formulas previously obtained in [18], [19], where the integral formulas for two cases \( \mathfrak{su}(p, 1) \) and \( \mathfrak{sp}(p, 1) \) are expressed in the form of double integrals on some family of nilpotent \( \mathbb{K}_R \)-orbits. In particular the reproducing kernels are expressed in simple forms for nilpotent orbits. In this paper we give a complete proof of these results for the case \( \mathfrak{sp}(p, 1) \) which is omitted in [20], together with that of the case \( \mathfrak{su}(p, 1) \) for the sake of completeness.

Concerning reproducing formulas, the results of Nagel-Rudin [12] and Rudin [13] are also known. Their results correspond to our formula for the case \( \mathfrak{g}_R = \mathfrak{su}(p, 1) \). Let \( \hat{H}_n(C^p) \) be the space of homogeneous polynomials \( f \) on \( C^p \cong \mathbb{R}^{2p} \) of degree \( n \) in the variables \( z_1, z_2, \cdots, z_p, \bar{z}_1, \bar{z}_2, \cdots, \bar{z}_p \) which satisfy \( \sum_{j=1}^p \bar{z}_j z_j = 0 \). For nonnegative integers \( k \) and \( l \) we denote by \( S^{k, l} \) the space of polynomials on \( C^p \) which have total degree \( k \) in the variables \( z_1, z_2, \cdots, z_p \) and total degree \( l \) in the variables \( \bar{z}_1, \bar{z}_2, \cdots, \bar{z}_p \). Set \( H^{k, l} = S^{k, l} \cap \hat{H}_{k+l}(C^p) \). Then the Lie group \( U(p) \) naturally acts on the space \( H_n(C^p) \) and \( H^{k, n-k} \) is a \( U(p) \)-invariant subspace of \( H_n(C^p) \). The sum \( H_n(C^p) = \bigoplus_{k=0}^n H^{k, n-k} \) gives the \( U(p) \)-irreducible decomposition (cf. [16]). And the reproducing formulas of \( H^{k, l} \) on the unit sphere \( \{ z \in C^p; |z| = 1 \} \) of \( C^p \) are explained in detail in [12], [13]. In this setting the element of \( H_n(C^p) \) corresponds to a harmonic polynomial on \( \mathfrak{p} \) and the unit sphere of \( C^p \) corresponds to one \( \mathbb{K}_R \)-orbit for the case \( \mathfrak{g}_R = \mathfrak{su}(p, 1) \).

The plan of this paper is roughly stated as follows: In §1 we recall the definitions and some fundamental properties of harmonic polynomials on \( \mathfrak{p} \) which we use in this paper, mainly following the results stated in [20]. In §2 we review the principal results for the case \( \mathfrak{su}(p, 1) \), which is previously stated in [20]. In §3–§5 we consider the case \( \mathfrak{sp}(p, 1) \). In §3 we review some known results on harmonic polynomials on \( \mathfrak{p} \) when \( \mathfrak{g}_R = \mathfrak{sp}(p, 1) \). In §4 we give the \( \mathbb{K}_R \)-irreducible decomposition of harmonic polynomials on \( \mathfrak{p} \), which is the principal part of this paper and state the main theorem (Theorem 4.5) by using the properties of \( \mathbb{K}_R \)-irreducible components. Finally in Appendix, we determine the dimension of the \( \mathbb{K}_R \)-irreducible component.

Thus, we obtain the reproducing kernels on each \( \mathbb{K}_R \)-orbit for all cases of classical real rank 1. To obtain integral formulas of harmonic polynomials in cases of classical real rank 2 is our next theme.

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1. Harmonic polynomials on \( \mathfrak{p} \).

In this section we fix several notations which we use in this paper, and recall the definitions and the known results on harmonic polynomials.

Let \( \mathfrak{g} \) be a complex semisimple Lie algebra, \( \mathfrak{g}_R \) be a noncompact real form of \( \mathfrak{g} \). \( \mathfrak{g}_R = \mathfrak{fr} + \mathfrak{pr} \) be a Cartan decomposition of \( \mathfrak{g}_R \) and \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) be its complexification. We put \( G = \text{exp} \mathfrak{g} \) and \( K_\theta = \{ g \in G; \theta g = g \theta \} \), where the involution \( \theta : \mathfrak{g} \rightarrow \mathfrak{g} \) is defined by \( \theta = 1 \) on \( \mathfrak{k} \) and \( \theta = -1 \) on \( \mathfrak{p} \). Let \( K \) be the identity component of \( K_\theta \). Then we have \( K = \text{exp} \mathfrak{k} \).

Now we define harmonic polynomials on \( \mathfrak{p} \) as follows. We denote by \( S \) and \( S_\theta \) the spaces of polynomials on \( \mathfrak{p} \) and homogeneous polynomials on \( \mathfrak{p} \) of degree \( n \), respectively. For \( f \in S \) and \( g \in K_\theta \), we define \( g f \in S \) by \( (g f)(X) = f(g^{-1}X) \) \( (X \in \mathfrak{p}) \). We denote by \( J \) the ring of \( K \)-invariant polynomials on \( \mathfrak{p} \) and put \( J_+ = \{ f \in J; f(0) = 0 \} \). It is known that \( J \) is also \( K_\theta \)-invariant. According to the definition in [5], a polynomial \( f \in S \) is called harmonic if and only if \( (\partial P)f = 0 \) for any \( P \in J_+ \). We denote by \( \mathcal{H}_n \) the space of homogeneous harmonic polynomials on \( \mathfrak{p} \) of degree \( n \). In the following we put \( \mathbb{Z}_+ = \{ 0, 1, 2, \cdots \} \). The following results are well known:
Theorem 1.1 (cf. [1], [5]). (i) For any \( n \in \mathbb{Z}_+ \) we have
\[
S_n = (J_+ S)_n \oplus \mathcal{H}_n,
\]
where we put \((J_+ S)_n = S_n \cap J_+ S\).

(ii) We put \( N = \{ X \in \mathfrak{p} ; P(X) = 0 \text{ for any } P \in J_+ \} \) and let \( h(X,Y) \) be a nondegenerate symmetric bilinear form on \( \mathfrak{p} \). Then \( \mathcal{H}_n \) is generated by \( \{ h( , Z)^n ; Z \in N \} \).

(iii) Let \( \mathfrak{g} \) be a \( K_\mathfrak{p} \)-orbit in \( \mathfrak{p} \) of maximal dimension. Then the restriction mapping \( f \longrightarrow f|_0 \) is a bijection from \( \mathcal{H}_n \) onto \( \mathcal{H}_n|_0 \).

For further properties on harmonic polynomials on \( \mathfrak{p} \), see [1], [5].

From now we consider the case where \( \mathfrak{g}_R \) is a classical simple Lie algebra with real rank 1, i.e. \( \mathfrak{g}_R = \mathfrak{so}(p, 1) \) \((p \geq 2)\), \( \mathfrak{su}(p, 1) \) \((p \geq 1)\) or \( \mathfrak{sp}(p, 1) \) \((p \geq 1)\). Let \( K_\mathfrak{R} \) be the adjoint group of \( \mathfrak{t}_\mathfrak{R} \): \( K_\mathfrak{R} = \exp \mathfrak{t}_\mathfrak{R} \). Then it is known that \( K_\mathfrak{R} \) acts on the space \( \mathcal{H}_n \), and we denote by
\[
\mathcal{H}_n = \bigoplus_{k=0}^{N(n)} \mathcal{H}_{n,k}
\]
the \( K_\mathfrak{R} \)-irreducible decomposition of \( \mathcal{H}_n \), where \( N(n) + 1 \) is the number of \( K_\mathfrak{R} \)-irreducible components. Now we assume that \( \mathcal{H}_{n,k} \not\cong \mathcal{H}_{m,l} \) if \((n,k) \neq (m,l)\). Then under this condition, the following results are proved in the previous paper [20].

Theorem 1.2 ([20] Theorem 1.3). Up to a non-zero constant there exists a unique function \( \tilde{H}_{n,k}(X,Y) \neq 0 \) \((0 \leq k \leq N(n))\) defined on \( \mathfrak{p} \times \mathfrak{p} \) such that
\[
\begin{align*}
(1.1) & \quad \tilde{H}_{n,k}( ,Y) \in \mathcal{H}_{n,k} \text{ for any } Y \in \mathfrak{p}, \\
(1.2) & \quad \tilde{H}_{n,k}(gX,gY) = \tilde{H}_{n,k}(X,Y) \text{ for any } g \in K_\mathfrak{R} \text{ and any } X,Y \in \mathfrak{p}, \\
(1.3) & \quad \tilde{H}_{n,k}(X,Y) = \underline{\tilde{H}_{n,k}(Y,X)} \text{ for any } X,Y \in \mathfrak{p}.
\end{align*}
\]

Theorem 1.3 ([20] Theorem 1.3). Let \( \tilde{H}_{n,k}(X,Y) \neq 0 \) \((0 \leq k \leq N(n))\) be a function which satisfies the conditions (1.1)–(1.3). Suppose \( X_0 \in \mathfrak{p} \) and \( \tilde{H}_{n,k}(X_0,X_0) \neq 0 \). Then for any \( f \in \mathcal{H}_{m,l} \) and \( X \in \mathfrak{p} \) the following reproducing formula of harmonic polynomials holds on each \( K_\mathfrak{R} \)-orbit \( K_\mathfrak{R} X_0 \):
\[
\delta_{n,m}\delta_{k,l} f(X) = \frac{\dim \mathcal{H}_{n,k}}{\tilde{H}_{n,k}(X_0,X_0)} \int_{K_\mathfrak{R}} f(gX_0)\tilde{H}_{n,k}(X,gX_0) dg.
\]

Here \( dg \) means the normalized Haar measure on \( K_\mathfrak{R} \).

Remark 1.4. To prove Theorem 1.2 and Theorem 1.3 we need the assumption \( \mathcal{H}_{n,k} \not\cong \mathcal{H}_{m,l} \) \((n,k) \neq (m,l)\). In the case \( \mathfrak{g}_R = \mathfrak{su}(p, 1) \) this fact is proved in Corollary of [16; p.241]. The proof for the case \( \mathfrak{g}_R = \mathfrak{sp}(p, 1) \) will be given in Proposition 4.2(ii) of this paper.

Remark 1.5. In the case \( \mathfrak{g}_R = \mathfrak{so}(p, 1) \) the above equality (1.4) is already known as a formula of classical harmonic polynomials on \( \mathbb{C}^p \) \((\simeq \mathfrak{p})\) and the above function \( \tilde{H}_{n,k}(X,Y) \) can be expressed explicitly in terms of the Legendre polynomial of degree \( n \) in dimension \( p \) (see, for example, [1], [2], [11], [17], [21]). When \( \mathfrak{g}_R = \mathfrak{su}(p, 1) \), the equality (1.4) is known as
a formula of polynomials on the space $H^{k,j}$ if $X_0 \in \mathfrak{p}_R$ and $Tr (X_0^2) = 2$ (cf. [12], [13]). But for the remaining cases of $\mathfrak{su}(p,1)$, including $\mathfrak{sp}(p,1)$ and $\mathfrak{so}(p,2)$, the function $\tilde{H}_{n,k}(X,Y)$ defined in [20] is expressed as a double integral of some inexplicit functions and is not so clear. In this paper we express $\tilde{H}_{n,k}(X,Y)$ as an integral of explicitly given polynomials on a single $K_R$-orbit of $\mathfrak{p}$ for two cases $\mathfrak{g}_R = \mathfrak{su}(p,1)$ and $\mathfrak{sp}(p,1)$.

2. Integral formulas of harmonic polynomials: The case of $\mathfrak{su}(p,1)$.

In this section we give the reproducing kernel of each irreducible subspace of $\mathcal{H}_n$ on $K_R$-orbits in $\mathfrak{p}$ for the case $\mathfrak{g}_R = \mathfrak{su}(p,1)$ ($p \geq 1$) (Theorem 2.1). The principal results for this case are already stated in [20]. The reproducing kernel $\tilde{H}_{n,k}(X,Y)$ takes a somewhat simpler form in case $X$ or $Y$ is contained in nilpotent orbits. Here we also give a proof of this fact.

In the case $\mathfrak{g}_R = \mathfrak{su}(p,1)$, we have

$$\mathfrak{t}_R = \left\{ \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix} : A \in \mathfrak{u}(p), \alpha \in \mathfrak{u}(1), \ Tr A + \alpha = 0 \right\},$$

$$\mathfrak{p}_R = \left\{ \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in \mathbb{C}^p \right\},$$

$$\mathfrak{t} = \left\{ \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix} : A \in \mathbb{M}(p,\mathbb{C}), \ Tr A + \alpha = 0 \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} : x, y \in \mathbb{C}^p \right\},$$

and $K_R = \text{Ad} S(U(p) \times U(1)) = \{ \text{Ad} \left( \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} : A \in U(p) \right) \}$. Let $B(\ , \ )$ be the Killing form on $\mathfrak{p}$. For $X = \left( \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right) \in \mathfrak{p}$, the polynomial

$$P(X) = (4p + 4)^{-1} B(X, X) = \frac{1}{2} Tr (X^2) = \ 'yx,$$

gives a generator of $J$. We put

$$\Sigma = \{ X \in \mathfrak{p} : P(X) = 0 \},$$

$$\Sigma_n = \{ X \in \mathfrak{p} : P(X) = 1 \},$$

and

$$\Sigma_R = \Sigma \cap \mathfrak{p}_R.$$

We denote by $\mathcal{H}_n = \{ f \in S_n : \sum_{j=1}^p \frac{\partial^2 f}{\partial x^j \partial y^j} = 0 \}$ the space of homogeneous harmonic polynomials on $\mathfrak{p}$ of degree $n$. For $X = \left( \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right) \in \mathfrak{p}$, we define the bijection $\Psi : \mathfrak{p} \rightarrow \mathbb{C}^{2p}$ by $\Psi(X) = \frac{1}{2} \left( \begin{pmatrix} x + y \\ i(y - x) \end{pmatrix} \right)$, and let $H_n(\mathbb{C}^{2p})$ be the space of homogeneous polynomials on $\mathbb{C}^{2p}$ of degree $n$ which satisfy $\sum_{j=1}^p \frac{\partial^2 f}{\partial x^j \partial y^j} = 0$. Then $f \in \mathcal{H}_n$ if and only if $f \circ \Psi^{-1} \in H_n(\mathbb{C}^{2p})$, and we have

$$\dim \mathcal{H}_n = \dim H_n(\mathbb{C}^{2p}) = \frac{2(n + p - 1)(n + 2p - 3)!}{n!(2p - 2)!}.$$  

Remark that the restriction mapping $\Psi : \Sigma_R \rightarrow S^{2p-1}$ is also bijective. This implies that $P_{n,2p} \left( \frac{\partial^2 f}{\partial x \partial y} \right) (P(X))^{n/2} (X \in \mathfrak{p}, Y \in \Sigma_R)$ is the reproducing kernel of $\mathcal{H}_n$ on $\Sigma_R$, where $P_{n,q}(t)$ is the Legendre polynomial of degree $n$ in dimension $q$ (cf. [8], [11], etc). Note
that the Legendre polynomial is related to the Gegenbauer function $C_n^\nu(t)$ by the equality

$$P_{n,q}(t) = \frac{(q-3)!!}{(n+q-3)!!}C_n^\nu(t).$$

In the rest of this section we assume $p \geq 2$. For the case $p = 1$, see Remark 2.3 at the end of this section. For $X = \left( \begin{array}{c} 0 \\ 0 \\ v \end{array} \right) \in \mathfrak{p}$ and $g = \text{Ad} \left( \begin{array}{cc} A & 0 \\ 0 & 1 \end{array} \right) \in K_\mathbb{R}$ ($A \in U(p)$) we have $gX = \left( \begin{array}{c} 0 \\ 0 \\ Ax \end{array} \right)$. We put

$$E_1 = \left( \begin{array}{cc} 0 & e_1 \\ e_1 & 0 \end{array} \right) \in \Sigma_\mathbb{R},$$

$$\tilde{E}_{r,q} = \left( \begin{array}{cc} 0 & re_1 \\ \left( \frac{1}{r} e_1 + qe_2 \right) & 0 \end{array} \right) \in \Sigma \quad (r > 0, q \geq 0),$$

$$\tilde{E}_r = \left( \begin{array}{cc} 0 & re_1 \\ \sqrt{1-r^2} e_2 & 0 \end{array} \right) \in \mathbb{N} \quad (0 \leq r \leq 1),$$

where $e_1 = \ell(1 \cdots 0)$, and $e_2 = \ell(01 \cdots 0)$. Remark that

$$K_\mathbb{R}E_1 = \Sigma_\mathbb{R}, \quad \mathfrak{p} = \mathbb{N} \cup \bigcup_{\lambda \in C \setminus \{0\}} \Lambda \Sigma,$$

and the $K_\mathbb{R}$-orbit decompositions of $\Sigma$ and $\mathbb{N}$ are given by

$$\Sigma = \bigcup_{q \geq 0, r \geq 0} K_\mathbb{R}\tilde{E}_{r,q} \quad \text{and} \quad \mathbb{N} = \bigcup_{\rho \geq 0, 0 \leq r \leq 1} K_\mathbb{R}(\rho \tilde{E}_r).$$

We put $\Lambda = \{(n,k); n \in \mathbb{Z}_+, 0 \leq k \leq n\}$. For $X = \left( \begin{array}{ccc} 0 & z \\ v & 0 \end{array} \right)$ and $Y = \left( \begin{array}{ccc} 0 & a \\ b & 0 \end{array} \right) \in \mathfrak{p}$ we put

$$\tilde{K}_{n,k}(X,Y) = (x \cdot \mathfrak{t}^k)(y \cdot \mathfrak{b}^{n-k}) \quad ((n,k) \in \Lambda),$$

where $z \cdot w = \ell zw$ for $z, w \in \mathbb{C}^p$. It is clear that

$$\tilde{K}_{n,k}(X,Y) = \tilde{K}_{n,k}(Y,X) \quad (X,Y \in \mathfrak{p}),$$

$$\tilde{K}_{n,k}(gX,gY) = \tilde{K}_{n,k}(X,Y) \quad (g \in K_\mathbb{R}),$$

$$\tilde{K}_{n,k}(\ ,Y) \in \mathcal{H}_n \quad (Y \in \mathbb{N}).$$

Let $\mathcal{H}_{n,k}$ be the subspace of $\mathcal{H}_n$ which is spanned by the elements $\tilde{K}_{n,k}(\ ,Y)$ ($Y \in \mathbb{N}$). From Theorem 14.4 in [16] we can easily see that $\mathcal{H}_n = \oplus_{k=0}^n \mathcal{H}_{n,k}$ gives the $K_\mathbb{R}$-irreducible decomposition of $\mathcal{H}_n$ and

$$\dim \mathcal{H}_{n,k} = \frac{(p+n-1)(k+p-2)!(n-k+p-2)!}{(p-1)!(p-2)!k!(n-k)!}.$$

Now we put $E_0 = \left( \begin{array}{ccc} 0 & c_2 \\ c_1 & 0 \end{array} \right)$, and by using $\tilde{K}_{n,k}$, we define a function $\tilde{H}_{n,k}(X,Z) (X,Z \in \mathfrak{p})$ by

$$\tilde{H}_{n,k}(X,Z) = \int_{K_\mathbb{R}} \tilde{K}_{n,k}(X,gE_0)\tilde{K}_{n,k}(gE_0,Z) dg,$$

where $dg$ is the normalized Haar measure on $K_\mathbb{R}$. For $f, h \in \mathcal{H}_n$, we define the $K_\mathbb{R}$-invariant inner product $(\ ,\ )$ by

$$(f,h) = \int_{K_\mathbb{R}} f(gE_0)h(gE_0) dg.$$
Then we see that \( \mathcal{H}_{n,k} \perp \mathcal{H}_{n,l} \) \((k \neq l)\). Therefore it is easy to show that \( \tilde{H}_{n,k} \in \mathcal{H}_{n,k} \). The following theorem asserts that the function \( \tilde{H}_{n,k} \) explicitly defined above gives the reproducing kernel of \( \mathcal{H}_n \).

**Theorem 2.1.** Let \( X_0 \in p \) and assume \( \tilde{H}_{n,k}(X_0, X_0) \neq 0 \) \( (n, k) \in \Lambda \). Let \( f \in \mathcal{H}_{m,l} \) and \( X \in p \). Then the following integral formula holds:

\[
\delta_{n,m} \delta_{k,l} f(X) = \frac{\dim \mathcal{H}_{n,k}}{\mathcal{H}_{n,k}(X_0, X_0)} \int_{K_R} f(gX_0) \tilde{H}_{n,k}(X, gX_0) dg.
\]

Especially if \( X \in N \) or \( Y \in N \), we have

\[
\tilde{H}_{n,k}(X, Y) = \tilde{K}_{n,k}(X, Y).
\]

And therefore the polynomial \( \tilde{K}_{n,k}(X, Y) \) itself gives a reproducing kernel on nilpotent orbits \( K_RX_0 \) \((X_0 \in N)\):

\[
\delta_{n,m} \delta_{k,l} f(X) = \frac{\dim \mathcal{H}_{n,k}}{\mathcal{H}_{n,k}(X_0, X_0)} \int_{K_R} f(gX_0) \tilde{K}_{n,k}(X, gX_0) dg.
\]

**Proof.** We can easily show that the function \( \tilde{H}_{n,k}(X, Y) \) satisfies the conditions (1.1)–(1.3) in Theorem 1.2, and hence we obtain the formula (2.1).

Now we show (2.2) and (2.3). For \( X, Y \in p \) we put

\[
F_{n,k}(X, Y) = \dim \mathcal{H}_{n,k} \int_{K_R} \tilde{H}_{n,k}(X, gE_0) \tilde{K}_{n,k}(gE_0, Y) dg.
\]

Then the function \( F_{n,k}(X, Y) \) also satisfies the conditions (1.1)–(1.3). Hence from Theorem 1.2 (i) there exists some \( c_{n,k} \in C \) such that

\[
F_{n,k}(X, Y) = c_{n,k} \tilde{H}_{n,k}(X, Y) \quad (X, Y \in p).
\]

On the other hand for \( Y \in N \) we have from (2.1)

\[
F_{n,k}(X, Y) = \tilde{H}_{n,k}(E_0, E_0) \tilde{K}_{n,k}(X, Y)
\]

because \( \tilde{K}_{n,k}(X, Y) \) belongs to \( \mathcal{H}_{n,k} \). The equalities (2.4) and (2.5) imply

\[
c_{n,k} \tilde{H}_{n,k}(X, Y) = \tilde{H}_{n,k}(E_0, E_0) \tilde{K}_{n,k}(X, Y).
\]

Since

\[
\tilde{K}_{n,k}(E_0, E_0) = 1 \neq 0,
\]

we have

\[
\tilde{H}_{n,k}(E_0, E_0) = \int_{K_R} | \tilde{K}_{n,k}(gE_0, E_0) |^2 dg \neq 0.
\]
Therefore from (2.6) we have \( c_{n,k} = 1 \) and hence by the property (1.3) the equality
\[
\tilde{H}_{n,k}(X, Y) = \tilde{H}_{n,k}(E_0, E_0)\tilde{K}_{n,k}(X, Y)
\]
holds if \( X \in N \) or \( Y \in N \). From this and (2.1) we have easily (2.2) and (2.3). Q.E.D.

**Remark 2.2.** We have
\[
\tilde{H}_{n,k}(X_0, X_0) = C \int_{K_R} |\tilde{H}_{n,k}(gX_0, E_1)|^2 \, dg \quad (X_0 \in \mathfrak{p}),
\]
where \( C = (\int_{K_R} |\tilde{K}_{n,k}(gX_0, E_1)|^2 \, dg)^{-1} \). Since \( \tilde{K}_{n,k}(\cdot, E_0) \neq 0 \) on \( \Sigma_R \), we have \( C > 0 \). Therefore the following two conditions (2.7) and (2.8) are equivalent.

(2.7) \[
\tilde{H}_{n,k}(X_0, X_0) = 0,
\]

(2.8) \[
\mathcal{H}_{n,k}|_{K_R \Sigma_R} = \{0\}.
\]

This implies that the assumption \( \tilde{H}_{n,k}(X_0, X_0) \neq 0 \) in Theorem 2.1 holds for any \((n, k) \in \Lambda\) if and only if \( X_0 \not\equiv \lambda K_R E_1 \) and \( X_0 \not\equiv \lambda K_R E_0 \) for any \( \lambda \in \mathbb{C} \).

**Remark 2.3.** We consider the case \( p = 1 \). For \( X = \left( \begin{array}{cc} 0 & z \\ t \bar{y} & 0 \end{array} \right) \) and \( Y = \left( \begin{array}{cc} 0 & x' \\ t' y & 0 \end{array} \right) \) \( \in \mathfrak{p} \) we put \( \tilde{H}_{n,1}(X, Y) = (x \cdot \bar{y})^n \) and \( \tilde{H}_{n,2}(X, Y) = (y \cdot \bar{y})^n \). We denote by \( \mathcal{H}_{n,k} \) the subspace of \( \mathcal{H}_n \) which is generated by \( \{\tilde{H}_{n,k}(\cdot, E_i)\} \) \((k = 1, 2)\). Then we have the \( K_R \)-irreducible decomposition \( \mathcal{H}_n = \mathcal{H}_{n,1} \oplus \mathcal{H}_{n,2} \). It is easy to show that \( \tilde{H}_{n,k}(X, Y) \) satisfies (1.1)–(1.3) in Theorem 1.2, and therefore Theorem 2.1 also holds in case \( p = 1 \).

### 3. Harmonic polynomials on \( \mathfrak{p} \) in the case \( \mathfrak{sp}(p, 1) \).

In the rest of this paper we consider the Lie algebra \( \mathfrak{sp}(p, 1) \) and give the explicit formula of the reproducing kernel of harmonic polynomials on each \( K_R \)-orbit (Theorem 4.5). In this case the expressions of matrices becomes much more complicated than the case of \( \mathfrak{su}(p, 1) \), because the complexification \( \mathfrak{sp}(p + 1, \mathbb{C}) \) of the real Lie algebra \( \mathfrak{sp}(p, 1) \) can not be realized as a subalgebra of the quaternion general linear Lie algebra \( \mathfrak{gl}(p + 1, \mathbb{H}) \). (Note that in the case \( \mathfrak{su}(p, 1) \), its complexification can be naturally identified with \( \mathfrak{sl}(p + 1, \mathbb{C}) \).)

The construction of the reproducing kernel is also complicated for the case \( \mathfrak{sp}(p, 1) \), and in this section we first settle the notations and state basic formulas on harmonic polynomials on \( \mathfrak{p} \) for the Lie algebra \( \mathfrak{sp}(p, 1) \). Since the Lie algebra \( \mathfrak{sp}(1, 1) \) is isomorphic to \( \mathfrak{so}(2, 1) \), we always assume \( p \geq 2 \) in the following argument. From now we put \( \mathfrak{g} = \mathfrak{sp}(p + 1, \mathbb{C}), \mathfrak{g}_R = \mathfrak{sp}(p, 1) \),

\[
\mathfrak{t}_R = \left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & a & b \\ -\bar{B} & 0 & \bar{A} \\ 0 & -\bar{b} & 0 & \bar{x} \end{pmatrix} : A \in \mathfrak{u}(p), a \in \mathfrak{u}(1), b \in \mathbb{C} \right\},
\]

\[
\mathfrak{p}_R = \left\{ \begin{pmatrix} 0 & x & 0 & 0 \\ t \bar{x} & 0 & y & 0 \\ 0 & \bar{y} & 0 & -\bar{y} \end{pmatrix} : x, y \in \mathbb{C}^p \right\}.
\]
Then we have
\[
\{\begin{pmatrix} A & 0 & B \\ 0 & \alpha & 0 \\ C & 0 & -\alpha \end{pmatrix}; \quad tB = B, tC = C \\
\alpha, \beta, \gamma \in C
\}
\]
and
\[
P_2(X) = \frac{1}{8(p+2)} B(X, X) = \frac{1}{4} \text{Tr}(X^2) = t_1 x + t_2 z
\]
gives a generator of \( J \) and \( \mathcal{H}_n \) is given by \( \mathcal{H}_n = \{ f \in S_n : \sum_{j=1}^{p} \left( \frac{\partial^2}{\partial x_j \partial y_j} + \frac{\partial^2}{\partial z_j \partial w_j} \right) f = 0 \} \).

For \( X \in \mathfrak{p} \) we define the bijective mapping \( \Psi : \mathfrak{p} \to \mathbb{C}^{4p} \) by \( \Psi(X) = \frac{1}{2} \begin{pmatrix} x + y \\ z + w \\ i(y - z) \\ i(w - z) \end{pmatrix} \).

We can see that \( f \in \mathcal{H}_n \) if and only if \( f \circ \Psi^{-1} \in H_n(\mathbb{C}^{4p}) \) and from this fact, we have
\[
\dim \mathcal{H}_n = \dim H_n(\mathbb{C}^{4p}) = \frac{2(n+2p-1)(n+4p-3)!}{n!(4p-2)!}
\]
We put
\[
\mathcal{N} = \{ X \in \mathfrak{p} ; P(X) = 0 \}, \\
\Sigma = \{ X \in \mathfrak{p} ; P(X) = 1 \},
\]
and
\[
\Sigma_R = \mathcal{N} \cap \mathfrak{p}_R.
\]
Remark that \( \Psi : \Sigma_R \simeq S^{4p-1} \) and \( \tilde{H}_n(X, Y) = P_{n,4p} \left( \frac{\text{Tr}_1 X Y}{\text{Tr} P(X)} \right) (P(X))^{n/2} (X \in \mathfrak{p}, Y \in \Sigma_R) \) gives the reproducing kernel on \( \Sigma_R \). Furthermore it is known that the restriction mapping \( f \to f |_{\mathcal{N}} \) is also a bijection from \( \mathcal{H}_n \) onto \( \mathcal{H}_n|_{\mathcal{N}} \).

Let \( g = \text{Ad} \begin{pmatrix} A & 0 & B \\ 0 & \alpha & 0 \\ 0 & -\beta & 0 \end{pmatrix} \in K_R \) and \( X = \begin{pmatrix} t_1 & t_2 & w \\ t_2 & 0 & -t_1 \\ 0 & w & 0 \end{pmatrix} \in \mathfrak{p} \). If we put
\[
\Phi(X) = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{C}^{4p},
\]
we have
\[
(3.1) \quad \Phi(gX) = \begin{pmatrix} A(x + \overline{y}w) + B(|y| + \overline{x}w) \\ A(-x + \overline{y}w) + B(|y| + \overline{x}w) \\ -B(x + \overline{y}w) + A(|y| + \overline{x}w) \\ A(-x + \overline{y}w) - B(|y| + \overline{x}w) \end{pmatrix}.
\]
In the following, we often simply write \( \alpha(g) = \alpha \) and \( \beta(g) = \beta \), though \( \alpha \) and \( \beta \) depend on \( g \in K_R \). We put

\[
\tilde{E}_r = \Phi^{-1} \left( \begin{array}{ccc} r e_1 & 0 & 0 \\ 0 & 0 & \sqrt{1 - r^2} e_2 \end{array} \right) \in N \quad (0 \leq r \leq 1),
\]

\[
\tilde{E}_{r,q} = \Phi^{-1} \left( \begin{array}{ccc} r e_1 & 0 & 0 \\ \frac{1}{2} e_1 + q e_2 & 0 & 0 \end{array} \right) \in \Sigma \quad (r > 0, q \geq 0).
\]

In addition we put \( E_1 = \tilde{E}_{1,0} \).

It is clear that \( p = N \cup \bigcup_{\lambda \in C \setminus \{0\}} \lambda \Sigma \). Remark that

\[
(3.2) \quad N = \bigcup_{q \geq 0, r \leq 1} K_R(q \tilde{E}_r), \quad \Sigma = \bigcup_{q \geq 0, r > 0} K_R \tilde{E}_{r,q}
\]

give the \( K_R \)-orbit decompositions of \( N \) and \( \Sigma \), respectively. For \( X = \Phi^{-1} \left( \begin{array}{ccc} x & y & z \\ w & v & w' \end{array} \right) \), \( Y = \Phi^{-1} \left( \begin{array}{ccc} x' & y' & z' \\ w' & v' & w'' \end{array} \right) \in p \), we put \( \langle X, Y \rangle = \frac{1}{2} \text{Tr} (t(X)Y) = x \cdot x' + y \cdot y' + z \cdot z' + w \cdot w' \). Then we can easily see that \( \langle , \rangle \) is \( K_R \)-invariant.

Next we put

\[
H_1 = \left\{ \text{Ad} \left( \begin{array}{ccc} A & 0 & B \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \in K_R \right\}
\]

and

\[
H_2 = \left\{ \text{Ad} \left( \begin{array}{ccc} I_p & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{array} \right) \in K_R \right\}.
\]

Then \( H_1 \) and \( H_2 \) are subgroups of \( K_R \), and for any \( g \in K_R \) there exist unique \( h_j \in H_j \) \((j = 1, 2)\) such that \( g = h_1 h_2 \). Furthermore, if \( g_j \in H_j \) \((j = 1, 2)\), we have \( g_1 g_2 = g_2 g_1 \). We denote by \( dh_j \) the normalized Haar measure on \( H_j \) and by \( C(H_j) \) the space of continuous functions on \( H_j \) \((j = 1, 2)\). Remark that if we put \( h_2 = \text{Ad} \left( \begin{array}{ccc} I_p & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{array} \right), \quad \alpha = pe^{i\varphi}, \beta = \sqrt{1 - \rho^2} e^{i\varphi} \), then for any \( f \in C(H_2) \) we have

\[
(3.3) \quad \int_{H_2} f(h_2) dh_2 = \frac{1}{2\pi^2} \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{1} \tilde{f}(\rho, \theta, \varphi) \rho \, d\rho \, d\varphi \, d\theta,
\]

where \( \tilde{f}(\rho, \theta, \varphi) = f(h_2) \).
For \( h_1 = \text{Ad} \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & 0 \end{pmatrix} \in H_1 \) we define the mapping \( \phi : H_1 \widetilde{E}_1 \to S^{4p-1} \) by

\[
\phi(h_1 \widetilde{E}_1) = \begin{pmatrix} \text{Re} a_1 \\ \text{Im} a_1 \\ \text{Re} (-b_1) \\ \text{Im} (-b_1) \end{pmatrix},
\]

where \( h_1 \widetilde{E}_1 = \Phi^{-1} \left( \begin{pmatrix} a_1 \\ -b \end{pmatrix} \right) \) and \( a_1 = A e_1, b_1 = B e_1 \). (If \( h_1, h'_1 \in H_1 \) satisfy \( h_1 \widetilde{E}_1 = h'_1 \widetilde{E}_1 \), then we can easily prove \( \phi(h_1 \widetilde{E}_1) = \phi(h'_1 \widetilde{E}_1) \). And this fact implies that the mapping \( \phi \) is well defined.) From the definition of \( H \) we see that \( \phi \) is bijective and the equality

\[
(3.4) \quad \int_{H_1} f(h_1 \widetilde{E}_1) dh_1 = \int_{S^{4p-1}} f \circ \phi^{-1}(s) ds
\]

holds for any \( f \in C(H_1) \), where \( ds \) is the normalized \( O(4p) \)-invariant measure on \( S^{4p-1} \).

For \( X = \Phi^{-1} \left( \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right), Y = \Phi^{-1} \left( \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \right) \in \mathfrak{p} \) we put

\[
K_2(X,Y) = (x \cdot x' + z \cdot z')(y \cdot y' + w \cdot w') + (x \cdot w' - z \cdot y')(y \cdot z' - w \cdot x'),
\]

\[
\tilde{K}_m(X,Y) = \frac{m!(2p-1)!}{m!(2p-1)!} \int_{K_R} \langle g \tilde{E}_1, Y \rangle^m \langle X, g \tilde{E}_1 \rangle^{n-m} dg,
\]

\[
\tilde{K}_{n,k}(X,Y) = \tilde{K}_{n-2k}(X,Y) \{ K_2(X,Y) \}^k
\]

(\( m, n \in \mathbb{Z}_+, k = 0, 1, \cdots, \lceil n/2 \rceil \)). These functions play an important role in constructing the function \( H_{n,k}(X,Y) \). Remark that the equalities

\[
(3.5) \quad \tilde{K}_{n,k}(X,Y) = \overline{K_{n,k}(Y,X)},
\]

\[
(3.6) \quad \tilde{K}_{n,k}(X,Y) = \tilde{K}_{n,k}(gX,gY)
\]

hold for any \( X, Y \in \mathfrak{p}, g \in K_R \).

4. Decomposition of the space \( \mathcal{K}_n \) and the integral formula for the case \( \mathfrak{sp}(p,1) \).

In this section we first show that \( \tilde{K}_{n,k}(\, ,Y) \in \mathcal{K}_n \) if \( Y \in \mathcal{X}_n \), and next by using this property, we define \( K_{R_0} \)-irreducible subspaces \( \mathcal{H}_{n,k} \) of \( \mathcal{H}_n \) \((k = 0, 1, \cdots, \lceil n/2 \rceil \)). And finally we state our main theorem for the case \( \mathfrak{sp}(p,1) \) (Theorem 4.5). As before we always assume \( p \geq 2 \).

First, for \( k = 0, 1, \cdots, \lceil n/2 \rceil \), we introduce the polynomial \( K_{n,k} \) to simplify the following calculations:

\[
K_{n,k}(X,Y) = \frac{1}{n - 2k + 1} (X,Y)^{n-2k} \{ K_2(X,Y) \}^k \quad (X,Y \in \mathfrak{p}).
\]
We can see that \( K_{n,k} \in \mathcal{H}_n \) if \( Y \in N \).

Now we prove that \( \tilde{K}_{n,k} \in \mathcal{H}_n \) (\( Y \in N \)). For this purpose we need the following

**Proposition 4.1.**

(i) For \( X = \Phi^{-1} \left( \begin{array}{c} x \\ y \\ z \\ w \end{array} \right) \), \( Y = \Phi^{-1} \left( \begin{array}{c} x' \\ y' \\ z' \\ w' \end{array} \right) \) \( \in p \) the following formula holds:

\[
\tilde{K}_m(X,Y) = \frac{1}{m+1} \sum_{m_1+m_2+2m_3=m} \frac{(m_1+m_2)!(m_2+m_3)!}{m_1!m_2!(m_3)!^2} (x \cdot \bar{z'} + z \cdot \bar{z'})^{m_1} \\
\times (y \cdot \bar{y'} + w \cdot \bar{w'})^{m_2} (z \cdot \bar{y'} - x \cdot \bar{w'})^{m_3} (y \cdot \bar{z'} - w \cdot \bar{x'})^{m_3}.
\]

(ii) There exist \( a_{m,q} \in \mathbb{R} \) \((q = 1, 2, \ldots, [m/2])\) such that

\[
\langle X, Y \rangle^m = (m+1)\tilde{K}_m(X,Y) + \sum_{q=1}^{[m/2]} a_{m,q} K_{m,q}(X,Y) \quad (X, Y \in p).
\]

(iii) There exist \( b_{m,q} \in \mathbb{R} \) \((q = 1, 2, \ldots, [m/2])\) such that

\[
\langle X, Y \rangle^m = (m+1)\tilde{K}_m(X,Y) + \sum_{q=1}^{[m/2]} b_{m,q} K_{m,q}(X,Y) \quad (X, Y \in p).
\]

**Proof.**

(i) Assume \( a, b \in \mathbb{C}^{4p} \) and \( a \cdot a = b \cdot b = 0 \). Then the following equality holds (see [11]):

\[
\int_{S^w_{v-1}} (s \cdot a)^m (s \cdot b)^m ds = \frac{m!(2p-1)!}{2^m(m+2p-1)!} (a \cdot b)^m.
\]

From this formula and from (3.1), (3.4) we have

\[
\tilde{K}_m(X,Y) = \frac{(m+2p-1)!}{m!(2p-1)!} \int_{K_n} \langle g\tilde{E}_1, Y \rangle^m \langle X, g\tilde{E}_1 \rangle^m dg \\
= \frac{(m+2p-1)!}{m!(2p-1)!} \int_{H_2} \left( \int_{H_1} \{ a_1 \cdot (\bar{a}x - \beta w) - b_1 \cdot (\bar{a}x + \beta y) \}^m \\
\times (\bar{a}x - \beta w) - b_1 \cdot (\alpha x + \beta y) \}^m dh_1 \right) dh_2 \\
= \frac{(m+2p-1)!}{m!(2p-1)!} \int_{H_2} \left( \int_{S^w_{v-1}} s \cdot \left( (\bar{a}x - \beta w) - (\bar{a}x + \beta y) \right) \right)^m \left( (\bar{a}x - \beta w) - (\bar{a}x + \beta y) \right) \right)^m dh_2.
\]

The last expression of (4.4) equals

\[
\int_{H_2} \{ |\alpha|^2 (x \cdot \bar{z'} + z \cdot \bar{z'}) + |\beta|^2 (w \cdot \bar{w'} + y \cdot \bar{y'}) \\
+ \alpha\beta (z \cdot \bar{y'} - x \cdot \bar{w'}) + \bar{\alpha}\beta (\bar{z} \cdot \bar{y'} - \bar{x} \cdot \bar{w'}) \}^m dh_2 \\
= \sum_{m_1+m_2+m_3+m_4=m} \frac{m!(m_1+m_2)!m_3!m_4!}{m_1!m_2!(m_3)!^2} \left( \int_{H_2} |\alpha|^{2m_1} |\beta|^{2m_2} (\alpha\beta)^{m_3} (\bar{\alpha}\bar{\beta})^{m_4} dh_2 \right) \\
\times (x \cdot \bar{z'} + z \cdot \bar{z'})^{m_1} (y \cdot \bar{y'} + w \cdot \bar{w'})^{m_2} (z \cdot \bar{y'} - x \cdot \bar{w'})^{m_3} (y \cdot \bar{z'} - w \cdot \bar{x'})^{m_4}.
\]
Putting $\alpha = te^{i\theta}$ and $\beta = \sqrt{1-t^2}e^{i\varphi}$, we have from (3.3)
\[
\int_{H_2} |\alpha|^{2m_1} |\beta|^{2m_2} (\alpha \beta)^{m_3} (\bar{\alpha} \bar{\beta})^{m_4} dh_2 = \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_1^{2m_1+m_3+m_4} (1-t^2)^{(2m_2+m_3+m_4)/2} (e^{i\theta} e^{i\varphi})^{m_3-m_4} dt \, d\theta \, d\varphi
\]
\[
= \delta_{m_3,m_4} (m_1+m_3)!(m_2+m_3)!(m+1)!
\]
Therefore we obtain (4.1).

(ii) We prove the following formulas by induction on $n$:
\[
(4.5) \quad \begin{align*}
\langle X,Y \rangle^{2n-1} &= (2n)\tilde{K}_{2n-1}(X,Y) + \sum_{q=1}^{n-1} a_{2n-1,q} K_{2n-1,q}(X,Y), \\
\langle X,Y \rangle^{2n} &= (2n+1)\tilde{K}_{2n}(X,Y) + \sum_{q=1}^n a_{2n,q} K_{2n,q}(X,Y), \\
(a_{2n-1,q}, a_{2n,q} \in \mathbb{R}, n = 1, 2, \cdots).
\end{align*}
\]
When $n = 1$, we have (4.5) because (4.1) gives
\[
2\tilde{K}_1(X,Y) = \langle X,Y \rangle,
\]
\[
3\tilde{K}_2(X,Y) = \langle X,Y \rangle^2 - K_2(X,Y).
\]
Assume that (4.5) is valid for $n = 1, 2, \cdots, k$. By this assumption and by (4.1), we obtain the following equality after some calculations:
\[
\langle X,Y \rangle^{2k+1} = \{ (2k+1)\tilde{K}_{2k}(X,Y) + \sum_{q=1}^k a_{2k,q} K_{2k,q}(X,Y) \} \langle X,Y \rangle
\]
\[
= \sum_{q=1}^k a'_{2k,q} K_{2k+1,q}(X,Y) + \sum_{m_1+m_2+m_3=2k} \frac{(m_1+m_3)!(m_2+m_3)!}{m_1!m_2!(m_3)!} \times (x \cdot \vec{z} + z \cdot \vec{x})^{m_1}(y \cdot \vec{y} + w \cdot \vec{w})^{m_2}(z \cdot \vec{z} - x \cdot \vec{x})^{m_3}(y \cdot \vec{y} - w \cdot \vec{w})^{m_3} \langle X,Y \rangle
\]
\[
= \sum_{q=1}^k a''_{2k,q} K_{2k+1,q}(X,Y) + (2k+2)\tilde{K}_{2k+1}(X,Y) + 2k\tilde{K}_{2k-1}(X,Y)K_2(X,Y),
\]
where $a'_{2k,q} = a_{2k,q}(2k-2q+2)(2k-2q+1)^{-1}$. By the assumption of induction we have
\[
2k\tilde{K}_{2k-1}(X,Y)K_2(X,Y) = K_2(X,Y) \{ \langle X,Y \rangle^{2k-1} - \sum_{q=1}^{k-1} a_{2k-1,q} K_{2k-1,q}(X,Y) \}
\]
\[
= 2kK_{2k+1,1}(X,Y) - \sum_{q=1}^{k-1} a_{2k-1,q} K_{2k+1,q+1}(X,Y).
\]
Hence there exist some $a_{2k+1,q} \in \mathbb{R}$ ($q = 1, 2, \cdots, k$) such that
\[
\langle X,Y \rangle^{2k+1} = (2k+2)\tilde{K}_{2k+1}(X,Y) + \sum_{q=1}^k a_{2k+1,q} K_{2k+1,q}(X,Y).
\]
In the same way we can show the second equality of (4.5) for $n = k+1$. 
(iii) Using (4.2), we can prove (4.3) easily. Q.E.D.

From (4.2) there exist \( a_{n-2k,q} \in \mathbb{R} \) \((q = 1, 2, \cdots, \lfloor n/2 \rfloor - k)\) such that

\[
(n - 2k + 1)\tilde{K}_{n-2k}(X,Y) = \langle X,Y \rangle^{n-2k} - \sum_{q=1}^{\lfloor n/2 \rfloor - k} a_{n-2k,q}K_{n-2k,q}(X,Y) \quad (X,Y \in p).
\]

From the definitions of \( \tilde{K}_{n,k}(X,Y) \) and \( K_{n,k}(X,Y) \) and from this formula, there exist \( c_{n,q} \in \mathbb{R} \) \((q = k, k+1, \cdots, \lfloor n/2 \rfloor)\) such that

\[
\tilde{K}_{n,k}(X,Y) = \sum_{q=k}^{\lfloor n/2 \rfloor} c_{n,q}K_{n,q}(X,Y).
\]

Hence we see that \( \tilde{K}_{n,k}(X,Y) \in \mathcal{H}_n \) because \( K_{n,k}(X,Y) \in \mathcal{H}_n \) \((Y \in \mathbb{N})\).

We denote by \( \mathcal{H}_{n,k} \) the subspace of \( \mathcal{H}_n \) which is generated by \( \{ \tilde{K}_{n,k}(X,Z); Z \in \mathbb{N} \} \).

Then from (3.6) it is clear that the space \( \mathcal{H}_{n,k} \) is \( K_{\mathbb{R}} \)-invariant. From now we put \( E_0 = \Phi^{-1} \begin{pmatrix} \varepsilon_1 \\ 0 \\ 0 \\ \varepsilon_2 \end{pmatrix} \in \mathbb{N} \). To show our main theorem, we must prepare the following proposition.

**Proposition 4.2.** (i) For any \( X, Y \in p \) we have

\[
\int_{K_{\mathbb{R}}} \tilde{K}_{n,l}(gE_0, Y)\tilde{K}_{n,k}(X, gE_0)dg = 0 \quad (l \neq k).
\]

(ii) \( \mathcal{H}_n = \bigoplus_{k=0}^{\lfloor n/2 \rfloor} \mathcal{H}_{n,k} \) gives the \( K_{\mathbb{R}} \)-irreducible decomposition of \( \mathcal{H}_n \). Furthermore, \( \mathcal{H}_{n,k} \) and \( \mathcal{H}_{m,l} \) are not equivalent as \( K_{\mathbb{R}} \)-representation spaces if \((n, k) \neq (m, l)\).

To prove this proposition, we need the following

**Lemma 4.3.** (i) For any \( h_2 \in H_2 \) and \( X, Y \in p \) it is valid that \( K_2(h_2X, Y) = K_2(X, Y) \).

(ii) If \( n, m \in \mathbb{Z}_+ \) and \( n > m \), we have for any \( X, Y \in p \)

\[
\int_{H_2} (h_2X, E_0)^n \langle Y, h_2E_1 \rangle^n dh_2 = 0,
\]

and

\[
\int_{H_2} \tilde{K}_n(h_2E_0, X)\tilde{K}_m(Y, h_2E_0)dh_2 = 0.
\]

**Proof.** If we put \( \Phi(X) = \begin{pmatrix} z \\ p \\ w \end{pmatrix} \in C^4p \) and \( h_2 = \text{Ad} \begin{pmatrix} I_p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_p \end{pmatrix} \) \((\alpha, \beta \in \mathbb{C}, \alpha\bar{\beta} + \beta\bar{\alpha} = 1)\), from (3.1) we have

\[
\Phi(h_2X) = \begin{pmatrix} x + \beta w \\ \alpha y + \beta z \\ \bar{\alpha}z - \beta y \\ -\beta x + \alpha w \end{pmatrix},
\]

\[
(4.7)
\]

\[
(4.8)
\]

\[
(4.9)
\]
By using (4.9), it is easy to show (i). We will prove (ii). From (4.9) there exist some $t, s, r, q, \mu, \nu \in \mathbb{C}$ such that

$$\langle h_2 X, E_0 \rangle^m = (\alpha t + \beta q + \alpha \mu + \beta \nu)^m,$$

and

$$\langle Y, h_2 \tilde{E}_1 \rangle^n = (\alpha t + \beta s)^n.$$

These formulas give that

$$\int_{H_2} \langle h_2 X, E_0 \rangle^m \langle Y, h_2 \tilde{E}_1 \rangle^n dh_2$$

$$= \sum_{k=0}^{n} \sum_{m_1+m_2+m_3+m_4=m} C_{m_1,m_2,m_3,m_4,n,k}(t, s, r, q, \mu, \nu) \int_{H_2} \alpha^{m_1+k} \beta^{m_2} \alpha^{m_3} \beta^{n-k+m_4} dh_2,$$

where $C_{m_1,m_2,m_3,m_4,n,k}(t, s, r, q, \mu, \nu)$ is a polynomial of $t, s, r, q, \mu, \nu$. Putting $\alpha = \rho e^{i\theta}$ and $\beta = \sqrt{1 - \rho^2} e^{i\phi}$, we have

$$(4.10) \quad \int_{H_2} \alpha^{m_1+k} \beta^{m_2} \alpha^{m_3} \beta^{n-k+m_4} dh_2$$

$$= \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left\{ \rho^{m_1+k+m_3} (1 - \rho^2)^{(m_2+m_4+n-k)/2} (e^{i\phi})^{m_1+k-m_3} \right.$$}

$$\left. \times (e^{i\phi})^{m_2-m_4-n+k} \rho \right\} d\phi d\theta d\phi.$$

If $n > m$, we have $m_1 + k - m_3 \neq 0$ or $m_2 - m_4 - n + k \neq 0$ because

$$(m_1 + k - m_3) - (m_2 - m_4 - n + k) = n + m_1 - m_3 + m_4 - m_2 \geq n - m > 0.$$

Therefore we obtain (4.7) from (4.10). From the definition of $\tilde{K}_{n}(\cdot,\cdot)$ we have for some $C_{n,m} \in \mathbb{R}$

$$(4.11) \quad \int_{H_2} \tilde{K}_{n}(h_2 E_0, X) \tilde{K}_{m}(Y, h_2 E_0) dh_2$$

$$= C_{n,m} \int_{H_2} \int_{K_R} \int_{K_R} \langle g\tilde{E}_1, X \rangle^n \langle h_2 E_0, g\tilde{E}_1 \rangle^n \langle g_0\tilde{E}_1, h_2 E_0 \rangle^m \langle Y, g_0\tilde{E}_1 \rangle^m dg dg_0 dh_2.$$

We put $g = g_{2g_1}$ ($g_i \in H_i, i = 1, 2$). By changing variables and by using the property $k_1k_2 = k_2k_1 (k_i \in H_i, i = 1, 2)$ we have from (4.7)

$$\int_{H_2} \langle h_2 E_0, g\tilde{E}_1 \rangle^n \langle g_0\tilde{E}_1, h_2 E_0 \rangle^m dh_2$$

$$= \int_{H_2} \langle E_0, h_2^{-1} g_2 g_1 \tilde{E}_1 \rangle^n \langle h_2^{-1} g_0 \tilde{E}_1, E_0 \rangle^m dh_2$$

$$= \int_{H_2} \langle g_1^{-1} E_0, h_2 \tilde{E}_1 \rangle^n \langle h_2 g_2^{-1} g_0 \tilde{E}_1, E_0 \rangle^m dh_2 = 0 \ (n > m).$$

Hence (4.11) implies (4.8). Q.E.D.
Proof of Proposition 4.2. (i) From (3.6) we have

\[(4.12) \quad \int_{\mathcal{K}_n} \tilde{K}_{n,l}(gE_0, Y) \tilde{K}_{n,k}(X, gE_0) dg \]
\[= \int_{H_1} \int_{H_2} \tilde{K}_{n,l}(h_1 h_2 E_0, Y) \tilde{K}_{n,k}(X, h_1 h_2 E_0) dh_2 dh_1 \]
\[= \int_{H_1} \int_{H_2} \tilde{K}_{n,l}(h E_0, h^{-1} Y) \tilde{K}_{n,k}(h^{-1} X, h E_0) dh_2 dh_1. \]

Assume \( k > l \). Then from (4.8) it is valid that for any \( X, Y \in \mathfrak{p} \)

\[(4.13) \quad \int_{H_2} \tilde{K}_{n,l}(h_2 E_0, Y_1) \tilde{K}_{n,k}(X_1, h_2 E_0) dh_2 \]
\[= \{ \tilde{K}_2(E_0, Y_1) \}^j \{ \tilde{K}_2(X_1, E_0) \}^k \]
\[\times \int_{H_2} \tilde{K}_{n-2k}(h_2 E_0, Y_1) \tilde{K}_{n-2k}(X_1, h_2 E_0) dh_2 = 0. \]

Therefore, by (4.12) and (4.13) we have (4.6). When \( k < l \), we obtain (4.6) because

\[\int_{\mathcal{K}_n} \tilde{K}_{n,k}(X, gE_0) \tilde{K}_{n,l}(gE_0, Y) dg = \int_{\mathcal{K}_n} \tilde{K}_{n,k}(gE_0, X) \tilde{K}_{n,l}(Y, gE_0) dg = 0.\]

(ii) We define the inner product of \( L^2(K_\mathfrak{R} E_0) \) by

\[(f, h) = \int_{\mathcal{K}_n} f(gE_0) \overline{h(gE_0)} dg \]

for \( f, h \in L^2(K_\mathfrak{R} E_0) \). Then from (4.6) we have \( \mathcal{H}_n \perp \mathcal{H}_{n,l} \) for \( k \neq l \) with respect to the inner product \((\ , \ )\). To prove \( \mathcal{H}_n = \bigoplus_{k=0}^{[n/2]} \mathcal{H}_{n,k} \), we have only to show that the number of \( K_\mathfrak{R} \)-irreducible components of \( \mathcal{H}_n \) is \([n/2] + 1\) because \( \mathcal{H}_{n,k} \neq \{0\} \) and \( \mathcal{H}_{n,k} \perp \mathcal{H}_{n,l} \) for \( k \neq l \). We denote by \( S^n(C^{2p} \otimes C^2) \) the \( n \)-th symmetric tensor product space of \( C^{2p} \otimes C^2 \). Then the sum

\[(4.14) \quad S^n(C^{2p} \otimes C^2) = \bigoplus_{\lambda} S_\lambda(C^{2p}) \otimes S_\lambda(C^2) \]

gives the irreducible decomposition of \( S^n(C^{2p} \otimes C^2) \) with respect to the natural action of \( GL(2p, \mathbb{C}) \times GL(2, \mathbb{C}) \), where \( S_\lambda(C^{2p}) \) and \( S_\lambda(C^2) \) denote the \( GL \)-irreducible representation space corresponding to the partition \( \lambda = (\lambda_1, \lambda_2) \) \( (\lambda_1 \geq \lambda_2 \geq 0, \lambda_1 + \lambda_2 = n) \). Using the branching rule from \( GL(2p, \mathbb{C}) \) to \( Sp(p) \) stated in [4: p.507], we can see that \( S_\lambda(C^2) \) is always irreducible as an \( Sp(1) \)-module and \( S_\lambda(C^{2p}) \) splits into \( \lambda_2 + 1 \) \( Sp(p) \)-irreducible components with highest weight \( (\lambda_1 - \lambda_2 + k) \varepsilon_1 + k \varepsilon_2 = (\lambda_1 - \lambda_2) \Lambda_1 + k \Lambda_2 \) \( (k = 0 \sim \lambda_2) \), where we use the usual numbering. Since \( \lambda_2 \) moves from 0 to \([n/2] \), it follows that the number of \( K_\mathfrak{R} \)-irreducible components of (4.14) is \( 1 + 2 + \cdots + ([n/2] + 1) \), which is equal to the number of \( K_\mathfrak{R} \)-irreducible subspaces of \( \mathcal{H}_n \). Let \( J_m \) be the space of \( K_\mathfrak{R} \)-invariant homogeneous polynomials of degree \( m \). In this case we have \( J_{2m-1} = \{0\} \) and \( \dim J_{2m} = 1 \) \( (m \in \mathbb{Z}_+) \). Then, from the formula \( \mathcal{H}_n = \bigoplus_{k=0}^{[n/2]} \mathcal{H}_{n,k} \) (cf. Theorem1.1 (i)) we can easily show that the number of \( K_\mathfrak{R} \)-irreducible subspaces of \( \mathcal{H}_n \) is \([n/2] + 1\) and this shows \( \mathcal{H}_n = \bigoplus_{k=0}^{[n/2]} \mathcal{H}_{n,k} \).
Next we show that \( \mathcal{H}_{n,k} \) and \( \mathcal{H}_{n,t} \) are not \( K_R \)-equivalent if \((n, k) \neq (m, l)\). By using the results \( S_n = \bigoplus_{k=0}^{n} \mathcal{H}_k \) and \( S_d(C^{2p}) \) is a sum of \( Sp(p) \)-irreducible components with highest weight \((\lambda_1 - \lambda_2)\Lambda_1 + k\Lambda_2\), we can show that \( \mathcal{H}_n \) is a sum of \( Sp(p) \times Sp(1) \)-irreducible components with highest weight \( \{(n - 2k)\Lambda_1 + k\Lambda_2\} \otimes (n - 2k)\Lambda_1 \) \((k = 0 \sim [n/2])\). From this fact we can easily see that \( \mathcal{H}_{n,k} \simeq \mathcal{H}_{n,t} \) if and only if \( n = m \) and \( k = l \), because two irreducible representations are equivalent if and only if their highest weights coincide.

Q.E.D.

**Remark 4.4.** The irreducible decomposition of \( S_n \) and the generators of irreducible components of this representation are also stated in [3], though the number of irreducible components in [3] was misprinted. However the generators given in [3] are not fitted to our purpose, and we give here a new proof for the sake of completeness.

Now we put \( \Lambda = \{(n, k) : n \in \mathbb{Z}_+, 0 \leq k \leq [n/2]\} \). Under these preliminaries we define the function \( \tilde{H}_{n,k}(X, Y) \) on \( p \times p \) as follows:

\[
(4.15) \quad \tilde{H}_{n,k}(X, Y) = \int_{K_R} \tilde{K}_{n,k}(X, gE_0)\tilde{K}_{n,k}(gE_0, Y)dg.
\]

From the definition it is clear that \( \tilde{H}_{n,k}(\cdot, Y) \in \mathcal{H}_{n,k} \) for any \( Y \in p \). Therefore we can easily show that \( \tilde{H}_{n,k}(\cdot, Y) \) satisfies the conditions (1.1)-(1.3) in Theorem 1.2. Then we can show the following theorem completely in the same way as in the case of \( su(p, 1) \) (Theorem 2.1).

**Theorem 4.5.** Let \( X_0 \in p \) and assume that \( \tilde{H}_{n,k}(X_0, X_0) \neq 0 \) \((\forall (n, k) \in \Lambda)\). Then for any \( f \in \mathcal{H}_{m,l} \) and \( X \in p \) we have

\[
(4.16) \quad \delta_{n,m} \delta_{k,l} f(X) = \frac{\dim \mathcal{H}_{n,k}}{H_{n,k}(X_0, X_0)} \int_{K_R} f(gX_0)\tilde{H}_{n,k}(X, gX_0)dg.
\]

Especially for any \( X_0 \in \mathcal{N} \) and \( f \in \mathcal{H}_{n,k} \) we have

\[
(4.17) \quad \tilde{H}_{n,k}(X, X_0) = \tilde{K}_{n,k}(X, X_0)
\]

and

\[
(4.18) \quad \delta_{n,m} \delta_{k,l} f(X) = \frac{\dim \mathcal{H}_{n,k}}{K_{n,k}(X_0, X_0)} \int_{K_R} f(gX_0)\tilde{K}_{n,k}(X, gX_0)dg.
\]

**Remark 4.6.** For any \( Z_0 \in \Sigma_R \) we have

\[
\tilde{H}_{n,k}(Z_0, Z_0) = \int_{K_R} |\tilde{K}_{n,k}(Z_0, gE_0)|^2 dg = \int_{K_R} |\tilde{K}_{n,k}(gZ_0, E_0)|^2 dg.
\]

Since \( \tilde{K}_{n,k}(E_0, E_0) = 1 \), we have \( \tilde{K}_{n,k}(X, E_0) \neq 0 \) on \( p \). From this we see \( \tilde{K}_{n,k}(X, E_0) \mid_{\Sigma_R} \neq 0 \) and \( \int_{K_R} |\tilde{K}_{n,k}(gZ_0, E_0)|^2 dg \neq 0 \) because \( \tilde{K}_{n,k}(\cdot, E_0) \in \mathcal{H}_n \). Therefore we can see that \( \tilde{H}_{n,k}(Z_0, Z_0) \neq 0 \) and \( \frac{\tilde{H}_{n,k}(X, Y)}{\tilde{H}_{n,k}(Z_0, Z_0)} \) satisfies (1.7) in [20].
Remark 4.7. We have $\tilde{H}_{n,k}(X_0,X_0) \neq 0$ for any $(n, k) \in \Lambda$ if and only if $X_0 \notin \lambda K_R \tilde{E}_1$ and $X_0 \notin \lambda K_R \tilde{E}_0$ for any $\lambda \in \mathbb{C}$.

Remark 4.8. To write down $\tilde{H}_{n,k}(X,Y)$ for the cases $su(p,1)$ and $sp(p,1)$ in a simple form by using some special functions is our subject.

Appendix.

In this section we will get the dimension of $\mathcal{H}_{n,k}$ for the case $sp(p,1)$.

Proposition A.1. When $\mathfrak{g}_R = sp(p,1)$ $(p \geq 2)$, we have

$$\dim \mathcal{H}_{n,k} = \frac{(n - 2k + 1)^2}{(n - k + 1)! (2p - 3)! (2p - 1)!} (2p + n - 1) (2p + n - k - 2)! (2p + k - 3)!.$$\hspace{1cm} (A.1)

Furthermore the highest weight of $\mathcal{H}_{n,k}$ is $\{(n - 2k)\Lambda_1 + k\Lambda_2 \ominus (n - 2k)\Lambda_1 (k = 0 \sim \lfloor n/2 \rfloor)$.

To prove this proposition we use the following lemma.

Lemma A.2 (cf. [19] Theorem 2.2). Assume $p \geq 2$. For any $f \in \mathcal{H}_{n}$ and any $X \in \mathfrak{p}$ we have

$$f(X) = \dim \mathcal{H}_{n} \int_0^1 \rho(t) \left( \int_{K_R} f(g\tilde{E}_t)(X,g\tilde{E}_t)^n dg \right) dt, \hspace{1cm} (A.2)$$

where we put

$$\rho(t) = 2^{4p-3} \frac{\Gamma(2p - \frac{1}{2})}{\sqrt{\pi}(2p - 3)!} t^{4p-5}(1 - t^2)^{2p-3}(2t^2 - 1)^2 \quad (0 \leq t \leq 1).$$

For the proof of this lemma see [19].

Proof of Proposition A.1. We can see that there exist $a_{n,q} \in \mathbb{R}$ $(q = 1, 2, \cdots, \lfloor n/2 \rfloor - k)$ such that

$$K_{n,k}(X,Y) = \tilde{K}_{n,k}(X,Y) + \sum_{q=1}^{\lfloor n/2 \rfloor - k} a_{n,q} \tilde{K}_{n,q+k}(X,Y) \quad (X,Y \in \mathfrak{p}) \hspace{1cm} (A.3)$$

by (4.3), (4.18) and (A.3) give that

$$(\dim \mathcal{H}_{n,k})^{-1} f(X) = \int_{K_R} f(gE_0) \tilde{K}_{n,k}(X,gE_0) dg \hspace{1cm} (A.4)$$

$$= \int_{K_R} f(gE_0) K_{n,k}(X,gE_0) dg,$$
because \( \tilde{K}_{n,k}(E_0, E_0) = 1 \). From (A.2) and (4.3) we have for any \( X \in \mathfrak{p} \) and \( Y \in \mathbb{N} \)

\[
(A.5) \quad (\dim \mathcal{H}_n)^{-1} \tilde{K}_{n,k}(X, Y) = \int_0^1 \rho(t) \left( \int_{K_n} \tilde{K}_{n,k}(g\tilde{E}_1, Y)(X, g\tilde{E}_1)^n dg \right) dt \\
= B_{n,k} \int_0^1 \rho(t) \left( \int_{K_n} \tilde{K}_{n,k}(g\tilde{E}_1, Y)\tilde{K}_{n,k}(X, g\tilde{E}_1) dg \right) dt \\
= A_{n,k} B_{n,k} \left( \dim \mathcal{H}_{n,k} \right)^{-1} \tilde{K}_{n,k}(X, Y) \\
= A_{n,k} B_{n,k} \int_{K_n} \tilde{K}_{n,k}(gE_0, Y)\tilde{K}_{n,k}(X, gE_0) dg \\
= A_{n,k} \int_{K_n} \tilde{K}_{n,k}(gE_0, Y)(X, gE_0)^n dg,
\]

where

\[
A_{n,k} = \int_0^1 \tilde{K}_{n,k}(\tilde{E}_i, \tilde{E}_i) \rho(t) dt
\]

and

\[
\langle X, Y \rangle^n = \sum_{q=0}^{[n/2]} B_{n,q} \tilde{K}_{n,q}(X, Y) \quad (X, Y \in \mathfrak{p}).
\]

Since

\[
\tilde{K}_{n,k}(\tilde{E}_i, \tilde{E}_i) = \begin{cases} 
\frac{t^{2k}(1-t^2)^k(1-t^2)^{n-2k+1}-t^{2(n-2k+1)}}{2^{-n}(n-2k+1)(1-2t^2)} & (t \neq \frac{1}{\sqrt{2}}), \\
\frac{t^{2k}(1-t^2)^k}{2^{-n}} & (t = \frac{1}{\sqrt{2}}),
\end{cases}
\]

we get

\[
A_{n,k} = 2^{4p-3} \frac{\Gamma(2p - \frac{3}{2})(2p + n - k - 2)!}{\sqrt{\pi}(2p - 3)!} \frac{2p + k - 3)!}{(4p + n - 3)!}.
\]

By (A.5) we get for any \( f \in \mathcal{H}_{n,k} \)

\[
(A.6) \quad (\dim \mathcal{H}_n)^{-1} f(X) = A_{n,k} \int_{K_n} f(gE_0)(X, gE_0)^n dg.
\]

Now we introduce the following polynomial to simplify the calculations:

\[
h_{n,k}(X) = (X, \tilde{E}_1)^{n-2k} \{ K_2(X, E_0) \}^k \quad (X \in \mathfrak{p}).
\]

Then we have \( h_{n,k} \in \mathcal{H}_n \). By using (4.6) we can see that
\[ \int_{K_R} h_{n,k}(gE_0) \tilde{K}_{n,l}(X,gE_0) dg = 0 \quad (k \neq l, X \in \mathfrak{p}) \]
and this and (4.18) show that \( h_{n,k} \) belongs to \( \mathcal{H}_{n,k} \). Hence (A.4) and (A.6) imply
\[
(\dim \mathcal{H}_{n,k})^{-1} = \int_{K_R} h_{n,k}(gE_0) K_{n,k}(E_0, gE_0) dg
\]
and
\[
(\dim \mathcal{H}_n)^{-1} = A_{n,k} \int_{K_R} h_{n,k}(gE_0)(E_0, gE_0)^n dg
\]
because \( h_{n,k}(E_0) = 1 \). In order to compute \( \dim \mathcal{H}_{n,k} \), we compare the values of the right hand sides of these two formulas. By some calculations we obtain
\[
\int_{K_R} h_{n,k}(gE_0) K_{n,k}(E_0, gE_0) dg = \frac{1}{n - 2k + 1} \int_{H_1} |K_2(X, E_0)|^{2k} \int_{H_2} (h_2X, \tilde{E}_1)^{n-2k} (E_0, h_2X)^{n-2k} dh_2 dh_1
\]
\[
= \frac{1}{2(n - 2k + 1)^2} \int_{H_1} (|x_1|^2 + |w_1|^2)^{n-2k} |K_2(X, E_0)|^{2k} dh_1
\]
and
\[
\int_{K_R} h_{n,k}(gE_0)(E_0, gE_0)^n dg = \int_{H_1} K_2(X, E_0)^{k} \int_{H_2} (h_2X, \tilde{E}_1)^{n-2k} (E_0, h_2X)^{n} dh_2 dh_1
\]
\[
= \frac{n!}{2k!(n - k + 1)!} \int_{H_1} (|x_1|^2 + |w_1|^2)^{n-2k} |K_2(X, E_0)|^{2k} dh_1,
\]
where we put \( X = \Phi^{-1} \left( \begin{array}{c} x \\ y \\ z \\ w \end{array} \right) = h_1E_0 \) and \( x_i = x \cdot e_i, w_i = w \cdot e_i \) (\( i = 1, 2 \)). Hence we obtain
\[
\dim \mathcal{H}_{n,k} = A_{n,k} \frac{n!(n - 2k + 1)^2}{k!(n - k + 1)!} \dim \mathcal{H}_n.
\]
From this we get (A.1).

In the proof of Proposition 4.2 (ii) we showed that \( \mathcal{H}_n \) is a direct sum of \( K_R \)-irreducible components with highest weight \( \{ (n - 2k)\Lambda_1 + k\Lambda_2 \} \otimes (n - 2k)\Lambda_1 \) (\( k = 0 \sim [n/2] \)). By using Weyl’s dimension formula, we know that the dimension of the irreducible component corresponding to \( \{ (n - 2k)\Lambda_1 + k\Lambda_2 \} \otimes (n - 2k)\Lambda_1 \) just coincides with \( (A.1) \). Hence the highest weight of \( \mathcal{H}_{n,k} \) is given by \( \{ (n - 2k)\Lambda_1 + k\Lambda_2 \} \otimes (n - 2k)\Lambda_1 \).

Q.E.D.
References


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