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ON PRE-COXETER ALGEBRAS

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Abstract. In this paper we show that the class of PC-algebras and the class of B-algebras with condition (D) are Smarandache disjoint, and show that an algebra $(X;*,0)$ is a Coxeter algebra if and only if it is a PC-algebra with (N). Moreover, we show that there is no non-trivial quadratic PC-algebras on a field with $|X| \geq 3$.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([5, 6]). Y. B. Jun, E. H. Roh and H. S. Kim ([7]) introduced a new notion, called a BH-algebra, i.e., (I), (II) and (V) $x*y=0$ and $y*x=0$ imply $x=y$, which is a generalization of $BCH/BCI/BCK$-algebras. J. Neggers and H. S. Kim ([10]) introduced and investigated a class of algebras, called a *B-algebra* which is related to several classes of algebras of interest such as $BCH/BCI/BCK$-algebras. Furthermore, they demonstrated a rather interesting connection between B-algebras and groups. P. J. Allen et al. ([1]) included several new families of Smarandache-type P-algebras and studied some of their properties in relation to the properties of previously defined Smarandache-types. Recently, Kim et al. ([8]) introduced the notion of a (pre-)Coxeter algebra and showed that a Coxeter algebra is equivalent to an abelian group all of whose elements have order 2, i.e., a Boolean group. Moreover, they proved that the class of Coxeter algebras and the class of B-algebras of odd order are Smarandache disjoint. In this paper we show that the class of PC-algebras and the class of B-algebras with condition (D) are Smarandache disjoint, and show that an algebra $(X;*,0)$ is a Coxeter algebra if and only if it is a PC-algebra with (N). Moreover, we show that there is no non-trivial quadratic PC-algebras on a field with $|X| \geq 3$.

2. Preliminaries

In this section we refer notions and theorems discussed in [8]. A *Coxeter algebra* ([8]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(B1) $x*x=0$,

(B2) $x*0=x$,

(C) $(x*y)*z=x*(y*z)$,

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for any $x, y, z \in X$. An example of a Coxeter algebra is a Klein 4-group.

Theorem 2.1. *If $(X; *, 0)$ is a Coxeter algebra, then it is an abelian group all of whose elements have order 2, i.e., a Boolean group, and conversely.*

J. Neggers and H. S. Kim introduced and investigated a class of algebras, called a B-algebra, which is related to several classes of algebras such as $BCH/BCI/BCK$-algebras. A *B-algebra* ([10]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms: (B1), (B2) and (B) $(x * y) * z = x * (z * (0 * y))$, for any $x, y, z \in X$.

Proposition 2.2. *If $(X; *, 0)$ is a Coxeter algebra, then it is a B-algebra.*

An algebra $(X; *, 0)$ is called a *pre-Coxeter algebra* (shortly, *PC-algebra*) if it satisfies the axioms (B1), (B2), (PC1) $x * y = y * x$, (PC2) $x * y = 0 \Longrightarrow x = y$, for any $x, y \in X$.

Example 2.3. Let $X := [0, \infty)$. If we define $x * y := |\, x - y \,|, x, y \in X$, then $(X; *, 0)$ is a pre-Coxeter algebra, but not a Coxeter algebra, since $(1 * 2) * 3 = 2$, but $1 * (2 * 3) = 0$.

Proposition 2.4. *Every Coxeter algebra is a pre-Coxeter algebra.*

Proposition 2.5. *Let $(X; *, 0)$ be a Coxeter algebra. Then $x * (x * y) = y$, for any $x, y \in Y$.*

3. pre-Coxeter algebras and B-algebras in Smarandache settings

Let $(X, *)$ be a binary system/algebra. Then $(X, *)$ is a *Smarandache-type P-algebra* if it contains a subalgebra $(Y, *)$, where Y is non-trivial, i.e., $|\, Y \,| \geq 2$, or Y contains at least two distinct elements, and $(Y, *)$ is itself of type P. Thus, we have *Smarandache-type semigroups* (the type P-algebra is a semigroup), *Smarandache-type groups* (the type P-algebra is a group), *Smarandache-type abelian groups* (the type P-algebra is an abelian group). A Smarandache semigroup in the sense of Kandasamy is in fact a Smarandache-type group (see [11]). Smarandache-type groups are of course a larger class than Kandasamy's Smarandache semigroups since they may include non-associative algebras as well.

Given algebra types $(X, *)$ (type-P_1) and $(X, \circ)$ (type-P_2), we shall consider them to be *Smarandache disjoint* ([1]) if the following two conditions hold:

(A) If $(X, *)$ is a type-P_1-algebra with $|\, X \,| > 1$ then it cannot be a Smarandache-type-P_2-algebra $(X, \circ)$;

(B) If $(X, \circ)$ is a type-P_2-algebra with $|\, X \,| > 1$ then it cannot be a Smarandache-type-P_1-algebra $(X, *)$.

Theorem 3.1. ([8]) *The class of Coxeter algebras and the class of B-algebras of odd order are Smarandache disjoint.*

Lemma 3.2. *If $(X; *, 0)$ is a pre-Coxeter algebra with (B), then $(X; *, 0)$ is a Coxeter algebra.*

Proof. For any $x, y, z \in X$, we have

$$
\begin{aligned}
(x * y) * z &= x * (z * (0 * y) && [(B)] \\
&= x * (z * (y * 0)) && [(PC1)] \\
&= x * (z * y) && [(B2)] \\
&= x * (y * z), && [(PC1)]
\end{aligned}
$$

proving the lemma. □

Proposition 3.3. *The class of PC-algebras and the class of B-algebras of odd order are Smarandache disjoint.*

Proof. Assume that a PC-algebra $(X;*,0)$ contains a B-algebra $(Y;*,0)$ of odd order where $|Y| \geq 2$. By applying Lemma 3.2, we obtain that Y is a Coxeter algebra. It follows from Theorem 2.3 that Y can not be a non-trivial a B-algebra of odd order, a contradiction.

Conversely, Assume that a B-algebra $(X;*,0)$ of odd order contains a PC-algebra $(Y;*,0)$ where $|Y| \geq 2$. Then $(Y;*,0)$ satisfies the condition (B). By Lemma 3.2 it is a Coxeter algebra, which leads to a contradiction by Theorem 3.1. □

A B-algebra $(X;*,0)$ is said to have the *condition* (D) if $x*(0*x) \neq 0$ for any $x \neq 0$ in X.

Example 3.4. Let $X := \{0,1,2\}$ be a set with the following table:

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then it is a B-algebra ([10]) with condition (D).

Example 3.5. Let X be the set of all real numbers except for a negative integer $-n$. Define a binary operation $*$ on X by

$$x*y := \frac{n(x-y)}{n+y}.$$

Then $(X;*,0)$ is a B-algebra ([10]), but it does not have condition (D), since $(-2n)*(0*(-2n)) = 0$.

Proposition 3.6. *Let $(X;*,0)$ be a PC-algebra. Then X can not be a Smarandache-type B-algebra with condition (D).*

Proof. Assume that $(Y;*,0)$ is both a B-algebra with condition (D) and a PC-algebra with $Y \subset X$, $|Y| \geq 2$. Then $0 = a*a = a*(a*0) = a*(0*a)$ by applying (B1), (B2) and (PC1), which leads to a contradiction. □

Proposition 3.7. *Let $(X;*,0)$ be a B-algebra with condition (D). Then X can not be a Smarandache-type PC-algebra.*

Proof. Similar to Proposition 3.7. □

It follows from Propositions 3.6 and 3.7 that:

Theorem 3.8. *The class of PC-algebras and the class of B-algebras with condition (D) are Smarandache disjoint.*

4. PC-algebras and $BF/BF_2/Coxeter$-algebras

A. Walendziak ([12]) introduced the notion of BF-algebras, and showed a very good diagram to understand the address of various algebras which are related to BF-algebras.

Definition 4.1. An algebra $(X;*,0)$ is called a *BF-algebra* ([3]) if it satisfies (B1),(B2) and (BF) $0*(x*y)=y*x$, for any $x,y\in X$.

Proposition 4.2. *Every PC-algebra is a BF-algebra.*

Proof. Let $(X;*,0)$ be a PC-algebra. Then, for any $x,y\in X$, we have

$$\begin{aligned} 0*(x*y) &= (x*y)*0 && [(PC1)] \\ &= x*y && [(B2)] \\ &= y*x, && [(PC1)] \end{aligned}$$

proving that X is a BF-algebra. □

A BF-algebra $(X;*,0)$ is called a *BF_2-algebra* ([12]) if it satisfies (V) $x*y=0$ and $y*x=0$ imply $x=y$.

Corollary 4.3. *If $(X,*)$ is a PC-algebra, then $(X,*)$ is also a BF_2-algebra.*

Proof. It can be easily proved by (PC2). □

Proposition 4.4. *If a PC-algebra $(X;*,0)$ satisfies the condition*

(N) $\quad (x*z)*(y*z)=x*y,$

then it is a Coxeter algebra.

Proof. Given $x,y,z\in X$, we have

$$\begin{aligned} (x*y)*z &= ((x*y)*y)*(z*y) && [(N)] \\ &= ((x*y)*(y*0))*(z*y) && [(B2)] \\ &= ((x*y)*(0*y))*(z*y) && [(PC1)] \\ &= (x*0)*(z*y) && [(N)] \\ &= x*(z*y) && [(B2)] \\ &= x*(y*z) && [(PC1)], \end{aligned}$$

proving the proposition. □

Lemma 4.5. *If $(X;*,0)$ is a Coxeter algebra, then $(x*y)*x=y$ for any $x,y\in X$.*

Proof. Given $x,y\in X$, we have $(x*y)*x=(x*y)*(x*0)=(x*y)*[x*(y*y)]=(x*y)*[(x*y)*y]=[(x*y)*(x*y)]*y=0*y=y$, proving the lemma. □

Proposition 4.6. *If $(X;*,0)$ is a Coxeter algebra, then it satisfies the condition (N).*

Proof. By applying Lemma 4.5, we have $(x*z)*(y*z)=x*(z*(y*z))=x*((z*y)*z)=x*y$, proving the proposition. □

Theorem 4.7. *An algebra $(X;*,0)$ is a Coxeter algebra if and only if it is a PC-algebra with (N).*

Proof. It is a consequence of Propositions 4.4 and 4.6. □

5. Quadratic PC-algebras

Let X be a field with $|X| \geq 3$. An algebra $(X,*)$ is said to be *quadratic* if $x*y$ is defined by

$$x * y := a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6,$$

where $a_1, \cdots, a_6 \in X$, for any $x, y \in X$. A PC-algebra $(X,*)$ is said to be a *quadratic PC-algebra* if it is quadratic.

Theorem 5.1. *Let X be a field with $|X| \geq 3$. Then there is no non-trivial quadratic PC-algebras.*

Proof. Let

$$x * y := Ax^2 + By^2 + Cxy + Dx + Ey + F \tag{1}$$

where $A, B, C, D, E, F \in X$. Consider $(B1)$. Given $x \in X$, we have

$$e = x * x = (A + B + C)x^2 + (D + E)x + F. \tag{2}$$

Then we have $A + B + C = 0$, $D + E = 0$, $F = 0$. If we consider $(B2)$, then $x = x * 0 = Ax^2 + Dx$, i.e., $A = 0, D = 1$. Hence $x*y = By^2 - Bxy + x - y = (x-y)(1-By)$. Consider $(PC1)$. Then we have $(x - y)(1 - By) = (y - x)(1 - Bx)$, i.e., $(x - y)(2 - B(x + y)) = 0$. From this, either $x = y$ or $2 = B(x + y)$.

We claim that $B \neq 0$. If we assume that $B = 0$, then $x*y = x-y$, and $x*0 = x, 0*x = -x$. By $(PC1)$, we obtain $2x = 0$, i.e., $X = \{0\}$, a contradiction.

In case of $x = y$, we have $x * y = 0$, a trivial case. Assume $2 = B(x + y)$. Then $y = -x + \frac{2}{B}$ and hence $x * y = (x - y)(1 - By) = 2Bx^2 - 4x + \frac{2}{B}$. By applying $(B2)$, $x = x*0 = 2Bx^2 - 4x + \frac{2}{B}$ for any $x \in X$, a contradiction. This means that there is no non-trivial quadratic polynomials satisfying the condition $(PC1)$ in X, proving the theorem. □

Corollary 5.2. *There is no non-trivial quadratic Coxeter algebra on the field X with $|X| \geq 3$.*

Proof. It can be easily proved by Theorem 5.1 and Theorem 4.7. □

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