CANTOR SETS IN LOCALLY COMPACT GROUPS

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Abstract. In this paper we introduce a modified version of Aleksandrov Theorem on non-discrete Hausdorff locally compact groups. This also provides us a method to construct Cantor type sets in any positive left Haar measure subset.

1. Introduction

Cantor sets widely appear as invariant sets and attractors of many chaotic dynamical systems. Therefore Cantor sets in manifolds play a role in the dynamics analysis and understanding topological nature of invariant sets of systems, e.g. [8, 9]. This signifies the importance of study of Cantor sets in a more general spaces such as locally compact groups. Concepts of perfect and Cantor sets were initiated by Georg Cantor, who introduced the accumulation point set of a set in 1872. He also constructed Cantor ternary set, and therefore Cantor set has been named after him. Note that in this paper a non-empty set which is nowhere dense, compact and perfect is called Cantor set, and we are just heading to deal with non-discrete Hausdorff locally compact groups. Constructing perfect and Cantor sets was extended to separable complete metric spaces by P. S. Aleksandrov in 1916. He proved that any uncountable Borel set, later extended to analytic sets, in a separable complete metric space contained a nonempty perfect subset. In fact he introduced an appropriate homeomorphism between uncountable Borel sets, analytic sets, and the unit interval of real line. Then, simply by considering Cantor sets in the unit interval and transferring them back into the original space provides desired perfect and Cantor sets, c.f. [1, 4, 5, 10, 11, 12]. Most research and study of Cantor sets has been focused on the real line and complete separable metric spaces and one cannot find many discussions and results on locally compact groups. Naturally, a question arises: what categories of sets contain a Cantor set?

Let us first recall that even on the real line there are uncountable sets with no non-empty perfect subset, so-called totally imperfect set, c.f. [1, §1, Exercise 22.8, and §3, Lemma 35]. Aleksandrov Theorem strikes the mind that there may be no totally imperfect set in the category of uncountable Borel sets. However, this is not true in a general locally compact group, see Remark 2.4. In fact, even an uncountable closed subset of a locally compact group may be a totally imperfect set. Therefore we are interested in some constraints on Borel or analytic sets under which the category would not have totally imperfect set and, even more, any set in the category would have some Cantor sets as its subsets. Further, let us recall the fact that any Borel and analytic set is measurable due to any outer metric measure. Thereby this leads us, instead, to think of category of measurable sets with positive left Haar measure.

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In order to deal with this problem, we introduce condensation derivative, a Borel derivative, as its iterations on a suitable set approach to a perfect set, c.f. [3] and references therein for relevant results in separable complete metric space cases. In the sequel we are concerned with the sufficient conditions under which the reached perfect set would be nonempty, nowhere dense and compact. We try to evaluate the impact of left Haar measure to provide sufficient conditions which lead to Cantor sets. We prove that the Condensation derivative is a measure-preserving and invariant set function on certain subsets of space. Then it enables us to introduce Aleksandrov Theorem on locally compact groups, indicating that any positive left Haar measure set contains a Cantor subset. This facilitates us to extend Cantor's idea to construct Cantor sets in any set of finite and positive left Haar measure. This also provides an efficient tool to discuss the measure of the obtained Cantor sets.

2. Condensation derivative and perfect sets

Through out this section we introduce condensation derivative and present some preliminary results related to perfect sets. Note that in this paper G is always considered to be a non-discrete Hausdorff locally compact group and $X \subseteq G$.

A Borel derivative on 2^X is a Borel map $D : 2^X \to 2^X$ which is monotone on the closed subsets of X, i.e., $D(H) \subseteq H$ for any closed set H. For instance Cantor-Bendixson derivative is a Borel derivative, which is defined as follows:

$$D(K) = K',$$

where K' denotes the set of accumulation points of K, c.f. [10]. For an illuminating presentation of Borel derivatives see Kechris [7].

For an ordinal number α and a Borel derivative $D : 2^X \to 2^X$, the α-th iterated derivative $D^\alpha : 2^X \to 2^X$ is defined inductively as follows:

$$\begin{array}{lcll} D^0(K) & = & K, & \\ D^{\alpha+1}(K) & = & D(D^\alpha(K)), & \text{and} \\ D^\alpha(K) & = & \bigcap_{\beta \prec \alpha} D^\beta(K) & \text{for limit ordinal number } \alpha. \end{array}$$

Each D^α is a Borel map, c.f [2] where the Borel complexity of the iterations is investigated. Condensation derivative is an example of Borel derivative:

Definition 2.1. A point $p \in X$ is called a condensation point of $A \subseteq X$ if any neighborhood of p contains uncountably many points from A. We call the set of all condensation points of A as condensation derived set (CDS) from A and denote CD for condensation derivative set function which maps any set to its CDS.

CDSs are always closed and CD is a Borel derivative. CD of a second-countable space excludes at most countably many points and is a perfect set. This is called Cantor-Bendixon Theorem. It, however, may be a non perfect set or a void set for non second-countable cases, see Remark 2.4.

If $\{F_\gamma | \gamma \in \Gamma\}$ is a family of perfect sets, then $\overline{\bigcup_{\gamma\in\Gamma} F_\gamma}$ is always perfect. Therefore definition of perfect kernel is well defined:

Definition 2.2. The maximal perfect subset of the closure of a set is called its perfect kernel.

We define the α-th iterated condensation derivative, for any ordinal number α, the same as what is defined for Borel derivatives.

Theorem 2.3. *Let X be a closed subset of G and $A \subseteq X$.*

1. *Then there is an ordinal number* α_0 *which*

$$P = \mathrm{CD}^{\alpha_0}(A) = \mathrm{CD}^{\alpha}(A), \text{ where } \alpha \succeq \alpha_0,$$

and P *is the perfect kernel of* A*. In other words, for any perfect set* H *in* X *which*

$$H \subseteq A \cup \mathrm{CD}(A) \subseteq \overline{A},$$

we have

$$H \subseteq P = \mathrm{CD}^{\alpha_0}(A).$$

2. *Let* G *be first-countable space and* X *be* σ*-compact, then* $P = \mathrm{CD}(A)$ *is the perfect kernel of* A*, for any set* $A \subseteq X$*, and* $P = \mathrm{CD}^{\alpha}(P)$ *for any ordinal number* α*. In particular, when* A *is an uncountable set, then* $P \neq \emptyset$*.*

Proof. It is straightforward that $\{\mathrm{CD}^{\alpha}(A)\}_{\alpha}$ is a descending chain of closed sets. Choose an ordinal number α_0 at which the cardinal number of its predecessor ordinal numbers is $\mathrm{Card}(\mathcal{P}(A))$, where $\mathcal{P}(A)$ denotes the power set of A. It is easy to see that $\mathrm{CD}^{\alpha}(A) = \mathrm{CD}^{\alpha_0}(A)$, for any $\alpha \succeq \alpha_0$. Now we need to prove that a set P is perfect if and only if $\mathrm{CD}(P) = P$. Suppose that there exists a limit point $x \in P$ which is not a condensation point. Then, there is a countable and relatively open set G such that $x \in G \subseteq P$. Choose another relatively open set N in P such that $x \in N$ and $N \subseteq \overline{N} \subset G$. Then, $\overline{N}$ is a nonempty countable perfect set, which contradicts with the fact that there is no nonempty countable perfect set in a Hausdorff locally compact group (space). This completes the proof for Part (1). Proving Part (2) is straightforward from Cantor-Bendixon Theorem and the fact that any first-countable locally compact group is metrizable and therefore X is 2nd-countable. □

Remark 2.4. At the first glance, one might naively think that the chain in Theorem 2.3 is just a single nonempty perfect set. It, however, is not true for a general locally compact group. In fact, the hypothesis for the "first-countability" of G and "σ-compactness" of X in Part (2), is crucial. Although it is easy to see that in a metric locally compact group, any CDS of an uncountable closed set A is perfect and $\mathrm{CD}(A) = \mathrm{CD}^2(A)$. It, however, may be a void set. Indeed, a non-discrete metric locally compact group may contain an uncountable closed set, say A, satisfying $\mathrm{CD}(A) = \emptyset$ and therefore it is a totally imperfect closed set, see Proposition 2.5. It is not very difficult to generalize the idea used in Proposition 2.5 to construct a non-discrete locally compact group, non-metrizable of course, such that the cardinal of mentioned chain in Theorem 2.3 is very large and the deriving sets would be totally imperfect set. This topic, however, is out of the scope of this paper, and thus is not further discussed.

Any nonempty CDS in a metric locally compact group is an uncountable perfect set. There is also no totally imperfect set with positive left Haar measure, see Theorem 3.3. There, however, exists a non-discrete metric locally compact group which contains uncountable closed totally imperfect sets.

Proposition 2.5. *There exists a non-discrete metric locally compact group which contains an uncountable closed set which is totally imperfect set, in particular its* CDS *is empty.*

Proof. Let us first construct a non-discrete metric locally compact group. Denote Ω_1 for the first uncountable ordinal number and define the additive group G of all real valued functions defined on $[1, \Omega_1]$. The group G is an ordered set, $(G, \prec)$, if we define $f \succ g$ when there exists an $\alpha_0 \in [1, \Omega_1]$ such that

$$f(\alpha_0) \succ g(\alpha_0) \text{ and } f(\alpha) = g(\alpha), \text{ for all } \alpha \prec \alpha_0.$$

Then $(G, \prec)$ is a totally ordered set and G equipped with order topology is a non-discrete additive group. Let f_a denote a function defined by

$$f_a(\alpha) = \begin{cases} 0 & \text{for} \quad \alpha \prec \Omega_1, \\ a & \text{when} \quad \alpha = \Omega_1. \end{cases}$$

Then, $\{(f_{-1/n}, f_{1/n})\}_{n=1}^{\infty}$ is a countable local neighborhood base at 0. It is easy to see that for any real numbers a and b, where $a \prec b$, we have $\overline{(f_a, f_b)} = [f_a, f_b]$. Because $[f_a, f_b]$ is also compact for any a and b, then $\{(f_{-1/n}, f_{1/n})\}_{n=1}^{\infty}$ is a relatively compact local neighborhood base at 0. Now for any $f \in G$ denote

$$f + (f_{-1/n}, f_{1/n}) = \Big\{g \in G | g = f + h, \text{ where } h \in (f_{-1/n}, f_{1/n})\Big\}$$

and consider the family of

$$\big\{f + (f_{-1/n}, f_{1/n}) | n \in \mathbb{N}\big\}.$$

This is a countable and relatively compact local neighborhood base around f. Therefore G is a first-countable locally compact group. Thus, G is metrizable by [6, §8, Theorem 8.3]. Now we claim that the set of all constant functions in G, say A, is an uncountable closed set which contains no perfect subset. In fact for any real number a and constant function f^a, where $f^a(\alpha) = a$ for any $\alpha \in [1, \Omega_1]$, $f^a + (f_{-1/2}, f_{1/2})$ is an open set and

$$f^a \in f^a + (f_{-1/2}, f_{1/2}).$$

However, $f^a + (f_{-1/2}, f_{1/2})$ does not contain any other constant function. Therefore, A is a closed set and $\mathrm{CD}(A) = \emptyset$. Hence, by Theorem 2.3, the perfect kernel of A is the void set, and the proof is complete. □

3. Constructing Cantor sets in locally compact groups via its left Haar measure

Now, we present the main result of this paper. Let us denote λ for the left Haar measure defined on G. The following theorem could be extended into the more general spaces. It, however, is irrelevant to the purpose of this paper and will be discussed in the more relevant treatise works, see e.g. [4].

Theorem 3.1. *Let $F \subseteq G$ be a closed set whose left Haar measure is finite and positive. Then, there is an ordinal number α_0 such that $P = \mathrm{CD}^{\alpha_0}(F)$ is a nonempty perfect set, $P = \mathrm{CD}(P)$ and $\lambda(P) = \lambda(F) \succ 0$. Besides, P is the perfect kernel of set F.*

Proof. For a closed set $F \subseteq X$, denote $P_1 = \mathrm{CD}(F)$, and assume that $a = \lambda(P_1) \prec \lambda(F) = b$. Then,

$$\lambda(F \setminus P_1) = b - a \succ 0.$$

Thus, there exists a compact set K such that $K \subseteq F \setminus P_1$, and $\lambda(K) \succ 0$. K is an uncountable compact set because any left Haar measure of a countable set in a non-discrete locally compact group is zero. Hence,

$$\mathrm{CD}(K) \neq \emptyset.$$

Since $K \subseteq F$ thereby

$$\mathrm{CD}(K) \subseteq \mathrm{CD}(F) = P_1.$$

Furthermore, because K is a closed set, we have

$$\mathrm{CD}(K) \subseteq K \subseteq F \setminus P_1,$$

which is a contradiction. Therefore, $a = \lambda(P_1) \succeq \lambda(F) = b$. However, we have $P_1 \subseteq F$ since F is a closed set. Thus, $\lambda(P_1) = \lambda(F)$ for any closed set F with a finite and positive left Haar measure.

Now let us first assume that there exists an ordinal number α such that

$$\lambda(\mathrm{CD}^{\alpha}(F)) \prec \lambda(F).$$

Consider α_0 as the least ordinal number which satisfies this. Therefore, α_0 must be a limit ordinal number. Let

$$P_{\alpha_0} = \mathrm{CD}^{\alpha_0}(F) \quad \text{and} \quad a = \lambda(P_{\alpha_0}) \prec \lambda(F) = b.$$

Then, by a similar argument we can show that there exists a compact set K, $K \subseteq F \setminus P_{\alpha_0}$, such that $\lambda(K) \succ 0$. Because

$$K \subseteq F \setminus P_{\alpha_0} = \bigcup_{\alpha \prec \alpha_0} F \setminus \mathrm{CD}^{\alpha}(F),$$

K is compact and $\{F \setminus \mathrm{CD}^{\alpha}(F) |\ \alpha \prec \alpha_0\}$ is a family of relatively open sets in F, there exists a finite number of ordinal numbers, $\alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_n \prec \alpha_0$, such that

$$K \subseteq \bigcup_{i=1}^{n} F \setminus \mathrm{CD}^{\alpha_i}(F) = F \setminus \mathrm{CD}^{\alpha_n}(F).$$

Therefore, $K \bigcap \mathrm{CD}^{\alpha_n}(F) = \emptyset$ and $\lambda(\mathrm{CD}^{\alpha_n}(F)) \prec \lambda(F)$. But $\alpha_n \prec \alpha_0$ contradicts with the choice of α_0 which is the least ordinal number with this property. Thus, we have

$$\lambda(F) = \lambda\big(\mathrm{CD}^{\alpha}(F)\big),$$

for any ordinal number α. Now, by Theorem 2.3, there is an ordinal number α_0 for which $P = \mathrm{CD}^{\alpha_0}(F)$ is a perfect invariant set for CD and $\lambda(P) = \lambda(F) \succ 0$. The proof is complete. □

Remark 3.2. The closeness of set F in Theorem 3.1 cannot be substituted with being a F_σ-set, even when the space is the real line. For instance, let $K \subsetneq [0,1]$ be a Cantor set with positive Lebesgue measure, and consider F to be a F_σ-set such that

$$F \subseteq (Q^c \bigcap [0,1]) \setminus K,$$

and

$$\lambda(F) = \lambda(Q^c \cap [0,1] \setminus K) \prec 1.$$

Then, $\mathrm{CD}(F) = [0,1]$. Thus,

$$\lambda(\mathrm{CD}(F)) = 1 \succ \lambda(F), \text{ where } F \text{ is a } F_\sigma.$$

The following theorem is a version of Aleksandrov Theorem on locally compact groups. It indicates that any positive left Haar measure set contains compact perfect sets of positive left Haar measure.

Theorem 3.3. *Let A be a measurable subset of a locally compact group with a finite and positive left Haar measure, say $\lambda(A) = \alpha \succ \theta \succ 0$. Then, there exists an uncountable perfect compact set $K_\theta \subseteq A$ such that $\lambda(P_\theta) = \theta$.*

Proof. For any real number θ, $0 \prec \theta \prec \alpha$, there exists a compact set $H \subseteq A$ such that $\lambda(H) \succ \theta$. Now, we need to find a compact subset P_θ from H with $\lambda(P_\theta) = \theta$. To achieve this, consider the ordered set

$$(\Lambda = \{K \subseteq H | \lambda(K) \succeq \theta \quad \text{and} \quad K \text{ is perfect and compact}\}, \preceq)$$

where

$$K_1 \preceq K_2, \text{ if } K_1 \supseteq K_2.$$

By Theorem 3.1, the ordered set Λ is non-empty. Based on Zorn's Lemma we claim that any maximal set K from Λ has the measure of θ. Consider an arbitrary chain $\{K_\iota\} \subseteq \Lambda$, then $B = \bigcap_\iota \{K_\iota\}$ is a nonempty compact set. If $\lambda(B) \prec \theta$, then there is an open set U such that

$$B \subseteq U \quad \text{and} \quad \lambda(U) \prec \theta.$$

Because $\bigcap_\iota \{U^c \cap K_\iota\} = \emptyset$, there is a finite number of ι_i such that

$$\bigcap_{i=1}^{n} \{U^c \cap K_{\iota_i}\} = \emptyset.$$

Thus, there exists a $K_{\iota_i} \subseteq U$ such that

$$\theta \preceq \lambda(K_{\iota_i}) \preceq \lambda(U) \prec \theta$$

which is a contradiction.

For our convenience, let us denote α_A for the least ordinal number satisfying Theorem 2.3 due to set A. Therefore,

$$\lambda(B) = \lambda(\mathrm{CD}^{\alpha_B}(B)) \succeq \theta.$$

In other words,

$$K_\iota \preceq P_B = \mathrm{CD}^{\alpha_B}(B) \in \Lambda$$

is the least upper bound for the chain $\{K_\iota\}$. Therefore, by Zorn's Lemma, Λ has a maximal set P_θ. Assume that $\lambda(P_\theta) \succ \theta$ and choose a point $x \in P_\theta$. Because $\lambda(\{x\}) = 0$, for any $\varepsilon \succ 0$ such that $\varepsilon \prec \lambda(P_\theta) - \theta$, there is an open set V with $x \in V$ and $\lambda(V) \prec \varepsilon$. Thus, if $P_\theta^V = P_\theta \backslash V$, we have

$$\mathrm{CD}^{\alpha_{P_\theta^V}}(P_\theta^V) = \lambda(P_\theta^V) \succ \theta, \quad \text{and } \mathrm{CD}^{\alpha_{P_\theta^V}}(P_\theta^V) \subset P_\theta,$$

and therefore,

$$\mathrm{CD}^{\alpha_{P_\theta^V}}(P_\theta^V) \in \Lambda \quad \text{and} \quad \mathrm{CD}^{\alpha_{P_\theta^V}}(P_\theta^V) \succeq P_\theta$$

which is a contradiction. Therefore,

$$\lambda(P_\theta) = \theta \succ 0 \quad \text{and} \quad P_\theta \subseteq A$$

is an uncountable compact perfect set, and the proof is complete. □

Remark 3.4. Note that P_θ in Theorem 3.3 can be chosen as a Cantor set, when A is nowhere dense or A contains a zero measure subset which is dense in *interiour*$(\overline{A})$, e.g. when X is separable.

Theorem 3.1 indicates that any positive left Haar-measure closed set in a non-discrete locally compact group has the same measure with its perfect kernel. Therefore, totally imperfect sets in non-discrete locally compact groups are null-left Haar measure set. Theorem 3.3, on the other hand, implies that when a non-discrete locally compact group is separable or contains a dense subset whose measure is less than the measure of space, then we can construct a Cantor set. This leads us to look for an efficient method for constructing Cantor sets in positive left Haar measure sets of non-discrete locally compact groups without these concerns. Indeed, we have extended Aleksandrov Theorem such that any finite and positive left Haar measure set contains a Cantor set, with no constraint. The interesting point is that Cantor's method can be directly applied to prove this modified version of Aleksandrov Theorem for Cantor sets on locally compact groups.

Theorem 3.5. *Let A be a subset of a locally compact group with a finite and positive left Haar measure, say $\lambda(A) = \alpha$. Then, for any real number β, $\alpha \succ \beta \succeq 0$, there is a Cantor set $P_\beta \subseteq A$ such that $\lambda(P_\beta) = \beta$.*

Proof. Consider $0 \prec \alpha - \beta = \sum_{i=1}^{\infty} \gamma_i$, where γ_i are positive numbers. By Theorem 3.3, for $\gamma_1 \succ 0$ there is a perfect compact set $K^0 \subset A$ such that $\lambda(K^0) = \alpha/2 - \gamma_1/2$. Then, we have

$$\lambda(A \setminus K^0) = \alpha/2 + \gamma_1/2 \succ \alpha/2 - \gamma_1/2.$$

Let $\alpha/2 - \gamma_1/2 \prec \theta \prec \alpha/2$. Then, there exists an open set $N_0 \supset K^0$ such that $\lambda(N_0) \prec \theta$. Now, because G is locally compact, there is a relatively compact open set N^0 such that

$$K^0 \subseteq N^0 \subseteq \overline{N^0} \subseteq N_0.$$

Thus, $\lambda(A \setminus \overline{N^0}) \succ \alpha/2$ and therefore there is a compact set

$$K^1 \subset A \setminus \overline{N^0} \subseteq \left(\overline{N^0}\right)^c = N^1$$

such that

$$\lambda(K_1) = \alpha/2 - \gamma_1/2 \prec \alpha/2.$$

Thereby we have separated compact perfect sets K^0 and K^1, i.e.

$$K^0 \subseteq N^0, \quad K^1 \subseteq N^1, \quad N^0 \bigcap N^1 = \emptyset,$$

where N^0 and N^1 are open sets, and $\lambda(K^0 \bigcup K^1) = \alpha - \gamma_1$. By the same procedure we find the compact sets

$$K^{00}, K^{01} \subset K^0, \text{ and } K^{10}, K^{11} \subset K^1$$

which are mutually separated, and so

$$\lambda\Big(\bigcup_{i,j\in\{0,1\}} K^{ij}\Big) = \alpha - \gamma_1 - \gamma_2 \text{ and } \lambda(K^{ij}) = \alpha/4 - \gamma_1/4 - \gamma_2/4.$$

By induction we construct the separated compact perfect sets:

$$K^{i_1 i_2 \cdots i_n}, \quad \text{where } \; i_j \in \{0,1\}, \quad j = 1, 2, \ldots n,$$

such that

$$K^{i_1 i_2 \cdots i_n} \subset K^{i_1 i_2 \cdots i_{n-1}}, \quad \lambda(\bigcup K^{i_1 i_2 \cdots i_n}) = \alpha - \sum_{i=1}^{n} \gamma_i$$

and

$$\lambda(K^{i_1 i_2 \cdots i_n}) = \alpha/2^n - \gamma_1/2^n \cdots - \gamma_n/2^n.$$

It is easy to see that

$$F = \bigcap_{n=1}^{\infty} \bigcup_{i_j \in\{0,1\}} K^{i_1 i_2 \cdots i_n} \neq \emptyset$$

is a compact set and $\lambda(F) = \beta \succ 0$. Theorem 3.3 indicates that there exists compact perfect set $P_\beta \subseteq F$ such that

$$\lambda(P_\beta) = \lambda(F) = \beta.$$

We claim that P_β is nowhere dense, and in fact so is F. Otherwise, F contains an open set. Consider N to be a connected component of this open set, therefore $N \subseteq F$ and $\lambda(N) \succ 0$. Because

$$\lambda(K^{i_1 i_2 \cdots i_n}) \prec \alpha/2^n,$$

thus for a sufficient large number n, N is not included in $K^{i_1 i_2 \cdots i_n}$ for any $i_1 i_2 \cdots i_n \in \{0,1\}^n$.

Since $K^{i_1 i_2 \cdots i_n}$, $i_1 i_2 \cdots i_n \in \{0,1\}^n$, are mutually separated, N is not connected. This contradicts with the choice of N. □

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