# ON THE TWO-BOUNDARY FIRST-PASSAGE TIME FOR A CLASS OF MARKOV PROCESSES

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Abstract. The first-passage time problem through two time-dependent boundaries for one-dimensional Gauss-Markov processes is considered, both for fixed and for random initial states. The first passage time probability density functions are proved to satisfy a system of continuous-kernel integral equations that can be numerically solved by an accurate and computationally simple algorithm. A condition on the boundaries of the process is given such that this system reduces to a single non-singular integral equation. Closed-form results are also obtained for classes of double boundaries that are intimately related to certain symmetry properties of the considered processes. Finally, the double-sided problem is considered.

## 1 Introduction and Mathematical Background

The first passage time (FPT) problem through generally time-dependent boundaries has been the object of numerous investigations over the past three decades, partly motivated by the role that it plays in biology, engineering, physics, psychology and in many other applied fields (cf., for instance, [1], [2], [3], [4], [7], [9], [14]). As is well-known, exact solutions for FPT problems are only available in very few special cases, and results based on numerical computations or simulations are in general scarce and fragmentary.

It should be emphasized that particularly relevant to theoretical neurobiology and population biology fields are FPT problems. Indeed, as discussed for instance in [12], and successively pinpointed in [13], the FPT pdf can be taken as the theoretical counterpart of the neuronal firing density function in the area of stochastic models of single neurons activity. Moreover, within in the context of population dynamics, extinction can be viewed to occur when for the first time the population size attains some small critical mass. Hence, again a first passage time problem may be suitable to model population growth with an eye at the very crucial expects of population extinction. The present paper aims to provide a contribution towards the solution of theoretical and computational aspects of first passage time problems for the wide class of Gauss-Markov processes with special attention to those situation in which sample paths are restricted within a strip characterized by a pair of assigned time functions, under the more general assumption that their starting point can be a random variable with preassigned probability distribution.

We recall that in [7] a new second-kind Volterra integral equation was obtained to determine the FPT pdf through a time-dependent boundary for Gauss-Markov (G-M) processes, both for fixed and for random initial states, and a new computationally simple, speedy and accurate method was proposed to construct the FPT probability density function (pdf). However, there are instances in which it would be desirable to possess similar simple and efficient procedures to evaluate FPT pdf's in the presence of double time-dependent boundaries for the class of G-M processes.

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As mentioned above, in this paper we consider the two-boundary FPT problem for the class of G-M processes, and prove that the FPT pdf's through the lower and through the upper boundaries respectively, are the solutions of continuous-kernel Volterra integral equations. Our arguments have been suggested by those originally proposed for time-homogeneous diffusion processes (cf. [4]) for the case of two boundaries and by those proposed for G-M processes (cf. [7]) in the case of a single boundary. Furthermore, a condition on the boundaries such that the system of integral equations for the G-M process reduces to a single non-singular integral equation for the FPT pdf is determined. Use of the symmetry properties of the transition pdf of G-M processes is then made to determine closed form expressions for the FPT pdf's and for the transition pdf in the presence of the two absorbing boundaries.

The FPT problem through either boundary, in which the initial state of the process is a random variable, is then considered, and an appropriate system of integral equations is determined. Such FPT problems appear to be particularly relevant in contexts such as population dynamics and neuronal modeling in which the initial population size or the reset value of the membrane potential are usually unknown.

The proposed integral equations will be seen to be particularly suited to provide accurate and speedy numerical evaluations of FPT densities. As for the one-boundary FPT problem, our direct numerical method bypasses some serious difficulties that generally originate if one numerically evaluates FPT densities for the Wiener process and then make use of a space-time transformation to obtain FPT densities of G-M processes.

Hereafter, the necessary preliminary background and notation is provided.

Let $\{X(t), t \in T\}$, where $T$ is a continuous parameter set, be a real, continuous G-M process with the following properties:

*(i)* $m(t) := E[X(t)]$ is continuous in $T$;

*(ii)* the covariance $c(s,t) := E\{[X(s) - m(s)]\,[X(t) - m(t)]\}$ is continuous in $T \times T$;

*(iii)* $\{X(t)\}$ is non-singular except possibly at the end points of $T$, i.e. if $T = [a, b]$, $\{X(t)\}$ has a non-singular normal distribution except possibly at $t = a$ or $t = b$, where $X(t)$ could be equal to $m(t)$ with probability one.

We shall make a systematic use of certain well-known properties of G-M processes as outlined in [11]. In particular,

*(a)* a Gaussian process is Markov if and only if its covariance is of the form

$$c(s,t) = h_1(s)\,h_2(t), \qquad s \le t, \tag{1.1}$$

where

$$r(t) = \frac{h_1(t)}{h_2(t)} \tag{1.2}$$

is a monotonically increasing function by virtue of the Cauchy-Schwarz inequality, and $h_1(t)\,h_2(t) > 0$ because of the assumed non-singularity of the process in the interior of $T$.

*(b)* The transition pdf $f(x, t \mid y, \tau)$ of a G-M process is a normal density characterized

respectively by mean and variance:

$$(1.3)\qquad \begin{aligned} &E[X(t) \mid X(\tau)=y] = m(t) + \frac{h_2(t)}{h_2(\tau)}\,\big[y - m(\tau)\big] \\ &\mathrm{Var}[X(t) \mid X(\tau)=y] = h_2(t)\,\Big[h_1(t) - \frac{h_2(t)}{h_2(\tau)}\,h_1(\tau)\Big]. \end{aligned} \qquad (t,\tau \in T, \tau < t)$$

Let $S_1(t)$ and $S_2(t)$ denote arbitrary $C^1(T)$-class functions such that *(i)* $S_1(t) < S_2(t)$, $\forall t \in T$ and *(ii)* $S_1(t_0) < X(t_0) \equiv x_0 < S_2(t_0),\ t_0 \in T$. For all $t \ge t_0,\ t, t_0 \in T$, we shall focus our attention on the random variables:

$$(1.4)\qquad \begin{aligned} &\mathcal{T}_{x_0}^{(1)} = \inf_{t\ge t_0}\big\{t : X(t) < S_1(t);\ X(\vartheta) < S_2(\vartheta), \forall\vartheta \in (t_0,t)\big\}, \qquad X(t_0)=x_0 \\ &\qquad\qquad \text{(FPT through the lower boundary)} \\ &\mathcal{T}_{x_0}^{(2)} = \inf_{t\ge t_0}\big\{t : X(t) > S_2(t);\ X(\vartheta) > S_1(\vartheta), \forall\vartheta \in (t_0,t)\big\}, \qquad X(t_0)=x_0 \\ &\qquad\qquad \text{(FPT through the upper boundary)} \\ &\mathcal{T}_{x_0} = \inf_{t\ge t_0}\big\{t : X(t) \notin \big(S_1(t), S_2(t)\big)\big\}, \qquad X(t_0)=x_0 \\ &\qquad\qquad \text{(first-exit time)} \end{aligned}$$

and denote by $g_1(t \mid x_0,t_0)$, $g_2(t \mid x_0,t_0)$ and $g(t \mid x_0,t_0)$, respectively, their pdf's:

$$(1.5)\qquad \begin{aligned} g_1(t \mid x_0,t_0) &= \frac{\partial}{\partial t} P\big(\mathcal{T}_{x_0}^{(1)} < t\big), \\ g_2(t \mid x_0,t_0) &= \frac{\partial}{\partial t} P\big(\mathcal{T}_{x_0}^{(2)} < t\big), \\ g(t \mid x_0,t_0) &= \frac{\partial}{\partial t} P\big(\mathcal{T}_{x_0} < t\big) \equiv g_1(t \mid x_0,t_0) + g_2(t \mid x_0,t_0). \end{aligned}$$

Hence, $P(\mathcal{T}_{x_0}^{(1)} < t)$ $[P(\mathcal{T}_{x_0}^{(2)} < t)]$ is the probability that $X(t)$ crosses for the first time $S_1(t)$ $[S_2(t)]$ at some time preceding $t$ before crossing $S_2(t)$ $[S_1(t)]$, whereas $P(\mathcal{T}_{x_0} < t)$ is the probability that $X(t)$ crosses for the first time either $S_1(t)$ or $S_2(t)$ before time $t$. Since $X(t)$ is Markov, for any $x \notin \big(S_1(t), S_2(t)\big)$ and $S_1(t_0) < x_0 < S_2(t_0)$ the following compatibility relation holds:

$$(1.6)\qquad \begin{aligned} f(x,t \mid x_0,t_0) = \int_{t_0}^{t} \Big\{ & g_1(\tau \mid x_0,t_0)\, f[x,t \mid S_1(\tau),\tau] \\ & + g_2(\tau \mid x_0,t_0)\, f[x,t \mid S_2(\tau),\tau] \Big\}\, d\tau. \end{aligned}$$

Setting $x = S_1(t)$ and $x = S_2(t)$, respectively, in (1.6) one obtains a system of two integral equations in the unknowns $g_1$ and $g_2$. Its solution is made complicated by the circumstance that $f[S_j(t),t|S_i(\tau),\tau]$ $(i,j = 1,2)$ exhibits a singularity as $\tau \uparrow t$. Hence, the problem of determining $g_1$ and $g_2$ from (1.6) via numerical methods is not at all trivial: efficient numerical algorithms are desirable, especially if one wishes to deal with the case of a time-varying threshold.

For all $t, t_0 \in T$ and $t > t_0$, let

$$(1.7)\qquad \begin{aligned} \beta(x,t|x_0,t_0) := \frac{\partial}{\partial x}\, P\Big\{ & X(t) < x; S_1(\vartheta) < X(\vartheta) < S_2(\vartheta),\ \forall\vartheta < t \mid X(t_0) = x_0 \Big\}, \\ & S_1(t) < x < S_2(t),\ S_1(t_0) < x_0 < S_2(t_0), \end{aligned}$$

be the transition pdf in the presence of absorbing boundaries at $S_1(t)$ and $S_2(t)$. Then,

$$\int_{S_1(t)}^{S_2(t)} \beta(x,t|x_0,t_0)\,dx = 1 - \int_{t_0}^{t} g(\vartheta \mid x_0,t_0)\,d\vartheta, \qquad S_1(t_0) < x_0 < S_2(t_0). \tag{1.8}$$

Any Gaussian process with covariance as in (1.1) can be represented in terms of the standard Wiener process $\{W(t), t \geq 0\}$ as

$$X(t) = m(t) + h_2(t)\,W\big[r(t)\big], \tag{1.9}$$

and is therefore Markov [8]. Hence, the possibility of constructing the FPT pdf of a G-M process $X(t)$ in terms of preassigned FPT pdf's of the standard Wiener process $W(t)$ is based on the following relations stemming out of (1.9):

$$\begin{aligned} g_1(t \mid x_0,t_0) &= \frac{dr(t)}{dt}\,\gamma_1\big[r(t) \mid x_0^*, r(t_0)\big],\\ g_2(t \mid x_0,t_0) &= \frac{dr(t)}{dt}\,\gamma_2\big[r(t) \mid x_0^*, r(t_0)\big],\\ g(t \mid x_0,t_0) &= \frac{dr(t)}{dt}\,\gamma\big[r(t) \mid x_0^*, r(t_0)\big], \end{aligned} \tag{1.10}$$

where $r(t)$ is defined in (1.2), $\gamma_1(\vartheta \mid x_0^*, \vartheta_0)$ $[\gamma_2(\vartheta \mid x_0^*, \vartheta_0)]$ is the FPT pdf of $W(\vartheta)$ through the lower [upper] boundary $S_1^*(\vartheta)$ $[S_2^*(\vartheta)]$ at time $\vartheta$ starting from $x_0^*$ at time $\vartheta_0$, whereas $\gamma(\vartheta \mid x_0^*, \vartheta_0)$ is the first-exit time of $W(\vartheta)$ from $(S_1^*(\vartheta), S_2^*(\vartheta))$, with

$$x_0^* = \frac{x_0 - m[r^{-1}(\vartheta_0)]}{h_2[r^{-1}(\vartheta_0)]}, \qquad S_j^*(\vartheta) = \frac{S_j[r^{-1}(\vartheta)] - m[r^{-1}(\vartheta)]}{h_2[r^{-1}(\vartheta)]} \qquad (j = 1,2). \tag{1.11}$$

Results on the FPT pdf's for the standard Wiener process can thus be used via (1.10) to yield the FPT pdf of any continuous G-M process. However, such a procedure often exhibits the serious drawback of implying inconvenient time dilations (cf., for instance, Example 2.1 in [7]).

In Section 2 we prove that the FPT densities (1.5) can be obtained by solving a simple system of continuous-kernel integral equations, thus overcoming the time-dilation difficulty implied by the transformation method. In Section 3 a condition is determined on the boundaries such that the system of integral equations reduces to a single non-singular integral equation. In Section 4, making use of symmetry properties of the G-M process, we obtain a family of suitable boundaries and initial states for which the transition densities in the presence of two boundaries and the first-exit time pdf are obtained in closed-form. In Section 5 the FPT problem through either boundary, in which the initial state of the process is a random variable, is then considered, and an appropriate system of integral equations is obtained. Finally, in Section 6 a computationally simple algorithm, based on the repeated Simpson rule, is proposed to determine the FPT pdf's.

## 2 FPT densities

We start proving the following

**Theorem 2.1** *Let $S_1(t)$, $S_2(t)$, $m(t)$, $h_1(t)$, $h_2(t)$ be $C^1(T)$ functions, with $S_1(t) < S_2(t)$, $\forall t \in T$ and $S_1(t_0) < x_0 < S_2(t_0)$. Then, $g_1(t \mid x_0,t_0)$ and $g_2(t \mid x_0,t_0)$ satisfy the following* *non-singular integral equations:*

$$
(2.1)\quad
\begin{aligned}
g_1(t \mid x_0,t_0) &= 2\,\Psi_1(t \mid x_0,t_0)\\
&\quad -2\int_{t_0}^{t}\Big\{g_1(\tau \mid x_0,t_0)\,\Psi_1[t \mid S_1(\tau),\tau]\ + g_2(\tau \mid x_0,t_0)\,\Psi_1[t \mid S_2(\tau),\tau]\Big\}\,d\tau,\\
g_2(t \mid x_0,t_0) &= -2\,\Psi_2(t \mid x_0,t_0)\\
&\quad +2\int_{t_0}^{t}\Big\{g_1(\tau \mid x_0,t_0)\,\Psi_2[t \mid S_1(\tau),\tau]\ + g_2(\tau \mid x_0,t_0)\,\Psi_2[t \mid S_2(\tau),\tau]\Big\}\,d\tau,
\end{aligned}
$$

*where*

$$
(2.2)\quad
\begin{aligned}
\Psi_j(t \mid y,\tau) = \bigg\{&\frac{S_j'(t)-m'(t)}{2} - \frac{S_j(t)-m(t)}{2}\,\frac{h_1'(t)\,h_2(\tau)-h_2'(t)\,h_1(\tau)}{h_1(t)\,h_2(\tau)-h_2(t)\,h_1(\tau)}\\
&-\frac{y-m(\tau)}{2}\,\frac{h_2'(t)\,h_1(t)-h_2(t)\,h_1'(t)}{h_1(t)\,h_2(\tau)-h_2(t)\,h_1(\tau)}\bigg\}\,f[S_j(t),t \mid y,\tau] \quad (j=1,2).
\end{aligned}
$$

**Proof.** Let $\gamma_1(\vartheta \mid x_0^*,\vartheta_0)$ $\big[\gamma_2\big(\vartheta \mid x_0^*,\vartheta_0\big)\big]$ denote the FPT pdf of the standard Wiener process through the lower [upper] boundary $S_1^*(\vartheta)$ $[S_2^*(\vartheta)]$ at the time $\vartheta$ starting from initial state $x_0^*$ at time $\vartheta_0$, where $S_j^*(\vartheta)$ $(j=1,2)$ and $x_0^*$ are defined in (1.11). As proved in [4], if $S_1^*(\vartheta_0) < x_0^* < S_2^*(\vartheta_0)$ the FPT pdf's $\gamma_1$ and $\gamma_2$ are solutions of the following integral equations:

$$
(2.3)\quad
\begin{aligned}
\gamma_1(\vartheta \mid x_0^*,\vartheta_0) &= -2\,\Phi_1(\vartheta \mid x_0^*,\vartheta_0)\\
&\quad +2\int_{\vartheta_0}^{\vartheta}\Big\{\gamma_1(\xi \mid x_0^*,\vartheta_0)\,\Phi_1[\vartheta \mid S_1^*(\xi),\xi] + \gamma_2(\xi \mid x_0^*,\vartheta_0)\,\Phi_1[\vartheta \mid S_2^*(\xi),\xi]\Big\}\,d\xi\\
\gamma_2(\vartheta \mid x_0^*,\vartheta_0) &= 2\,\Phi_2(\vartheta \mid x_0^*,\vartheta_0)\\
&\quad -2\int_{\vartheta_0}^{\vartheta}\Big\{\gamma_1(\xi \mid x_0^*,\vartheta_0)\,\Phi_2[\vartheta \mid S_1^*(\xi),\xi] + \gamma_2(\xi \mid x_0^*,\vartheta_0)\,\Phi_2[\vartheta \mid S_2^*(\xi),\xi]\Big\}\,d\xi
\end{aligned}
$$

where

$$
(2.4)\qquad \Phi_j[\vartheta \mid y,\xi] = \frac{1}{2}\Big[\frac{dS_j^*(\vartheta)}{d\vartheta} - \frac{S_j^*(\vartheta)-y}{\vartheta-\xi}\Big]\,f_W[S_j^*(\vartheta),\vartheta \mid y,\xi] \quad (j=1,2)
$$

and where $f_W(x,\vartheta \mid y,\xi)$ denotes the transition pdf for the standard Wiener process. Multiplying equation (2.3) by $\dfrac{dr(t)}{dt}$, setting $\vartheta = r(t)$ and $\vartheta_0 = r(t_0)$ and recalling (1.10) there follows:

$$
\begin{aligned}
g_1(t \mid x_0,t_0) &= -2\,\frac{dr(t)}{dt}\,\Phi_1\big[r(t)\mid x_0^*, r(t_0)\big]\\
&\quad +2\,\frac{dr(t)}{dt}\int_{t_0}^{t}\Big[g_1(\tau\mid x_0,t_0)\,\Phi_1\big\{r(t)\mid S_1^*[r(\tau)], r(\tau)\big\}\\
&\qquad +g_2(\tau\mid x_0,t_0)\,\Phi_1\big\{r(t)\mid S_2^*[r(\tau)], r(\tau)\big\}\Big]\,d\tau,
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
g_2(t \mid x_0,t_0) &= 2\,\frac{dr(t)}{dt}\,\Phi_2\big[r(t)\mid x_0^*, r(t_0)\big]\\
&\quad -2\,\frac{dr(t)}{dt}\int_{t_0}^{t}\Big[g_1(\tau\mid x_0,t_0)\,\Phi_2\big\{r(t)\mid S_1^*[r(\tau)], r(\tau)\big\}\\
&\qquad +g_2(\tau\mid x_0,t_0)\,\Phi_2\big\{r(t)\mid S_2^*[r(\tau)], r(\tau)\big\}\Big]\,d\tau.
\end{aligned}
$$

Setting $\vartheta = r(t)$, $y = x_0^*$, $\xi = r(t_0)$ in (2.4) and making use of (1.2) and (1.11), one has:

$$
\Phi_j\big[r(t)\mid x_0^*, r(t_0)\big] = \Big[\frac{dr(t)}{dt}\Big]^{-1}\Psi_j\big[t\mid x_0,t_0\big], \quad (j=1,2) \tag{2.6}
$$

where $\Psi_j \quad (j=1,2)$ are defined in (2.2). Similarly, setting $\vartheta = r(t)$, $y = S_j^*[r(\tau)]$ $(j=1,2)$, $\xi = r(\tau)$ in (2.4) and making use (1.2) and (1.11), one obtains:

$$
\Phi_j\big[r(t)\mid S_i^*[r(\tau)], r(\tau)\big] = \Big[\frac{dr(t)}{dt}\Big]^{-1}\Psi_j\big[t\mid S_i(\tau),\tau\big] \quad (i,j=1,2). \tag{2.7}
$$

Substituting (2.6) and (2.7) in (2.5), Equations (2.1) immediately follow. Finally, since $r(t)$ is a monotonically increasing function in the parameter set $T$,

$$
\begin{aligned}
\lim_{\tau\uparrow t}\Psi_j\big[t\mid S_i(\tau),\tau\big] &= \frac{dr(t)}{dt}\lim_{\tau\uparrow t}\Phi_j\big\{r(t)\mid S_i^*[r(\tau)], r(\tau)\big\}\\
&= \frac{dr(t)}{dt}\lim_{\xi\uparrow\vartheta}\Phi_j\big[\vartheta\mid S_i^*(\xi),\xi\big] = 0, \qquad (i,j=1,2),
\end{aligned}
\tag{2.8}
$$

which proves the non-singularity of (2.1). This completes the proof. □

**Theorem 2.2** *Let $T=[a,b]$. Under the assumptions of Theorem 2.1, if*

$$
\begin{aligned}
&\lim_{t\uparrow b} r(t) = +\infty,\\
&P\big\{S_1(t)\le X(t) < S_2(t)\mid X(t_0)=x_0\big\}\neq 1 \qquad \forall t\in T,
\end{aligned}
\tag{2.9}
$$

*there holds:*

$$
\int_{t_0}^{b} g(t\mid x_0,t_0)\,dt = 1. \tag{2.10}
$$

**Proof.** After setting

$$
k_j(t) = \frac{m'(t)-S_j'(t)}{2} + \frac{S_j(t)-m(t)}{2}\,\frac{h_2'(t)}{h_2(t)} \qquad (j=1,2), \tag{2.11}
$$

(2.2) can be written as:

$$\Psi_j(t \mid y, \tau) = \frac{d}{dt} F[S_j(t), t \mid y, \tau] + k_j(t)\, f[S_j(t), t \mid y, \tau] \qquad (j = 1, 2), \tag{2.12}$$

where $F(x, t \mid y, \tau) := P(X(t) < x \mid X(\tau) = y)$ denotes the probability distribution function of the G-M process $X(t)$. Expressing $g$ as the sum of $g_1$ and $g_2$, from (2.1) one obtains:

$$\begin{aligned} g(t \mid x_0, t_0) = 2 \Big[\Psi_1(t \mid x_0, t_0) - \Psi_2(t \mid x_0, t_0)\Big] \\ -2 \int_{t_0}^{t} \Big[ g_1(\tau \mid x_0, t_0) \Big\{ \Psi_1[t \mid S_1(\tau), \tau] - \Psi_2[t \mid S_1(\tau), \tau] \Big\} \\ + g_2(\tau \mid x_0, t_0) \Big\{ \Psi_1[t \mid S_2(\tau), \tau] - \Psi_2[t \mid S_2(\tau), \tau] \Big\} \Big]\, d\tau. \end{aligned} \tag{2.13}$$

Integrating both sides of (2.13) with respect to $t$ in $(t_0, b)$, making use of (1.6) and (2.12) and recalling that

$$\lim_{t \to t_0} F[S_1(t), t|x_0, t_0] = 0, \quad \lim_{t \to t_0} F[S_2(t), t|x_0, t_0] = 1,$$
$$\lim_{t \to \tau} F[S_1(t), t|S_1(\tau), \tau] = \frac{1}{2}, \quad \lim_{t \to \tau} F[S_2(t), t|S_1(\tau), \tau] = 1,$$
$$\lim_{t \to \tau} F[S_1(t), t|S_2(\tau), \tau] = 0, \quad \lim_{t \to \tau} F[S_2(t), t|S_2(\tau), \tau] = \frac{1}{2},$$

one has:

$$\begin{aligned} \int_{t_0}^{b} g(t \mid x_0, t_0) dt = 1 - \lim_{t \to b} \Big( F[S_2(t), t \mid x_0, t_0] - F[S_1(t), t \mid x_0, t_0] \Big) \\ - \int_{t_0}^{b} d\tau g_1(\tau \mid x_0, t_0) \Big\{ \lim_{t \to b} \Big( F[S_1(t), t \mid S_1(\tau), \tau] - F[S_2(t), t \mid S_1(\tau), \tau] \Big) \Big\} \\ - \int_{t_0}^{b} d\tau g_2(\tau \mid x_0, t_0) \Big\{ \lim_{t \to b} \Big( F[S_1(t), t \mid S_2(\tau), \tau] - F[S_2(t), t \mid S_2(\tau), \tau] \Big) \Big\}. \end{aligned} \tag{2.14}$$

Since $X(t)$ is G-M, recalling the first of (2.9) one obtains:

$$\lim_{t \to b} F[S_1(t), t \mid x_0, t_0] = \lim_{t \to b} F[S_1(t), t \mid S_1(\tau), \tau] = \lim_{t \to b} F[S_1(t), t \mid S_2(\tau), \tau] = A$$
$$\lim_{t \to b} F[S_2(t), t \mid x_0, t_0] = \lim_{t \to b} F[S_2(t), t \mid S_1(\tau), \tau] = \lim_{t \to b} F[S_2(t), t \mid S_2(\tau), \tau] = B.$$

Hence, (2.14) leads to

$$(1 - B + A) \int_{t_0}^{b} g(t \mid x_0, t_0)\, dt = (1 - B + A).$$

Making use of the second of (2.9) one has $B - A \neq 1$, so that (2.10) follows. □

## 3 Reduction to a single integral equation

Under suitable assumptions on the boundaries of the G-M process, the determination of the first-exit time $g(t \mid x_0, t_0)$ can be reduced to the solution of a single non-singular Volterra integral equation in place of the system (2.1).

**Theorem 3.1** *Under the assumptions of Theorem 2.1, if $S_1(t)$ and $S_2(t)$ are such that*

$$S_1(t) + S_2(t) = 2\,m(t) + 2\,c\,h_2(t), \qquad (c \in \mathbb{R}), \tag{3.1}$$

*for all $t \in T$, then*

$$\begin{aligned} g(t \mid x_0, t_0) = {} & 2\Big[\Psi_1(t \mid x_0, t_0) - \Psi_2(t \mid x_0, t_0)\Big] \\ & -2\int_{t_0}^{t} g(\tau \mid x_0, t_0)\,\Big\{\Psi_1[t \mid S_1(\tau), \tau] - \Psi_2[t \mid S_1(\tau), \tau]\Big\}\, d\tau. \end{aligned} \tag{3.2}$$

**Proof.** Due to the G-M nature of $X(t)$, if (3.1) holds for all $t \in T$, one obtains:

$$\begin{aligned} & f\big[S_1(t), t \mid S_1(\tau), \tau\big] = f\big[S_2(t), t \mid S_2(\tau), \tau\big] \\ & f\big[S_1(t), t \mid S_2(\tau), \tau\big] = f\big[S_2(t), t \mid S_1(\tau), \tau\big]. \end{aligned} \qquad (t, \tau \in T;\ \tau < t) \tag{3.3}$$

Furthermore, making use of (3.1) and (3.3), from (2.2) it follows that:

$$\begin{aligned} & \Psi_1\big[t \mid S_1(\tau), \tau\big] = -\Psi_2\big[t \mid S_2(\tau), \tau\big], \\ & \Psi_1\big[t \mid S_2(\tau), \tau\big] = -\Psi_2\big[t \mid S_1(\tau), \tau\big], \end{aligned} \qquad (t, \tau \in T;\ \tau < t) \tag{3.4}$$

so that the following identity holds:

$$\Psi_1[t \mid S_1(\tau), \tau] - \Psi_2[t \mid S_1(\tau), \tau] = \Psi_1[t \mid S_2(\tau), \tau] - \Psi_2[t \mid S_2(\tau), \tau]. \tag{3.5}$$

Eq. (3.2) then follows from (2.13) by virtue of (2.8) and (3.5). □

If (3.1) holds, under suitable assumption on the initial state the determination of $g_1(t \mid x_0, t_0)$ and $g_2(t \mid x_0, t_0)$ can be obtained via the solution of a single non-singular Volterra integral equation.

**Theorem 3.2** *Under the assumptions of Theorem 2.1, if $S_1(t)$ and $S_2(t)$ are such that (3.1) holds for all $t \in T$, and if the pair $(x_0, t_0)$ is such that*

$$x_0 = m(t_0) + c\,h_2(t_0), \qquad (c \in \mathbb{R}), \tag{3.6}$$

*then*

$$g_1(t \mid x_0, t_0) = g_2(t \mid x_0, t_0). \tag{3.7}$$

**Proof.** ¿From (2.1) one has:

$$\begin{aligned} g_1(t \mid x_0, t_0) - g_2(t \mid x_0, t_0) = {} & 2\Big[\Psi_1(t \mid x_0, t_0) + \Psi_2(t \mid x_0, t_0)\Big] \\ & -2\int_{t_0}^{t} \Big[g_1(\tau \mid x_0, t_0)\,\Big\{\Psi_1[t \mid S_1(\tau), \tau] + \Psi_2[t \mid S_1(\tau), \tau]\Big\} \\ & + g_2(\tau \mid x_0, t_0)\,\Big\{\Psi_1[t \mid S_2(\tau), \tau] + \Psi_2[t \mid S_2(\tau), \tau]\Big\}\Big]\, d\tau. \end{aligned} \tag{3.8}$$

Making use of (3.1) and (3.6), for $S_1(t_0) < x_0 < S_2(t_0)$ one obtains $f\big[S_1(t), t \mid x_0, t_0\big] = f\big[S_2(t), t \mid x_0, t_0\big]$, so that

$$\Psi_1\big(t \mid x_0, t_0\big) = -\Psi_2\big(t \mid x_0, t_0\big). \tag{3.9}$$

Recalling (3.1) and making use of (3.9), Eq. (3.8) can be written as follows:

$$\begin{aligned} g_1(t \mid x_0, t_0) - g_2(t \mid x_0, t_0) = -2 \int_{t_0}^{t} \Big[ g_1(\tau \mid x_0, t_0) - g_2(\tau \mid x_0, t_0) \Big] \\ \times \Big[ \Psi_1[t \mid S_1(\tau), \tau] + \Psi_2[t \mid S_1(\tau), \tau] \Big] \, d\tau, \end{aligned}$$

that admits only of the trivial solution (cf., for instance, Tricomi [15]). Hence (3.7) holds. □

An immediate consequence of Theorem 3.2 is that

$$g(t \mid x_0, t_0) \equiv 2\, g_1(t \mid x_0, t_0) \equiv 2\, g_2(t \mid x_0, t_0)$$

satisfies the single integral equation (3.2).

We stress that in general the determination of the FPT pdf's for the case of two boundaries requires the solution of the system of integral equations (2.1). However, under the assumptions of the Theorems 3.1 and 3.2 the problem is greatly simplified since the system of integral equations reduces to a single non-singular Volterra integral equation. This is also a noteworthy simplification for computation purposes since in this case the numerical method of [7] can be implemented.

## 4 Closed-form results

In this section we shall make use of symmetry properties of the G-M process $X(t)$ to obtain a family of suitable boundaries and initial state for which the transition densities in the presence of two boundaries and the first-exit time pdf are obtained in closed-form.

We start remarking that the transition pdf of a G-M process characterized by conditional mean and variance (1.3) possesses the following symmetry properties:

$$f(x, t \mid x_0, t_0) = \frac{\phi(x, t)}{\phi(x_0, t_0)}\, f[\psi(x, t), t \mid \psi(x_0, t_0), t_0] \tag{4.1}$$

and

$$\phi(x, t)\, f[\psi(x, t), t \mid x_0, t_0] = f(x, t \mid x_0, t_0)\, \exp\Big\{ - \frac{2\,[x - z(t)]\,[x_0 - z(t_0)]}{h_1(t)\, h_2(t_0) - h_1(t_0)\, h_2(t)} \Big\}, \tag{4.2}$$

where

$$\begin{aligned} \psi(x, t) &= 2\, z(t) - x \\ \phi(x, t) &= \exp\Big\{ - \frac{2\, d_1\, [x - z(t)]}{h_2(t)} \Big\} \\ z(t) &= m(t) + d_1\, h_1(t) + d_2\, h_2(t), \end{aligned} \tag{4.3}$$

with $d_1, d_2 \in \mathbb{R}$. Using the terminology of [6], $z(t)$ will be called a "symmetry curve", and $\psi(x, t)$ and $\phi(x, t)$ the corresponding "symmetry functions".

The following theorem shows the existence of a closed form relation of the transition density in the presence of two suitable boundaries in terms of the free transition pdf.

**Theorem 4.1** *Under the assumptions of Theorem 2.1, let $S_1(t)$ and $S_2(t)$ be such that*

$$S_1(t) = m(t) + b\,h_1(t) + c_1\,h_2(t), \qquad S_2(t) = m(t) + b\,h_1(t) + c_2\,h_2(t), \tag{4.4}$$

*with $S_1(t) < S_2(t)$ for all $t \in T$, and let the pair $(x_0, t_0)$ satisfy*

$$x_0 = m(t_0) + b\,h_1(t_0) + c\,h_2(t_0), \tag{4.5}$$

*with $b, c, c_1, c_2 \in \mathbb{R}$ and $S_1(t_0) < x_0 < S_2(t_0)$. The transition pdf $\beta(x, t \mid x_0, t_0)$, in the presence of the absorbing boundaries (4.4) with the initial state $x_0$ such that (4.5) holds, is then:*

$$\beta(x,t \mid x_0,t_0) = f(x,t \mid x_0,t_0) \sum_{n=-\infty}^{+\infty} \Bigg[\exp\Big\{-\frac{2\,[x-u_n(t)]\,[x_0-u_n(t_0)]}{h_1(t)\,h_2(t_0)-h_1(t_0)\,h_2(t)}\Big\} - \exp\Big\{-\frac{2\,[x-v_n(t)]\,[x_0-v_n(t_0)]}{h_1(t)\,h_2(t_0)-h_1(t_0)\,h_2(t)}\Big\}\Bigg], \tag{4.6}$$

*where for all $t \in T$ and $n = 0, \pm 1, \pm 2, \ldots$ we have set:*

$$\begin{aligned} u_n(t) &= m(t) + b\,h_1(t) + [c + n\,(c_2 - c_1)]\,h_2(t),\\ v_n(t) &= m(t) + b\,h_1(t) + [c_2 - n\,(c_2 - c_1)]\,h_2(t). \end{aligned} \tag{4.7}$$

**Proof.** Note that (4.7) are symmetry curves. Denote by $\psi_{1,n}(x,t)$ and $\phi_{1,n}(x,t)$ the symmetry functions associated to $u_n(t)$ ($n = 0, \pm 1, \pm 2, \ldots$) and by $\psi_{2,n}(x,t)$ and $\phi_{2,n}(x,t)$ the symmetry functions associated to $v_n(t)$ ($n = 0, \pm 1, \pm 2, \ldots$). From (4.3) one obtains:

$$\begin{aligned} \psi_{1,n}(x,t) &= 2\,u_n(t) - x, \qquad \phi_{1,n}(x,t) = \exp\Big\{-\frac{2\,b\,[x-u_n(t)]}{h_2(t)}\Big\},\\ \psi_{2,n}(x,t) &= 2\,v_n(t) - x, \qquad \phi_{2,n}(x,t) = \exp\Big\{-\frac{2\,b\,[x-v_n(t)]}{h_2(t)}\Big\}, \end{aligned} \tag{4.8}$$

so that, by virtue of (4.2), Eq. (4.6) can be also written as:

$$\beta(x,t \mid x_0,t_0) = \sum_{n=-\infty}^{+\infty} \Big\{\phi_{1,n}(x,t)\,f[\psi_{1,n}(x,t),t \mid x_0,t_0] - \phi_{2,n}(x,t)\,f[\psi_{2,n}(x,t),t \mid x_0,t_0]\Big\}. \tag{4.9}$$

We now remark that the series in (4.6) and (4.9) are absolutely convergent and term by term differentiable. Because of (4.1), from (4.9) $\beta(x, t \mid x_0, t_0)$ is seen to satisfy the Fokker-Planck equation

$$\frac{\partial \beta(x,t|x_0,t_0)}{\partial t} = -\frac{\partial}{\partial x}\big[A_1(x,t)\,\beta(x,t|x_0,t_0)\big] + \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\big[A_2(t)\,\beta(x,t|x_0,t_0)\big], \tag{4.10}$$

with $A_1(x,t)$ and $A_2(t)$ given by

$$A_1(x,t) = m'(t) + [x - m(t)]\,\frac{h_2'(t)}{h_2(t)}, \qquad A_2(t) = h_2^2(t)\,r'(t). \tag{4.11}$$

Furthermore, by virtue of (4.1) and (4.5), it follows that

$$\lim_{t\downarrow t_0} \phi_{1,n}(x,t)\, f[\psi_{1,n}(x,t), t \mid x_0, t_0] = \begin{cases} \delta(x-x_0), & n=0 \\ 0, & n=\pm1,\pm2,\ldots \end{cases} \tag{4.12}$$
$$\lim_{t\downarrow t_0} \phi_{2,n}(x,t)\, f[\psi_{2,n}(x,t), t \mid x_0, t_0] = 0, \qquad n=\pm1,\pm2,\ldots,$$

so that the right-hand side of (4.9) is immediately seen to satisfy the initial delta-condition:

$$\lim_{t\downarrow t_0} \beta(x,t|x_0,t_0) = \delta(x-x_0). \tag{4.13}$$

Finally, use of (4.6) shows that the absorbing conditions on the boundaries are satisfied, i.e.

$$\beta[S_1(t), t \mid x_0, t_0] = \beta[S_2(t), t \mid x_0, t_0] = 0. \tag{4.14}$$

□

In the following theorem we obtain in closed form the first-exit time pdf $g(t \mid x_0, t_0)$ through the boundaries (4.4) with the initial state $x_0$ such that (4.5) holds.

**Theorem 4.2** *Under the assumptions of Theorem 4.1, there holds:*

$$\begin{aligned} g(t \mid x_0,t_0) &= \frac{h_2(t)}{r(t)-r(t_0)}\,\frac{dr(t)}{dt} \sum_{n=-\infty}^{+\infty} \exp\left\{-\frac{2\,n^2\,(c_2-c_1)^2}{r(t)-r(t_0)}\right\} \\ &\times\left\{\Big[c-c_1+2\,n\,(c_2-c_1)\Big]\,\exp\left\{-\frac{2\,n\,(c_2-c_1)\,(c-c_1)}{r(t)-r(t_0)}\right\} f[S_1(t),t \mid x_0,t_0] \right. \\ &\left. +\Big[c_2-c-2\,n\,(c_2-c_1)\Big]\,\exp\left\{\frac{2\,n\,(c_2-c_1)\,(c_2-c)}{r(t)-r(t_0)}\right\} f[S_2(t),t \mid x_0,t_0]\right\}, \end{aligned} \tag{4.15}$$

*where $r(t)$ is defined in (1.2).*

**Proof.** Recalling (1.8) and making use of (4.10) and (4.14), one has:

$$\begin{aligned} g(t \mid x_0,t_0) &= -\frac{\partial}{\partial t}\int_{S_1(t)}^{S_2(t)} \beta(x,t \mid x_0,t_0)\,dx \\ &= \frac{A_2(t)}{2}\left\{\frac{\partial}{\partial x}\beta(x,t \mid x_0,t_0)\Big|_{x=S_1(t)} - \frac{\partial}{\partial x}\beta(x,t \mid x_0,t_0)\Big|_{x=S_2(t)}\right\}. \end{aligned} \tag{4.16}$$

Making use of (4.6) and recalling (4.4), (4.5) and (4.7), from (4.16) one easily obtains (4.15).

□

Setting $b=0$ and $c_1+c_2=2\,c$ in (4.4) and (4.5), we note that assumptions (3.1) and (3.6) of Theorem 3.2 are satisfied, so that (4.15) is solution of the single integral equation (3.2) and $g_1(t \mid x_0,t_0) = g_2(t \mid x_0,t_0) = g(t \mid x_0,t_0)/2$.

Theorems 4.1 and 4.2 require an infinite superposition of symmetry curves. In the remaining part of this Section, by making use of two symmetry curves, we obtain simple closed form solutions for the transition densities in the presence of suitable boundaries and for the corresponding first-exit time pdf.

**Theorem 4.3** *For all $t \geq t_0$ and $t, t_0 \in T$, we set*

$$
\begin{aligned}
u(t) &= m(t) + b_1\, h_1(t) + c_1\, h_2(t),\\
v(t) &= m(t) + (2\,b - b_1)\, h_1(t) + (2\,c - c_1)\, h_2(t),
\end{aligned}
\tag{4.17}
$$

*with $b, c, b_1, c_1 \in \mathbb{R}$ and $u(t) < v(t)$, and denote*

$$
\Delta(t;t_0) = 1 - 4\,\alpha_1\,\alpha_2\, \exp\Big\{ -\frac{[v(t)-u(t)]\,[v(t_0)-u(t_0)]}{h_1(t)\,h_2(t_0) - h_1(t_0)\,h_2(t)} \Big\},
\tag{4.18}
$$

*with $\alpha_1 > 0, \alpha_2 > 0$ and $\lim_{t \to \sup T} \Delta(t;t_0) > 0$. Under the assumptions of Theorem 2.1, the transition pdf $\beta(x,t \mid x_0, t_0)$ in the presence of the pair of absorbing boundaries*

$$
\begin{aligned}
S_1(t;t_0) &= u(t) - \frac{h_1(t)\,h_2(t_0) - h_1(t_0)\,h_2(t)}{v(t_0)-u(t_0)} \ln\Big[\frac{1+\sqrt{\Delta(t;t_0)}}{2\,\alpha_1}\Big],\\
S_2(t;t_0) &= v(t) + \frac{h_1(t)\,h_2(t_0) - h_1(t_0)\,h_2(t)}{v(t_0)-u(t_0)} \ln\Big[\frac{1+\sqrt{\Delta(t;t_0)}}{2\,\alpha_2}\Big],
\end{aligned}
\tag{4.19}
$$

*with the initial state $x_0$ such that (4.5) holds, is*

$$
\begin{aligned}
\beta(x,t \mid x_0,t_0) = f(x,t \mid x_0,t_0)\, \Big[ 1 &- \alpha_1\, \exp\Big\{ -\frac{2\,[x-u(t)]\,[x_0-u(t_0)]}{h_1(t)\,h_2(t_0) - h_1(t_0)\,h_2(t)} \Big\}\\
&-\alpha_2\, \exp\Big\{ -\frac{2\,[x-v(t)]\,[x_0-v(t_0)]}{h_1(t)\,h_2(t_0) - h_1(t_0)\,h_2(t)} \Big\} \Big].
\end{aligned}
\tag{4.20}
$$

**Proof.** For the symmetry curves (4.17) we denote by $\psi_1(x,t)$ and $\phi_1(x,t)$ the symmetry functions corresponding to $u(t)$ and by $\psi_2(x,t)$ and $\phi_2(x,t)$ the symmetry functions corresponding to $v(t)$. From (4.3) one obtains:

$$
\begin{aligned}
\psi_1(x,t) &= 2\,u(t) - x, \qquad \phi_1(x,t) = \exp\Big\{ -\frac{2\,b_1\,[x-u(t)]}{h_2(t)} \Big\},\\
\psi_2(x,t) &= 2\,v(t) - x, \qquad \phi_2(x,t) = \exp\Big\{ -\frac{2\,(2\,b-b_1)\,[x-v(t)]}{h_2(t)} \Big\},
\end{aligned}
\tag{4.21}
$$

so that, by virtue of (4.2), Eq. (4.20) can be written as:

$$
\begin{aligned}
\beta(x,t \mid x_0,t_0) = f(x,t \mid x_0,t_0) &- \alpha_1\,\phi_1(x,t)\, f[\psi_1(x,t), t \mid x_0, t_0]\\
&-\alpha_2\,\phi_2(x,t)\, f[\psi_2(x,t), t \mid x_0, t_0].
\end{aligned}
\tag{4.22}
$$

Recalling (4.1), we note that the right-hand-side of (4.22) satisfies the Fokker-Planck equation (4.10). Furthermore, by virtue of (4.1) and (4.5), from (4.22) one sees that the initial delta-condition (4.13) is satisfied. Finally, from (4.20) one obtains:

$$
\begin{aligned}
\beta(x,t \mid x_0,t_0) = -\alpha_1\, f(x,t \mid x_0,t_0)\, &\exp\Big\{ \frac{2\,[x-u(t)]\,[x_0-u(t_0)]}{h_1(t)\,h_2(t_0) - h_1(t_0)\,h_2(t)} \Big\}\\
&\times \Big[ \exp\Big\{ -\frac{2\,[x-u(t)]\,[x_0-u(t_0)]}{h_1(t)\,h_2(t_0) - h_1(t_0)\,h_2(t)} \Big\} - \frac{1-\sqrt{\Delta(t;t_0)}}{2\,\alpha_1} \Big]\\
&\times \Big[ \exp\Big\{ -\frac{2\,[x-u(t)]\,[x_0-u(t_0)]}{h_1(t)\,h_2(t_0) - h_1(t_0)\,h_2(t)} \Big\} - \frac{1+\sqrt{\Delta(t;t_0)}}{2\,\alpha_1} \Big],
\end{aligned}
\tag{4.23}
$$

with $\Delta(t;t_0)$ defined in (4.18). Hence, the right-hand-side of (4.23) is identically zero at $x = S_i(t;t_0)$ $(i=1,2)$ and non negative for all $x \in (S_1(t;t_0), S_2(t;t_0))$ and $x_0 \in \big(u(t_0), v(t_0)\big)$. This completes the proof. □

The following theorem shows the existence of a simple closed-form relation of the first-exit time pdf $g(t \mid x_0, t_0)$ through the boundaries (4.19) and the initial state $x_0$ such that (4.5) holds, in terms of the free transition pdf.

**Theorem 4.4** *Under the assumptions of Theorem 4.3, one has:*

$$
\begin{aligned}
g(t \mid x_0, t_0) &= \frac{v(t_0)-u(t_0)}{2\,[r(t)-r(t_0)]}\,\frac{h_2(t)}{h_2(t_0)}\,\frac{dr(t)}{dt}\,\sqrt{\Delta(t;t_0)} \\
&\quad \times \Big\{ f[S_1(t;t_0), t \mid x_0, t_0] + f[S_2(t;t_0), t \mid x_0, t_0] \Big\},
\end{aligned}
\tag{4.24}
$$

*where $u(t)$, $v(t)$ and $\Delta(t;t_0)$ are defined in (4.17) and (4.18), respectively.*

**Proof.** The proof goes along the lines indicated in Theorem 4.2 □

The closed form expression (4.24) provides a tool to test the accuracy of numerical solutions of the system of integral equations (2.1).

## 5 Double-sided FPT densities

In this Section we shall focus on the up-down double sided (DS) FPT problem. By this terminology, we indicate the first-exit time problem from the strip $\big(S_1(t), S_2(t)\big)$ for the subset of sample paths of the G-M process $X(t)$ that originates at time $t_0 \in T$ at a state $X_0$, that is a random variable bounded from below by $\lim_{t\downarrow t_0} S_1(t)$ and from above by $\lim_{t\downarrow t_0} S_2(t)$.

If $-\infty < S_1(t_0) = \lim_{t\downarrow t_0} S_1(t) < \lim_{t\downarrow t_0} S_2(t) = S_2(t_0) < +\infty$, let

$$
\gamma_{\varepsilon_1,\varepsilon_2}(x_0,t_0) := \begin{cases} \dfrac{f(x_0,t_0)}{\displaystyle\int_{S_1(t_0)+\varepsilon_1}^{S_2(t_0)-\varepsilon_2} f(z,t_0)\,dz}, & S_1(t_0)+\varepsilon_1 < x_0 < S_2(t_0)-\varepsilon_2 \\[2ex] 0, & \text{elsewhere,} \end{cases}
\tag{5.1}
$$

be the pdf of $X_0$, where $\varepsilon_1$ and $\varepsilon_2$ are positive real numbers such that $S_2(t_0)-S_1(t_0) > \varepsilon_1+\varepsilon_2$ and where $f(x_0,t_0)$ denotes the pdf of $X(t_0)$. Hence,

$$
\int_{S_1(t_0)+\varepsilon_1}^{S_2(t_0)-\varepsilon_2} f(z,t_0)\,dz = \frac{1}{2}\bigg\{ \mathrm{Erf}\bigg[\frac{S_2(t_0)-\varepsilon_2-m(t_0)}{\sqrt{2\,h_1(t_0)\,h_2(t_0)}}\bigg] - \mathrm{Erf}\bigg[\frac{S_1(t_0)+\varepsilon_1-m(t_0)}{\sqrt{2\,h_1(t_0)\,h_2(t_0)}}\bigg]\bigg\}.
\tag{5.2}
$$

For all $t \ge t_0,\ t, t_0 \in T$, we are thus led to define the following random variables:

$$
\begin{aligned}
\mathcal{T}^{(1)}_{X_0} &= \inf_{t\ge t_0}\big\{t : X(t) < S_1(t);\ X(\vartheta) < S_2(\vartheta), \forall \vartheta \in (t_0,t)\big\} \qquad \text{(down FPT)},\\
\mathcal{T}^{(2)}_{X_0} &= \inf_{t\ge t_0}\big\{t : X(t) > S_2(t);\ X(\vartheta) > S_1(\vartheta), \forall \vartheta \in (t_0,t)\big\} \qquad \text{(up FPT)},\\
\mathcal{T}_{X_0} &= \inf_{t\ge t_0}\big\{t : X(t) \notin \big(S_1(t), S_2(t)\big)\big\}, \qquad \text{(up-down first-exit time)},
\end{aligned}
\tag{5.3}
$$

with

$$
\begin{aligned}
P\big(\mathcal{T}_{X_0}^{(1)} < t\big) &= \int_{S_1(t_0)+\varepsilon_1}^{S_2(t_0)-\varepsilon_2} P\big(\mathcal{T}_{x_0}^{(1)} < t\big)\,\gamma_{\varepsilon_1,\varepsilon_2}(x_0,t_0)\,dx_0,\\
P\big(\mathcal{T}_{X_0}^{(2)} < t\big) &= \int_{S_1(t_0)+\varepsilon_1}^{S_2(t_0)-\varepsilon_2} P\Big(\mathcal{T}_{x_0}^{(2)} < t\Big)\,\gamma_{\varepsilon_1,\varepsilon_2}(x_0,t_0)\,dx_0,\\
P\big(\mathcal{T}_{X_0} < t\big) &= \int_{S_1(t_0)+\varepsilon_1}^{S_2(t_0)-\varepsilon_2} P\big(\mathcal{T}_{x_0} < t\big)\,\gamma_{\varepsilon_1,\varepsilon_2}(x_0,t_0)\,dx_0.
\end{aligned}
\tag{5.4}
$$

Hence, from (5.4) one has $P\big(\mathcal{T}_{X_0} < t\big) = P\big(\mathcal{T}_{X_0}^{(1)} < t\big) + P\big(\mathcal{T}_{X_0}^{(2)} < t\big)$.

The inferior FPT density $g_{\varepsilon_1,\varepsilon_2}^{(1)}(t \mid t_0) := \partial P\big(\mathcal{T}_{X_0}^{(1)} < t\big)/\partial t$ is related to the conditioned FPT density $g_1(t \mid x_0,t_0)$ as follows:

$$
g_{\varepsilon_1,\varepsilon_2}^{(1)}(t \mid t_0) = \int_{S_1(t_0)+\varepsilon_1}^{S_2(t_0)-\varepsilon_2} g_1(t \mid x_0,t_0)\,\gamma_{\varepsilon_1,\varepsilon_2}(x_0,t_0)\,dx_0,
\tag{5.5}
$$

whereas the superior FPT density $g_{\varepsilon_1,\varepsilon_2}^{(2)}(t \mid t_0) := \partial P\big(\mathcal{T}_{X_0}^{(2)} < t\big)/\partial t$ and to the conditioned FPT density $g_2(t \mid x_0,t_0)$ are mutually related as follows:

$$
g_{\varepsilon_1,\varepsilon_2}^{(2)}(t \mid t_0) = \int_{S_1(t_0)+\varepsilon_1}^{S_2(t_0)-\varepsilon_2} g_2(t \mid x_0,t_0)\,\gamma_{\varepsilon_1,\varepsilon_2}(x_0,t_0)\,dx_0.
\tag{5.6}
$$

Furthermore, the DS first-exit time density $g_{\varepsilon_1,\varepsilon_2}(t \mid t_0) := \partial P\Big(\mathcal{T}_{X_0} < t\Big)/\partial t$ is related to the conditioned first-exit time density $g(t \mid x_0,t_0)$ as follows:

$$
g_{\varepsilon_1,\varepsilon_2}(t \mid t_0) = \int_{S_1(t_0)+\varepsilon_1}^{S_2(t_0)-\varepsilon_2} g(t \mid x_0,t_0)\,\gamma_{\varepsilon_1,\varepsilon_2}(x_0,t_0)\,dx_0 \equiv g_{\varepsilon_1,\varepsilon_2}^{(1)}(t \mid t_0) + g_{\varepsilon_1,\varepsilon_2}^{(2)}(t \mid t_0).
\tag{5.7}
$$

**Theorem 5.1** *Let $T = [a,b]$ and $t_0 \in T$. Then, there holds $P\big(\mathcal{T}_{X_0} < b\big) = 1$ for all positive real numbers $\varepsilon_1, \varepsilon_2$ such that $S_2(t_0) - S_1(t_0) > \varepsilon_1 + \varepsilon_2$ if and only if $P\big(\mathcal{T}_{x_0} < b\big) = 1$ for all $x_0 \in \big(S_1(t_0), S_2(t_0)\big)$.*

**Proof.** By taking the limit as $t \uparrow b$ in the last of (5.4), we obtain:

$$
P\big(\mathcal{T}_{X_0} < b\big) = \int_{S_1(t_0)+\varepsilon_1}^{S_2(t_0)-\varepsilon_2} P\big(\mathcal{T}_{x_0} < b\big)\,\gamma_{\varepsilon_1,\varepsilon_2}(x_0,t_0)\,dx_0.
\tag{5.8}
$$

Hence, the necessary part immediately follows from (5.8) and from the hypothesis $P\big(\mathcal{T}_{x_0} < b\big) = 1$ for all $x_0 \in \big(S_1(t_0), S_2(t_0)\big)$. To prove the sufficient part, we note that, if $P\big(\mathcal{T}_{X_0} < b\big) = 1$ for fixed positive $\varepsilon_1, \varepsilon_2$ such that $S_2(t_0) - S_1(t_0) > \varepsilon_1 + \varepsilon_2$, then (5.8) is equivalent to

$$
\int_{S_1(t_0)+\varepsilon_1}^{S_2(t_0)-\varepsilon_2} \Big[1 - P\big(\mathcal{T}_{x_0} < b\big)\Big]\,\gamma_{\varepsilon_1,\varepsilon_2}(x_0,t_0)\,dx_0 = 0.
\tag{5.9}
$$

Since $\gamma_{\varepsilon_1,\varepsilon_2}(x_0,t_0) > 0$ and $1 - P\big(\mathcal{T}_{x_0} < b\big) \geq 0$ for all $x_0 \in \big(S_1(t_0)+\varepsilon_1, S_2(t_0)-\varepsilon_2\big)$, it follows that $P\big(\mathcal{T}_{x_0} < b\big) = 1$ for all $x_0 \in \big(S_1(t_0)+\varepsilon_1, S_2(t_0)-\varepsilon_2\big)$. Hence, due to the arbitrariness of $\varepsilon_1$ and $\varepsilon_2$, one has $P\big(\mathcal{T}_{x_0} < b\big) = 1$ for all $x_0 \in \big(S_1(t_0), S_2(t_0)\big)$. □

The following theorem provides the basis for a numerical method to evaluate the DS first-exit time density $g_{\varepsilon_1,\varepsilon_2}(t \mid t_0)$.

**Theorem 5.2** *Let $S_1(t)$, $S_2(t)$, $m(t)$, $h_1(t)$, $h_2(t)$ be $C^1(T)$ functions, with $S_1(t) < S_2(t)$, $\forall t \in T$. Then, $g^{(1)}_{\varepsilon_1,\varepsilon_2}(t \mid t_0)$ and $g^{(2)}_{\varepsilon_1,\varepsilon_2}(t \mid t_0)$ satisfy the following non-singular integral equations:*

$$\begin{aligned} g^{(1)}_{\varepsilon_1,\varepsilon_2}(t \mid t_0) &= 2\,\Psi^{(1)}_{\varepsilon_1,\varepsilon_2}(t \mid t_0) \\ &\quad -2\int_{t_0}^{t} \Big\{ g^{(1)}_{\varepsilon_1,\varepsilon_2}(\tau \mid t_0)\,\Psi_1[t \mid S_1(\tau),\tau] \;+ g^{(2)}_{\varepsilon_1,\varepsilon_2}(\tau \mid t_0)\,\Psi_1[t \mid S_2(\tau),\tau] \Big\}\, d\tau, \\ g^{(2)}_{\varepsilon_1,\varepsilon_2}(t \mid t_0) &= -2\,\Psi^{(2)}_{\varepsilon_1,\varepsilon_2}(t \mid t_0) \\ &\quad +2\int_{t_0}^{t} \Big\{ g^{(1)}_{\varepsilon_1,\varepsilon_2}(\tau \mid t_0)\,\Psi_2[t \mid S_1(\tau),\tau] \;+ g^{(2)}_{\varepsilon_1,\varepsilon_2}(\tau \mid t_0)\,\Psi_2[t \mid S_2(\tau),\tau] \Big\}\, d\tau, \end{aligned} \tag{5.10}$$

*where $\Psi_1$ and $\Psi_2$ are defined in (2.2) and*

$$\begin{aligned} \Psi^{(j)}_{\varepsilon_1,\varepsilon_2}(t \mid t_0) &= \Big\{ \mathrm{Erf}\Big[\frac{S_2(t_0)-\varepsilon_2-m(t_0)}{\sqrt{2\,h_1(t_0)\,h_2(t_0)}}\Big] - \mathrm{Erf}\Big[\frac{S_1(t_0)+\varepsilon_1-m(t_0)}{\sqrt{2\,h_1(t_0)\,h_2(t_0)}}\Big]\Big\}^{-1} \\ &\times \Big[ \frac{h_1(t_0)}{h_1(t)}\,\big[h_2'(t)\,h_1(t) - h_2(t)\,h_1'(t)\big]\,\Big\{ f[S_j(t),t \mid S_2(t_0)-\varepsilon_2,t_0]\, f[S_2(t_0)-\varepsilon_2,t_0] \\ &\quad - f[S_j(t),t \mid S_1(t_0)+\varepsilon_1,t_0]\, f[S_1(t_0)+\varepsilon_1,t_0]\Big\} + \frac{1}{2}\, f[S_j(t),t] \\ &\times \Big\{ S_j'(t) - m'(t) - \frac{h_1'(t)}{h_1(t)}\,[S_j(t)-m(t)]\Big\}\,\Big\{\mathrm{Erf}[\varphi_{j,2}(t,t_0)] - \mathrm{Erf}[\varphi_{j,1}(t,t_0)]\Big\}\Big], \end{aligned} \tag{5.11}$$

$(j = 1,2)$

*with*

$$\begin{aligned} \varphi_{j,k}(t \mid t_0) &= \sqrt{\frac{h_1(t)}{2\,h_1(t_0)\big[h_1(t)\,h_2(t_0) - h_1(t_0)\,h_2(t)\big]}} \\ &\times \Big\{ S_k(t_0) - (-1)^k \varepsilon_k - m(t_0) - [S_j(t)-m(t)]\,\frac{h_1(t_0)}{h_1(t)} \Big\} \qquad (j,k = 1,2). \end{aligned} \tag{5.12}$$

**Proof.** We multiply both sides of equations (2.1) by $\gamma_{\varepsilon_1,\varepsilon_2}(x_0,t_0)$ and then integrate with respect to $x_0$ between $S_1(t_0)+\varepsilon_1$ and $S_2(t_0)-\varepsilon_2$. Equations (5.11) then follow after making use of (5.5) and (5.6) and after setting

$$\Psi^{(j)}_{\varepsilon_1,\varepsilon_2}(t \mid t_0) = \int_{S_1(t_0)+\varepsilon_1}^{S_2(t_0)-\varepsilon_2} \Psi_j[t \mid x_0,t_0]\,\gamma_{\varepsilon_1,\varepsilon_2}(x_0,t_0)\,dx_0 \qquad (j=1,2). \tag{5.13}$$

Finally, (5.11) follows from (5.13) by making use of (2.2) and (5.1) and after proving that

$$\begin{aligned} &\int_{S_1(t_0)+\varepsilon_1}^{S_2(t_0)-\varepsilon_2} \Psi_j[t \mid x_0,t_0]\, f(x_0,t_0)\,dx_0 \\ &\quad = \frac{1}{2}\Big\{ S_j'(t) - m'(t) - [S_j(t)-m(t)]\,\frac{h_1'(t)\,h_2(t_0) - h_2'(t)\,h_1(t_0)}{h_1(t)\,h_2(t_0) - h_2(t)\,h_1(t_0)}\Big\} A_j(t \mid t_0) \\ &\qquad -\frac{1}{2}\,\frac{h_2'(t)\,h_1(t) - h_2(t)\,h_1'(t)}{h_1(t)\,h_2(t_0) - h_2(t)\,h_1(t_0)}\, B_j(t \mid t_0) \end{aligned}$$

with

$$A_j(t \mid t_0) = \int_{S_1(t_0)+\varepsilon_1}^{S_2(t_0)-\varepsilon_2} f[S_j(t), t \mid x_0, t_0]\, f(x_0, t_0)\, dx_0$$
$$= \frac{1}{2}\, f[S_j(t), t] \Big\{ \mathrm{Erf}[\varphi_{j,2}(t, t_0)] - \mathrm{Erf}[\varphi_{j,1}(t, t_0)] \Big\} \qquad (j = 1, 2)$$

$$B_j(t \mid t_0) = \int_{S_1(t_0)+\varepsilon_1}^{S_2(t_0)-\varepsilon_2} [x_0 - m(t_0)]\, f[S_j(t), t \mid x_0, t_0]\, f(x_0, t_0)\, dx_0$$
$$= \frac{1}{2}\, f[S_j(t), t]\, \frac{h_1(t_0)}{h_1(t)}\, [S_j(t) - m(t)] \Big\{ \mathrm{Erf}[\varphi_{j,2}(t, t_0)] - \mathrm{Erf}[\varphi_{j,1}(t, t_0)] \Big\}$$
$$- \frac{h_1(t_0)\, [h_1(t)\, h_2(t_0) - h_2(t)\, h_1(t_0)]}{h_1(t)} \Big\{ f[S_j(t), t \mid S_2(t_0) - \varepsilon_2, t_0]$$
$$\times f[S_2(t_0) - \varepsilon_2, t_0] - f[S_j(t), t \mid S_1(t_0) + \varepsilon_1, t_0]\, f[S_1(t_0) + \varepsilon_1, t_0] \Big\}$$
$$(j = 1, 2).$$

This completes the proof. □

**Theorem 5.3** *Under the assumption of Theorem 5.1, if $S_1(t)$ and $S_2(t)$ are such that (3.1) holds for all $t \in T$, then*

$$g_{\varepsilon_1,\varepsilon_2}(t \mid t_0) = 2 \Big[ \Psi^{(1)}_{\varepsilon_1,\varepsilon_2}(t \mid t_0) - \Psi^{(2)}_{\varepsilon_1,\varepsilon_2}(t \mid t_0) \Big]$$
$$-2 \int_{t_0}^{t} g_{\varepsilon_1,\varepsilon_2}(\tau \mid t_0) \Big\{ \Psi_1[t \mid S_1(\tau), \tau] - \Psi_2[t \mid S_2(\tau), \tau] \Big\}\, d\tau. \qquad (5.14)$$

**Proof.** Since relation (3.1) of Theorem 3.1 holds, one is immediately led to Eq. (5.14) after multiplying both sides of (3.2) by $\gamma_{\varepsilon_1,\varepsilon_2}(x_0, t_0)$ and then integrating with respect to $x_0$ between $S_1(t_0) + \varepsilon_1$ and $S_2(t_0) - \varepsilon_2$. □

## 6 A Computational Method

In this Section we shall describe a straightforward numerical procedure to evaluate $g_1$, $g_2$ and $g$ and to estimate the related computational errors, by solving the system of integral equations (2.1) via an algorithm based on the repeated Simpson rule (cf. [5] and [10]).

For the sake of conciseness, in the sequel the following short-hand notation will be employed:

$$g_i(t) := g_i(t \mid x_0, t_0) \quad (i = 1, 2), \qquad g(t) := g_1(t) + g_2(t) \qquad (t_0 < t)$$
$$\Psi_i(t) := \Psi_i(t \mid x_0, t_0) \quad (i = 1, 2) \qquad (t_0 < t)$$
$$\Psi_{ij}(t \mid \tau) := \Psi_i[t \mid S_j(\tau), \tau] \quad (i, j = 1, 2) \qquad (t_0 < \tau \le t),$$

so that system (2.1) can be written as:

$$g_1(t) = 2\, \Psi_1(t) - 2 \int_{t_0}^{t} \Big[ g_1(\tau)\, \Psi_{11}(t \mid \tau) + g_2(\tau)\, \Psi_{12}(t \mid \tau) \Big]\, d\tau$$
$$(6.1)$$
$$g_2(t) = -2\Psi_2(t) + 2 \int_{t_0}^{t} \Big[ g_1(\tau)\, \Psi_{21}(t \mid \tau) + g_2(\tau)\, \Psi_{22}(t \mid \tau) \Big]\, d\tau.$$

Denoting by $p > 0$ the discretization step, setting $t = t_0 + k\,p$ $(k = 1, 2, \dots)$ and making use of the repeated Simpson rule, for the system (6.1) we obtain the following approximate solutions $\tilde{g}_1$, $\tilde{g}_2$ to $g_1$, $g_2$, respectively:

$$
\begin{aligned}
&\tilde{g}_1(t_0+p) = 2\,\Psi_1(t_0+p),\\
&\tilde{g}_1(t_0+kp) = 2\Psi_1(t_0+kp) - 2\,p\sum_{j=1}^{k-1} w_{k,j}\Big[\tilde{g}_1(t_0+jp)\Psi_{11}(t_0+kp \mid t_0+jp)\\
&\qquad\qquad + \tilde{g}_2(t_0+jp)\Psi_{12}(t_0+kp \mid t_0+jp)\Big] \qquad (k=2,3,\dots)\\
&\tilde{g}_2(t_0+p) = -2\,\Psi_2(t_0+p)\\
&\tilde{g}_2(t_0+kp) = -2\Psi_2(t_0+kp) + 2\,p\sum_{j=1}^{k-1} w_{k,j}\Big[\tilde{g}_1(t_0+jp)\Psi_{21}(t_0+kp \mid t_0+jp)\\
&\qquad\qquad + \tilde{g}_2(t_0+jp)\Psi_{22}(t_0+kp \mid t_0+jp)\Big] \qquad (k=2,3,\dots)
\end{aligned}
\tag{6.2}
$$

where the weights $w_{k,j}$ are specified as follows:

$$
\begin{aligned}
&w_{2n,2j-1} = \frac{4}{3} && (j=1,2,\dots,n;\ n=1,2,\dots),\\
&w_{2n,2j} = \frac{2}{3} && (j=1,2,\dots,n-1;\ n=2,3,\dots),\\
&w_{2n+1,2j-1} = \frac{4}{3} && (j=1,2,\dots,n-1;\ n=2,3,\dots),\\
&w_{2n+1,2j} = \frac{2}{3} && (j=1,2,\dots,n-2;\ n=3,4,\dots),\\
&w_{2n+1,2(n-1)} = \frac{17}{24} && (n=2,3,\dots),\\
&w_{2n+1,2n-1} = w_{2n+1,2n} = \frac{9}{8} && (n=1,2,\dots).
\end{aligned}
\tag{6.3}
$$

The sum $\tilde{g}_1 + \tilde{g}_2$ then provides an evaluation of $g$. We emphasize that the above outlined algorithm is an extension to the case of two boundaries of the algorithm proposed in [7] for the single boundary case.

The convergence of the above computational method is expressed by the following

**Theorem 6.1** *Let $p$ be the discretization step, $t_m = t_0 + N\,p$, with $N = 0, 1, \dots,$ and set*

$$
\begin{aligned}
&\Delta_{kp}^{(1)} := g_1(t_0+k\,p) - \tilde{g}_1(t_0+k\,p) && (k=1,2,\dots,N),\\
&\Delta_{kp}^{(2)} := g_2(t_0+k\,p) - \tilde{g}_2(t_0+k\,p) && (k=1,2,\dots,N),
\end{aligned}
\tag{6.4}
$$

*and*

$$
|\,\Delta_{kp}\,| := |\,\Delta_{kp}^{(1)}\,| + |\,\Delta_{kp}^{(2)}\,| \qquad (k=1,2,\dots,N)
\tag{6.5}
$$

*Then,*

$$
\lim_{p\to 0} |\,\Delta_{kp}\,| = 0 \qquad (k=1,2,\dots,N)
\tag{6.6}
$$

*for all fixed $kp$.*

**Proof.** For brevity, we limit ourselves to mentioning that the proof is a suitable variant of the proof of Theorem 4.1 of Di Nardo et al. [7], where a single Volterra integral equation of second kind was taken into account. The proof consists of showing that the absolute error $|\Delta_{kp}|$ is bounded from above as follows:

$$(6.7)\quad |\Delta_{kp}| \le 2Np\, e^{8\,M\,k\,p/3} \Big\{\omega\big[(\psi_{11}g_1+\psi_{12}g_2)_{kp}\,,2\,p/3\big]+\omega\big[(\psi_{21}g_1+\psi_{22}g_2)_{kp}\,,2\,p/3\big]\Big\}$$

where

$$M = M_1 + M_2$$

with

$$M_j = \max_{t_0\le\tau\le t\le t_m}\Big\{\big|\Psi_{j1}(t\mid\tau)\big|,\big|\Psi_{j2}(t\mid\tau)\big|\Big\} \qquad (j=1,2)$$

and

$$\omega\big[(\psi_{j1}\,g_1+\psi_{j2}\,g_2)_{kp}\,,2\,p/3\big] \equiv \sup_{\substack{\tau_1,\tau_2\in[t_0,t_0+kp]\\ |\tau_1-\tau_2|<2\,p/3}} \Big|g_1(\tau_1)\,\Psi_{j1}(t_0+kp\mid\tau_1)+g_2(\tau_1)\,\Psi_{j2}(t_0+kp\mid\tau_1)$$
$$-g_1(\tau_2)\Psi_{j1}(t_0+kp\mid\tau_2)-g_2(\tau_2)\Psi_{j2}(t_0+kp\mid\tau_2)\Big)\Big|, \qquad (j=1,2)$$

is the continuity modulus of $\psi_{j1}\,g_1+\psi_{j2}\,g_2$ $(j=1,2)$ in $[t_0,t_0+kp]$. Since the continuity modulus tends to 0 as $p\to 0$, (6.6) follows. Hence, the convergence of the computational procedure is insured. □

A noteworthy feature of the above algorithm is its being implementable after specifying the initial data $t_0, x_0$, the functions $m(t), h_1(t), h_2(t)$ that characterize the process, the boundaries $S_1(t)$, $S_2(t)$ and the discretization step $p$. Furthermore, it does not involve any heavy computation, neither requires use of any library subroutines, Monte Carlo methods or other special software packages.

Since (5.10) possesses the same kernel as the equations of systems (2.1), the numerical iterative procedure (6.2) is again applicable. Hence, if $\lim_{t\downarrow t_0} S_1(t)$ and $\lim_{t\downarrow t_0} S_2(t)$ are finite, the approximations $\tilde g^{(1)}_{\varepsilon_1,\varepsilon_2}$, $\tilde g^{(2)}_{\varepsilon_1,\varepsilon_2}$ of $g^{(1)}_{\varepsilon_1,\varepsilon_2}$, $g^{(2)}_{\varepsilon_1,\varepsilon_2}$ are obtained via (6.2) changing $\Psi_j$ and $\tilde g_j$ $(j=1,2)$ with $\Psi^{(j)}_{\varepsilon_1,\varepsilon_2}$ and $\tilde g^{(j)}_{\varepsilon_1,\varepsilon_2}$ $(j=1,2)$, respectively.

The following examples show the effectiveness of the proposed numerical procedure.

*(a)* Let $\{X(t), t\in\mathbb{R}\}$ be the G-M process with $m(t)=t/2$ and covariance $c(s,t)=s$ $(0\le s\le t)$, so that $h_1(t)=t$ and $h_2(t)=1$. We consider the FPT problem through the constant boundaries $S_1(t)=-1$ and $S_2(t)=1$ starting from zero at time 0. Figure 1 shows the computed FPT pdf's $\tilde g_1(t|0,0)$ (dash-dot line), $\tilde g_2(t|0,0)$ (dashed line) and $\tilde g(t|0,0)$ (solid line) obtained via (6.2) with the integration step $10^{-3}$. We note that in this case, by choosing $b=-1/2$, $c_1=-1$, $c_2=1$ and $c=0$, the assumptions of Theorem 4.1 are satisfied, so that the series expansion (4.15) of $g(t|0,0)$ holds. Table 1 shows the differences between the computed FPT pdf's $\tilde g(t/0,0)$ and $\widehat g(t/0,0)$ obtained via (6.2) and (4.15), respectively. In particular, the FPT $\tilde g(t|0,0)$ (column 2), the absolute error $\varrho_a(t)=|\tilde g(t|0,0)-\widehat g(t|0,0)|$ (column 3), the relative error $\varrho_r(t)=\varrho_a(t)/\widehat g(t|0,0)$ (column 4) and cumulative distribution $\tilde P(t)$ are listed for various values of $t$. In the series expansion (4.15) only the terms for $n=0,\pm1,\cdots,\pm10$ have been considered.

*(b)* Let $\{X(t), t\in\mathbb{R}\}$ be the G-M process with $m(t)=0$ and covariance $c(s,t)=s$ $(0\le s\le t)$, so that $h_1(t)=t$ and $h_2(t)=1$. As in example *(a)*, we again consider the FPT

| $t$ | $\tilde{g}(t)$ | $\varrho_a(t)$ | $\varrho_r(t)$ | $\tilde{P}(t)$ |
|---|---|---|---|---|
| 0.5 | 0.878567122E+00 | 2.2146829e-010 | 2.5207896e-010 | 0.340844516E+00 |
| 1.0 | 0.455136220E+00 | 3.3499625e-010 | 7.3603514e-010 | 0.664975202E+00 |
| 2.0 | 0.116985662E+00 | 3.1024069e-010 | 2.6519549e-009 | 0.913898849E+00 |
| 3.0 | 0.300646734E-01 | 1.4482082e-011 | 4.8169764e-010 | 0.977872477E+00 |
| 4.0 | 0.772645605E-02 | 1.7433615e-012 | 2.2563534e-010 | 0.994313348E+00 |
| 5.0 | 0.198565680E-02 | 9.3917452e-013 | 4.7297928e-010 | 0.998538562E+00 |
| 6.0 | 0.510302899E-03 | 1.8841363e-013 | 3.6921920e-010 | 0.999624418E+00 |
| 7.0 | 0.131145044E-03 | 3.2149752e-013 | 2.4514653e-009 | 0.999903478E+00 |
| 8.0 | 0.337035566E-04 | 8.1232493e-015 | 2.4102054e-010 | 0.999975194E+00 |
| 9.0 | 0.866162907E-05 | 1.8651411e-015 | 2.1533376e-010 | 0.999993625E+00 |
| 10.0 | 0.222599113E-05 | 1.6283018e-015 | 7.3149520e-010 | 0.999998362E+00 |

Table 1: For the same choices of Figure 1, the computed FPT pdf $\tilde{g}(t/0,0)$, the absolute error $\varrho_a(t)$, the relative error $\varrho_r(t)$ and the cumulative distribution $\tilde{P}(t)$ are listed for various values of $t$ with the integration step $10^{-3}$.

| $t$ | $\tilde{g}(t)$ | $\varrho_a(t)$ | $\varrho_r(t)$ | $\tilde{P}(t)$ |
|---|---|---|---|---|
| 0.5 | 0.836733751E-01 | 2.5539210e-011 | 3.0522504e-010 | 0.947198042E-02 |
| 1.0 | 0.218211245E+00 | 1.1038917e-010 | 5.0588213e-010 | 0.920285474E-01 |
| 2.0 | 0.208776449E+00 | 9.7014091e-011 | 4.6467929e-010 | 0.317521533E+00 |
| 3.0 | 0.155920336E+00 | 4.2558593e-010 | 2.7295088e-009 | 0.499277339E+00 |
| 4.0 | 0.114334008E+00 | 4.5387809e-010 | 3.9697558e-009 | 0.633373742E+00 |
| 5.0 | 0.837108606E-01 | 4.8515900e-011 | 5.7956518e-010 | 0.731604527E+00 |
| 6.0 | 0.612818313E-01 | 1.9858941e-011 | 3.2405919e-010 | 0.803519040E+00 |
| 7.0 | 0.448618187E-01 | 4.7526059e-011 | 1.0593877e-009 | 0.856164788E+00 |
| 8.0 | 0.328413962E-01 | 3.3956879e-011 | 1.0339658e-009 | 0.894704474E+00 |
| 9.0 | 0.240417631E-01 | 4.2870027e-011 | 1.7831482e-009 | 0.922917708E+00 |
| 10.0 | 0.175999328E-01 | 4.3515212e-011 | 2.4724647e-009 | 0.943571395E+00 |

Table 2: For the same choices of Figure 3, the computed FPT pdf $\tilde{g}(t/0,0)$, the absolute error $\varrho_a(t)$, the relative error $\varrho_r(t)$ and the cumulative distribution $\tilde{P}(t)$ are listed for various values of $t$ with the integration step $10^{-3}$.

problem through the constant boundaries $S_1(t) = -1$ and $S_2(t) = 1$ (cf. Figure 2). In this case relations (3.1) and (3.6) hold, so that $g_1(t|0,0) = g_2(t|0,0)$.

*(c)* Let $\{X(t), t \in \mathbb{R}\}$ be the G-M process with $m(t) = t/3$ and covariance $c(s,t) = s$ $(0 \le s \le t)$, so that $h_1(t) = t$ and $h_2(t) = 1$. We consider the FPT problem through the linear boundaries $S_1(t) = t/4 - 2$ and $S_2(t) = t/4 + 2$ starting from zero initial state at time 0. Figure 3 shows the computed FPT pdf's $\tilde{g}_1(t|0,0)$ (dash-dot line), $\tilde{g}_2(t|0,0)$ (dashed line) and $\tilde{g}(t|0,0)$ (solid line) obtained via (6.2) with the integration step $10^{-3}$.

Note that in this case, by choosing $b = -1/12$, $c_1 = -2$, $c_2 = 2$ and $c = 0$, the assumptions of Theorem 4.1 are satisfied, so that the series expansion (4.15) of $g(t|0,0)$ holds. Table 2 shows the differences between the computed FPT pdf's $\tilde{g}(t/0,0)$ and $\widehat{g}(t/0,0)$ obtained via (6.2) and (4.15), respectively. In the series expansion (4.15) we have considered only the terms for $n = 0, \pm 1, \cdots, \pm 10$.

*(d)* Let $\{X(t), t \in \mathbb{R}\}$ be the G-M process with $m(t) = 0$ and covariance $c(s,t) = s$ $(0 \le s \le t)$, so that $h_1(t) = t$ and $h_2(t) = 1$. As in the case *(c)*, we again consider the FPT problem through the constant boundaries $S_1(t) = t/4 - 2$ and $S_2(t) = t/4 + 2$ (cf. Figure 4).

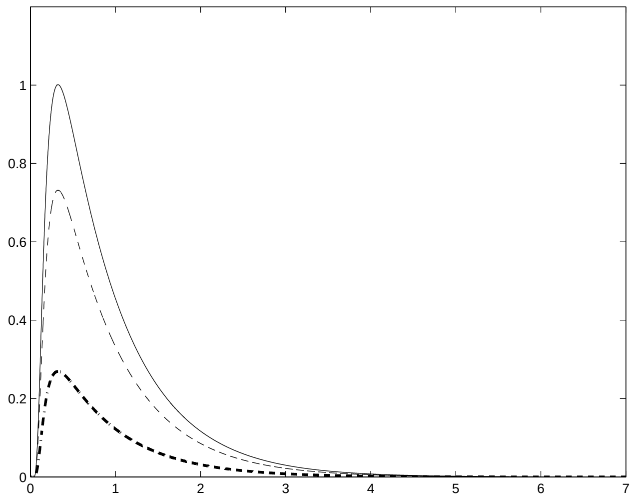


Figure 1: Plot of $\tilde{g}_1(t|0,0)$ (dash-dot line), $\tilde{g}_2(t|0,0)$ (dashed line) and $\tilde{g}(t|0,0)$ (solid line) through the boundaries $S_1(t) = -1$ and $S_2(t) = 1$ for the G-M process with $m(t) = t/2$ and $c(s,t) = s$.

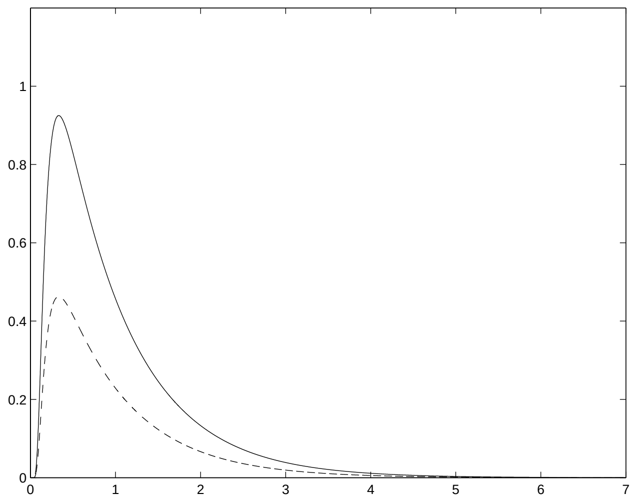


Figure 2: As in Figure 1 for the G-M process with $m(t) = 0$ and $c(s,t) = s$. The dashed line refers to $\tilde{g}_1(t|0,0)$ and $\tilde{g}_2(t|0,0)$, whereas the solid line indicates $\tilde{g}(t|0,0)$.

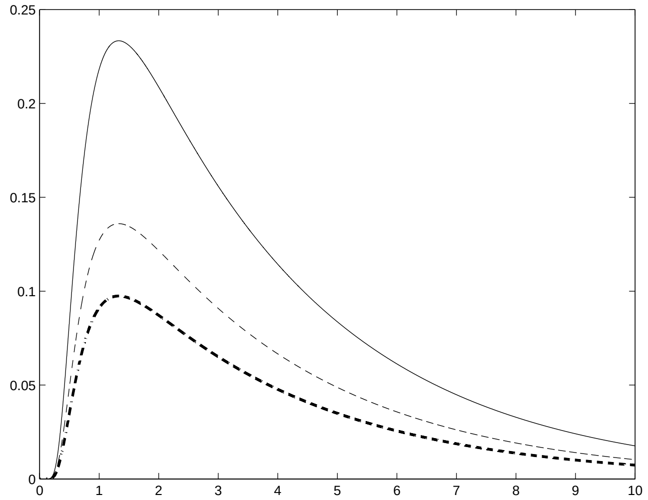


Figure 3: Plot of $\tilde{g}_1(t|0,0)$ (dash-dot line), $\tilde{g}_2(t|0,0)$ (dashed line) and $\tilde{g}(t|0,0)$ (solid line) through the boundaries $S_1(t) = t/4 - 2$ and $S_2(t) = t/4 + 2$ for the G-M process with $m(t) = t/3$ and $c(s,t) = s$.

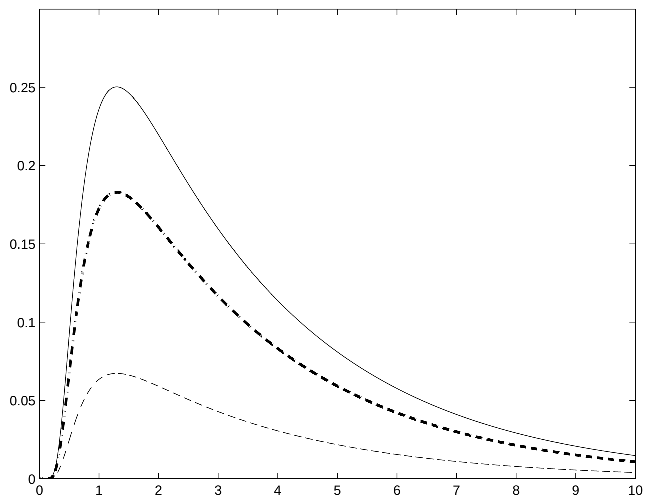


Figure 4: As in Figure 3 for the G-M process with $m(t) = 0$ and $c(s,t) = s$.

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