BCK/BCI-BIALGEBRAS

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Abstract. The notion of BCK/BCI-bialgebras and sub-bialgebras is introduced, and related properties are investigated. A characterization of \( X = pI(X_1) \uplus pI(X_2) \) is provided.

1 Introduction. A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers. Bialgebraic structures, for example, bisemigroups, bigroups, bigroupoids, biloops, birings, bisemirings, binary-rings, etc., are discussed in [6]. In this paper, we consider bialgebraic structures in BCK/BCI-algebras. We introduced the notion of BCK/BCI-bialgebras and sub-bialgebras, and investigate several properties. Using the notion of a commutative bigroup, we construct the concept of \( X = pI(X_1) \uplus pI(X_2) \), and vice versa.

2 Preliminaries. An algebra \((X; *, 0)\) of type \((2, 0)\) is called a \(\text{BCI-algebra}\) if it satisfies the following conditions:

\[
\begin{align*}
(I) \quad & (\forall x, y, z \in X) \left( (x * y) * (x * z) * (z * y) = 0 \right), \\
(II) \quad & (\forall x, y \in X) \left( (x * (x * y)) * y = 0 \right), \\
(III) \quad & (\forall x \in X) \left( x * x = 0 \right), \\
(IV) \quad & (\forall x, y \in X) \left( x * y = 0, y * x = 0 \Rightarrow x = y \right).
\end{align*}
\]

If a BCI-algebra \(X\) satisfies the following identity:

\[
(\forall x \in X) \ (0 * x = 0),
\]

then \(X\) is called a \(\text{BCK-algebra}\). In a BCK-algebra \(X\), the following identity holds.

\[
(\forall x, y, z \in X) \ ( (x * y) * z = (x * z) * y).
\]

A nonempty subset \(S\) of a BCK/BCI-algebra \(X\) is called a \(\text{subalgebra}\) of \(X\) if \(x * y \in S\) for all \(x, y \in S\). A BCK-algebra \(X\) is said to be \textit{positive implicative} if it satisfies the following identity:

\[
(\forall x, y, z \in X) \ ( (x * y) * z = (x * y) * (x * z)).
\]

A positive implicative BCK-algebra will be written by \(\pi\text{BCK}\)-algebra for short. A BCK-algebra \(X\) is said to be \textit{commutative} if \(x * (x * y) = y * (y * x)\) for all \(x, y \in X\). A commutative BCK-algebra will be written by \(\text{cBCK}\)-algebra for short. A BCI-algebra \(X\) is said to be \textit{\(p\)-semisimple} if its \(p\)-radical is trivial. In a \(p\)-semisimple BCI-algebra \(X\), we have the following axioms:

\[
\begin{align*}
2000 \ & \text{Mathematics Subject Classification. 06F35, 03G25.} \\
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\end{align*}
\]
\[(a2) \quad (\forall x, y \in X) \quad (x * (0 * y) = y * (0 * x)),\]
\[(a3) \quad (\forall x \in X) \quad (0 * (0 * x) = x).\]

We refer the reader to the book [5] for further information regarding BCK/BCI-algebras.

3 BCK/BCI-bialgebras

Definition 3.1. Let \(X = (X, *, \oplus, 0)\) be an algebra of type \((2, 2, 0)\). Then \(X = (X, *, \oplus, 0)\) is called a BCK-bialgebra (resp. BCI-bialgebra) if there exists two distinct proper subsets \(X_1\) and \(X_2\) of \(X\) such that

(i) \(X = X_1 \cup X_2\).

(ii) \((X_1, *, 0)\) is a BCK-algebra (resp. BCI-algebra).

(iii) \((X_2, \oplus, 0)\) is a BCK-algebra (resp. BCI-algebra).

Denote by \(X = K(X_1) \uplus K(X_2)\) (resp. \(X = I(X_1) \uplus I(X_2)\)) the BCK-bialgebra (resp. BCI-bialgebra). If \((X_1, *, 0)\) is a BCK-algebra (resp. BCI-algebra) and \((X_2, \oplus, 0)\) is a BCI-algebra (resp. BCK-algebra), then we say that \(X = (X, *, \oplus, 0)\) is a BCKI-bialgebra (resp. BCIK-bialgebra), and denoted by \(X = K(X_1) \uplus I(X_2)\) (resp. \(X = I(X_1) \uplus K(X_2)\)).

Example 3.2. (1) Let \(X = \{0, a, b, c, d\}\) and consider two proper subsets \(X_1 = \{0, a, b\}\) and \(X_2 = \{0, a, c, d\}\) of \(X\) together with Cayley tables respectively as follows:

\[
\begin{array}{c|ccc}
* & 0 & a & b \\
\hline
0 & 0 & 0 & b \\
a & a & 0 & 0 \\
b & b & a & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\oplus & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & a & 0 \\
c & c & c & 0 & 0 \\
d & d & c & a & 0 \\
\end{array}
\]

Then \((X_1, *, 0)\) and \((X_2, \oplus, 0)\) are BCK-algebras. Hence \((X, *, \oplus, 0)\) is a BCK-bialgebra, i.e., \(X = K(X_1) \uplus K(X_2)\).

(2) Let \(X = \mathbb{R}^+ \cup \{0, a, b, c\}\) where \(\mathbb{R}^+\) is the set of all positive real numbers. Define two binary operations ‘*’ and ‘\(\oplus\)’ as follows:

\[(\forall x, y \in \mathbb{R}^+ \cup \{0\}) \quad (x * y = \max\{x - y, 0\})\]

and

\[
\begin{array}{c|ccc}
\oplus & 0 & a & b \\
\hline
0 & 0 & 0 & 0 \\
a & a & 0 & a \\
b & b & b & 0 \\
c & c & b & a \\
\end{array}
\]

Then \((X_1 := \mathbb{R}^+ \cup \{0\}, *, 0)\) and \((X_2 := \{0, a, b, c\}, \oplus, 0)\) are BCK-algebras. Hence \((X, *, \oplus, 0)\) is a BCK-bialgebra, i.e., \(X = K(X_1) \uplus K(X_2)\).

(3) Let \(X = \{0, a, b, c, d\}\) and consider two proper subsets \(X_1 = \{0, a, b\}\) and \(X_2 = \{0, a, c, d\}\) of \(X\) together with Cayley tables respectively as follows:

\[
\begin{array}{c|ccc}
* & 0 & a & b \\
\hline
0 & 0 & 0 & 0 \\
a & a & 0 & 0 \\
b & b & a & 0 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\oplus & 0 & a & b & c \\
\hline
0 & 0 & 0 & c & c \\
a & a & 0 & c & c \\
c & c & c & 0 & 0 \\
d & d & c & a & 0 \\
\end{array}
\]
Since every BCK-algebra is a BCI-algebra, it is straightforward.

Proof. Since every BCK-algebra is a BCI-algebra, it is straightforward.

Proposition 3.3. We have

\[ X = K(X_1) \uplus I(X_2) \]

\[ X = K(X_1) \uplus K(X_2) \]

\[ X = I(X_1) \uplus I(X_2) \]

\[ X = I(X_1) \uplus K(X_2) \]

Note that any BCI-algebra need not be a BCK-algebra. Hence the converse of Proposition 3.3 is not true in general.

Definition 3.4. Let \( X = K(X_1) \uplus K(X_2) \) (resp. \( X = K(X_1) \uplus I(X_2) \), \( X = I(X_1) \uplus K(X_2) \), \( X = I(X_1) \uplus I(X_2) \)). A subset \( H \neq \emptyset \) of \( X \) is called a sub-bialgebra of \( X \) if there exist subsets \( H_1 \) and \( H_2 \) of \( X_1 \) and \( X_2 \), respectively, such that

(i) \( H_1 \neq H_2 \) and \( H = H_1 \cup H_2 \),

(ii) \( (H_1, \#, 0) \) is a subalgebra of \( (X_1, \#, 0) \),

(iii) \( (H_2, \circ, 0) \) is a subalgebra of \( (X_2, \circ, 0) \).

Example 3.5. Let \( X \) be a BCK-bialgebra in Example 3.2(1) and let \( H_1 = \{0, a\} \) and \( H_2 = \{0, c\} \). Then \( H_1 \neq H_2 \) and \( H_1 \) (resp. \( H_2 \)) is a subalgebra of \( X_1 \) (resp. \( X_2 \)). Hence \( H = \{0, a, c\} \) is a sub-bialgebra of \( X \). We can easily check that \( (H = \{0, a, c\}, \circ, 0) \) is a BCK-algebra. Note also that \( H_3 = \{0, d\} \) is a subalgebra of \( X_2 \) and \( H_1 \neq H_3 \). Thus \( G = \{0, a, d\} \) is a sub-bialgebra of \( X \). We can easily check that \( (G = \{0, a, d\}, \circ, 0) \) is not a BCK-algebra.

Remark 3.6. Let \( L \) be a sub-bialgebra of a BCK-bialgebra \( (X, \#, \circ, 0) \). Then \( L \) may not be a BCK-algebra under \( \# \) or \( \circ \) as seen in Example 3.5.

We provide a characterization of a sub-bialgebra.
Theorem 3.7. Let $X = K(X_1) \uplus K(X_2)$ (resp. $X = K(X_1) \uplus I(X_2)$, $X = I(X_1) \uplus K(X_2)$, $X = I(X_1) \uplus I(X_2)$) and let $H$ be a nonempty subset of $X$. Then $H$ is a sub-bialgebra of $X$ if and only if there exist two proper subsets $X_1$ and $X_2$ of $X$ such that

(i) $X = X_1 \cup X_2$, where $(X_1, *, 0)$ and $(X_2, \oplus, 0)$ are BCK-algebras (resp. $(X_1, *, 0)$ is a BCK-algebra and $(X_2, \oplus, 0)$ is a BCI-algebra, $(X_1, *, 0)$ is a BCI-algebra and $(X_2, \oplus, 0)$ is a BCK-algebra, $(X_1, *, 0)$ and $(X_2, \oplus, 0)$ are BCI-algebras),

(ii) $(H \cap X_1, *, 0)$ is a subalgebra of $(X_1, *, 0)$,

(iii) $(H \cap X_1, \oplus, 0)$ is a subalgebra of $(X_1, \oplus, 0)$.

Proof. We prove it for the case $X = K(X_1) \uplus K(X_2)$. For other cases, we can have desired results by the similar method. Assume that $H$ is a sub-bialgebra of $X$. Then $(H, *, \oplus, 0)$ is a BCK-bialgebra. Hence there exist two distinct proper subsets $H_1$ and $H_2$ of $H$ such that

- $H = H_1 \cup H_2$,
- $(H_1, *, 0)$ and $(H_2, \oplus, 0)$ are BCK-algebras.

Taking $H_1 = H \cap X_1$ and $H_2 = H \cap X_2$ imply that $(H_1 = H \cap X_1, *, 0)$ and $(H_2 = H \cap X_2, \oplus, 0)$ are subalgebras of $(X_1, *, 0)$ and $(X_2, \oplus, 0)$, respectively. Conversely, Let $H$ be a nonempty subset of a BCK-bialgebra $(X, *, \oplus, 0)$ satisfying conditions (i), (ii) and (iii).

It is sufficient to show that $(H \cap X_1) \cup (H \cap X_2) = H$. Now,

$$
(H \cap X_1) \cup (H \cap X_2) = (H \cap (X_1 \cup H)) \cup ((H \cap X_1) \cup X_2)
$$

$$
= (H \cup (X_1 \cup H)) \cap (H \cup X_2) \cap (X_1 \cup X_2)
$$

$$
= H \cap X_2
$$

This completes the proof.

\[\square\]

Denote by $X = piK(X_1) \uplus cK(X_2)$ the $X = K(X_1) \uplus K(X_2)$ in which $(X_1, *, 0)$ is a positive implicative BCK-algebra and $(X_2, \oplus, 0)$ is a commutative BCK-algebra. Denote by $X = iK(X_1) \uplus cK(X_2)$ the $X = K(X_1) \uplus K(X_2)$ in which $(X_1, *, 0)$ is an implicative BCK-algebra and $(X_2, \oplus, 0)$ is a commutative BCK-algebra. Note that

$$
X = iK(X_1) \uplus cK(X_2) \Rightarrow X = piK(X_1) \uplus cK(X_2) \Rightarrow X = K(X_1) \uplus K(X_2),
$$

but the converse is not true in general. In fact, in Example 3.2(1), we can see that the implication

$$
X = K(X_1) \uplus K(X_2) \Rightarrow X = piK(X_1) \uplus cK(X_2)
$$

does not hold.

Example 3.8. Let $X = \{0, x, y, a, b, c\}$ and consider two subsets $X_1 = \{0, a, b, c\}$ and $X_2 = \{0, x, y\}$ of $X$ with Cayley tables as follows:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

|   | 0 | x | y |
|---|---|---|
| 0 | 0 | 0 | 0 |
| x | x | 0 | 0 |
| y | y | x | 0 |

It is easy to check that $X = piK(X_1) \uplus cK(X_2)$, but $X \not= iK(X_1) \uplus cK(X_2)$. 
Lemma 3.9. [3] A BCK-algebra $X$ is positive implicative if and only if it satisfies the following identity:

$$(\forall x, y \in X) \ (x \ast y = (x \ast y) \ast y).$$

Lemma 3.10. [3] A BCK-algebra $X$ is commutative if and only if it is a semilattice with respect to $\land$.

Using Lemmas 3.9 and 3.10, we provide a condition for $X = K(X_1) \cup K(X_2)$ to be

$X = \pi K(X_1) \cup c K(X_2)$.

Theorem 3.11. Let $X = K(X_1) \cup K(X_2)$. Then $X = \pi K(X_1) \cup c K(X_2)$ if and only if the following conditions are true.

(i) $\ (\forall x, y \in X) \ (x \ast y = (x \ast y) \ast y),$

(ii) $X_2$ is a semilattice with respect to $\land \oplus$ which is given by

$$(\forall a, b \in X_2) \ (a \land \oplus b = b \oplus (b \oplus a)).$$

Lemma 3.12. [3] A BCK-algebra $X$ is commutative if and only if it satisfies the following identity:

$$(\forall x, y \in X) \ (A(x) \cap A(y) = A(\land y)),$$

where $A(x)$ is the initial section of $x$.

Applying Lemmas 3.9 and 3.12, we have a characterization of $X = \pi K(X_1) \cup c K(X_2)$.

Theorem 3.13. Let $X = K(X_1) \cup K(X_2)$. Then $X = \pi K(X_1) \cup c K(X_2)$ if and only if the following conditions are true.

(i) $\ (\forall x, y \in X) \ (x \ast y = (x \ast y) \ast y),$

(ii) $X_2$ is a semilattice with respect to $\land \oplus$ which is given by

$$(\forall a, b \in X_2) \ (A(a) \cap A(b) = A(a \land \oplus b)).$$

Definition 3.14. [4] A set $(G, +, \cdot)$ with two binary operations $+$ and $\cdot$ is called a bigroup if there exists two proper subsets $G_1$ and $G_2$ of $G$ such that $G = G_1 \cup G_2$, $(G_1, +)$ is a group, and $(G_2, \cdot)$ is a group. If both $(G_1, +)$ and $(G_2, \cdot)$ are commutative, then we say that $(G, +, \cdot)$ is a commutative bigroup.

Denote by $X = pI(X_1) \cup pI(X_2)$ the $X = I(X_1) \cup I(X_2)$ in which $(X_1, \ast, 0)$ and $(X_2, \oplus, 0)$ are $p$-semisimple BCI-algebras.

Lemma 3.15. [1] A BCI-algebra $X$ satisfies the identity

$$(\forall x, y \in X) \ (x \ast (x \ast y) = y)$$

if and only if it has a sum $+$ and $(X, +)$ is a commutative group.

Lemma 3.16. [2] In a BCI-algebra $X$, the following are equivalent.

(i) $\ (\forall x, y \in X) \ (x \ast (x \ast y) = y).$

(ii) $X$ is $p$-semisimple.
Theorem 3.17. If $X = pI(X_1) \cup pI(X_2)$, then $X$ has operations $+$ and $\bullet$ so that $(X, +, \bullet)$ is commutative bigroup.

Proof. If $X = pI(X_1) \cup pI(X_2)$, then $X = X_1 \cup X_2$, and $(X_1, *, 0)$ and $(X_2, \oplus, 0)$ are $p$-semisimple BCI-algebras. By means of Lemmas 3.15 and 3.16, $X$ has two operations $+$ and $\bullet$ so that $(X_1, *)$ and $(X_2, \oplus)$ are commutative groups, in which $+$ and $\bullet$ are given by $x + y = x * (0 * y)$ and $x \bullet y = x \oplus (0 \oplus y)$ for all $x, y \in X$. Hence $(X, +, \bullet)$ is a commutative bigroup.

Theorem 3.18. Let $(G, +, \bullet)$ be a commutative bigroup. If we define operations $\ast$ and $\oplus$ on $G$ as follows:

$$(\forall x, y \in G) (x \ast y = x - y) \text{ and } (\forall a, b \in G) (a \oplus b = a \bullet b^{-1}),$$

then $G = pI(G_1) \cup pI(G_2)$ for some $G_1, G_2 \subseteq G$.

Proof. If $(G, +, \bullet)$ is a commutative bigroup, then $G = G_1 \cup G_2$ for some $G_1, G_2 \subseteq G$, and $(G_1, +)$ and $(G_2, \bullet)$ are (commutative) groups. It is easy to prove that $(G_1, *, 0)$ and $(G_2, \oplus, 0)$ are $p$-semisimple BCI-algebras. Hence $G = pI(G_1) \cup pI(G_2)$.

References


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