A NOTE ON SPIRAL-LIKE POLYNOMIALS

ALAN GLUCHOFF AND FREDERICK HARTMANN

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Abstract. Let D be the unit disk $\{z : |z| < 1\}$. $f : D \to \mathbb{C}$, $f(z) = z + a_2z^2 + \dots$ be analytic in D. For $-\pi/2 < \alpha < \pi/2$ and $0 \le \lambda < 1$, we define $SP(\alpha, \lambda)$ to be the class of f as above such that $\Re[e^{-i\alpha}zf'(z)/f(z)] > \lambda$ for all $z \in D$. Furthermore, let P_n be the class of polynomials $p(z) = z + a_2z^2 + \cdots + a_nz^n = z(1 - z/z_1)\cdots(1 - z/z_{n-1})$, for $a_k \in \mathbb{C}, k = 2, \dots, n$ and $z_k \in C$, $k = 1, \dots, n-1$. For $a, b > 0$ we define $S(a, b)$ to be the class of f analytic in D for which $|[zf'(z)/f(z)] - a| < b$. Given an $n \ge 2$ and $R > 1$, we find a and b in terms of R and n such that if $p \in P_n$ has $|z_k| > R$ for $k = 1, \dots, n$ then $p \in S(a, b)$. The result is sharp. The result implies membership in $SP(\alpha, \lambda) \cap SP(-\alpha, \lambda)$ where α and λ also depend on R and n. The results improve theorems of T. Başgöze. A physical criterion is given on the set $\{z_k\}$ which implies membership of p in $SP(\alpha, 0)$, and the criterion is used to given an example of a $p \in [P_3 \cap SP(\alpha)] \setminus SP(-\alpha)$, where $\alpha = 0.4$ Connections are made with other classes of univalent functions.

1. Introduction & Preliminaries

Let D be the unit disk $\{z : |z| < 1\}$ in the complex plane $\mathbb{C}$, and let $f : D \to \mathbb{C}$, $f(z) = z + a_2z^2 + \dots$ be analytic in D. For $-\pi/2 < \alpha < \pi/2$ and $0 \le \lambda < 1$, we define $SP(\alpha, \lambda)$ to be the class of f as above such that $\Re[e^{-i\alpha}zf'(z)/f(z)] > \lambda$ for all $z \in D$. When $\lambda = 0$ we write $SP(\alpha, 0) = SP(\alpha)$; this is the so-called α spiral-like class [6, p. 52]. When $\alpha = 0$ we have the class $SP(0, \lambda) = St(\lambda)$, the class of functions starlike of order λ; these are in turn subclasses of the class $St = \{f : \Re[zf'(z)/f(z)] > 0\}$ of starlike functions. For all λ and α it may be shown that $SP(\alpha, \lambda)$ consist of univalent mappings of D. Clearly, for any fixed α as above, if $0 \le \lambda_1 < \lambda_2 < 1$, then $SP(\alpha, \lambda_2) \bigcap SP(-\alpha, \lambda_2) \subset SP(\alpha, \lambda_1) \bigcap SP(-\alpha, \lambda_1)$, and for any fixed λ as above, if $-\pi/2 < \alpha_1 < \alpha_2 < \pi/2$, then $SP(\alpha_2, \lambda) \bigcap SP(-\alpha_2, \lambda) \subset SP(\alpha_1, \lambda) \bigcap SP(\alpha_1, \lambda)$. In addition, for $a, b > 0$ we define $S(a, b)$ to be the class of f analytic in D for which $|[zf'(z)/f(z)] - a| < b$. Finally, for n an integer, $n > 1$, we let P_n be the class of polynomials $p(z) = z + a_2z^2 + \cdots + a_nz^n = z(1 - z/z_1)\cdots(1 - z/z_{n-1})$, for $a_k \in \mathbb{C}, k = 2, \dots, n$ and $z_k \in \mathbb{C}$, $k = 1, \dots, n - 1$.

In a series of papers [2], [3], [4], [5] T. Başgöze considered problems involving conditions on the zeroes of a polynomial $p \in P_n$ which insure that it is a univalent mapping. In particular, the last dealt with the problem of finding a value $R = R(\alpha, \lambda) > 0$ having the following property: if $|z_k| > R$ for $k = 1 \dots n - 1$, then $p \in P_n$ has $p \in SP(\alpha, \lambda)$. The $SP(\alpha, \lambda)$ classes obtained were shown to be best possible. (Başgöze stated his results differently, but the correspondence between his treatment and ours can be made by a trivial transformation.) These problems originated with the work of Alexander [1].

This note has two purposes. The first is to point out that Başgöze's results in [5] can be refined. Suppose that $R = R(\alpha, \lambda)$ is as above, and let p have $|z_k| > R$. It is easy to see that $p \in SP(-\alpha, \lambda)$ as well: letting $p^*(z) = z(1 - z/\overline{z}_1)\cdots(1 - z/\overline{z}_{n-1})$, we have that $p^* \in$

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$SP(\alpha,\lambda)$. But it follows that $\Re[\overline{\dfrac{e^{-i\alpha}z(p^*)'(z)}{(p^*)(z)}}] > \lambda$ for all $z \in D$, hence $\Re[\dfrac{e^{i\alpha}\overline{z}p(\overline{z})}{p(\overline{z})}] > \lambda$ for all $z \in D$. But this is clearly equivalent to $p \in SP(-\alpha,\lambda)$. Thus for R as above we actually have $p \in SP(\alpha,\lambda)\bigcap SP(-\alpha,\lambda)$. (For $\lambda = 0$ the class $SP(\alpha,0)\bigcap SP(-\alpha,0)$ is known as the set of functions strongly starlike of order α, see [7, p. 138], where its properties are discussed.) We shall improve on this observation by proving that for any $R > 1$, if $p \in P_n$ with $|z_k| > R$, then $p \in S(a,b)$, where a and b will be given in terms of R and n. This membership will in turn imply membership in $SP(\alpha,\lambda)\bigcap SP(-\alpha,\lambda)$ for appropriate values of α and λ. These results will be shown best possible, and are equvalent to Başgöze's for the cases considered.

Our main tool for this part is a corollary of the Walsh Coincidence Theorem. Since the theorem may not be familiar, it will be stated as originally given by Walsh. Our main theorem (Theorem 2) will follow from the corollary. It would also be possible to obtain Theorem 2 from more general results of Kasten [9], which involve the convolution theory of Ruscheweyh, but we prefer to derive it from this less abstract source.

The second purpose of this note is to introduce an equivalent physical interpretation for membership in $SP(\alpha)$ which allows the construction of examples of $p \in P_n \bigcap SP(\alpha)$ which are not strongly starlike of order α. As far as the authors are aware this is the first discussion of this kind of polynomial. Some connections are made with classes of univalent functions $S(a,b)$, $T(a,b)$, and $R(a,b)$ introduced in [8].

2. Walsh Coincidence Theorem

In this section we state a version of the Walsh Coincidence Theorem and derive the corollary which we need for our work. We state the result in full generality as it appears in [12, p. 164], though we will not need this much generality.

Theorem 1. *(Walsh Coincidence Theorem) Let f be a polynomial in z whose coefficients are linear in and symmetric in each of the sets $\{a_1, a_2, \cdots, a_k\}, \{b_1, b_2, \cdots, b_l\}, \cdots, \{q_1, q_2, \cdots, q_s\} \subset \mathbb{C}$. Each coefficient must be a linear combination of the elementary symmetric functions of each of these sets with coefficients linear combinations of the elementary symmetric functions of the other sets. These linear combinations may contain constant terms. Let these points $\{a_i\}, \{b_i\}, \cdots, \{q_i\}$ lie in circular regions $C_a, C_b, \cdots, C_q$. Then for any fixed values of these variables and of z we can always make all the a_i coincide in C_a, all the b_i coincide in C_b, etc., without altering the value of $f(z)$.*

Note: The C_i may be the interior or exterior of a bounded circle, or any half plane. *The C_i always include the boundary of the set.* It is shown in [10, p. 63] that Theorem 1 is equivalent to Grace's theorem on the zeroes of apolar polynomials. These theorems in turn are related to those used in [9], see the reference to Szegö's Theorem on page 92.

The following corollary is stated but not proved in Marden, p. 69.

Corollary 1. *Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n = a_n(z-\zeta_1)(z-\zeta_2)\cdots(z-\zeta_n)$, and $\beta \neq n$. Then if all the ζ_k lie in a circular region C, every zero Z of the polynomial $f_1(z) = -zf'(z) + \beta f(z)$ may be written in the form $Z = w$ or $Z = [\beta/(\beta - n)]w$, where w is a point of C.*

Proof: We note that $f_1(z) = (\beta-n)a_n z^n + (\beta-(n-1))a_1 z^{n-1} + \cdots + (\beta-1)a_{n-1}z + \beta a_n$. Thus we can apply Theorem 1 to f_1, where there is a single set of variables $\{\zeta_1, \zeta_2, \cdots \zeta_n\}$ and a single circular region C. The result is that if $f_1(Z) = 0$ then there is a w in C such that $Q_\beta(w, Z) = (\beta - n)Z^n + (-1)(\beta - (n-1))\binom{n}{1}wZ^{n-1} + (\beta - (n-2))\binom{n}{2}w^2Z^{n-2} + \cdots + (\beta - 1)(-1)^{n-1}\binom{n}{n-1}w^{n-1}Z + \beta(-1)^n w^n = 0$.

But we have $Q_\beta(w, Z) = [\beta(Z^n - \binom{n}{1}wZ^{n-1} + \binom{n}{2}w^2Z^{n-2} + \cdots + (-1)^{n-1}\binom{n}{n-1}w^{n-1}Z + (-1)^n w^n))] + [(-n)Z^n + (n-1)\binom{n}{1}z^{n-1} + \cdots + (-1)(-1)^{n-1}\binom{n}{n-1}w^{n-1}Z] = \beta(Z - w)^n - nZ(Z-w)^{n-1}$. □.

3. Main Results

Theorem 2. *Let $R > 1$, $n > 1$, and $p \in P_n$, where $|z_k| \geq R$, $k = 1 \cdots n-1$. Then $p \in S(\frac{R^2 - n}{R^2 - 1}, \frac{R(n-1)}{R^2 - 1})$. This result cannot be improved.*

Proof: Let $D_{R,n} = \{z : |z - \frac{R^2 - n}{R^2 - 1}| < \frac{R(n-1)}{(R^2 - 1)}\}$, and $B(x)$ be the boundary of $D_{R,n}$ parametrized by $x \in \mathbb{C}$ with $|x| = 1$. Note that $1 \in D_{R,n}$ for all R and n. Then since $\frac{zp'(z)}{p(z)} = 1$ for $z = 0$, we have that $\frac{zp'(z)}{p(z)} \in D_{R,n}$ for $|z| < 1$ if and only if $\frac{zp'(z)}{p(z)} \neq B(x)$ for all $|x| = 1, |z| < 1$. But it is easily seen that this occurs if and only if $F_x(z) = -z\left[\frac{p(z)}{z}\right]' + (-1 + B(x))\left[\frac{p(z)}{z}\right]$ has no zeroes in D for each $|x| = 1$. But by Corollary 1 applied to the circular region $\{z : |z| \geq R\}$ the zeroes of $F_x(z)$ are either of the form w_k for $|w_k| \geq R > 1$ or $\left[\frac{B(x) - 1}{B(x) - n}\right]w_k$ again for $|w_k| \geq R$. Clearly we need only consider the second case. But the boundary of $D_{R,n}$ is produced by $B(x) = \frac{R - nx}{R - x}$, and one has $\frac{B(x) - 1}{B(x) - n} = \frac{x}{R}$. Since $|w_k| \geq R$, the second case gives no zero of F_x in D. Finally, an easy calculation shows that if $q(z) = z(1 - z/R)^{n-1}$, then $\frac{zq'(z)}{q(z)} = \frac{R - nz}{R - z}$, which shows q is best possible. □

Note: (1) The classes $S(\frac{R^2 - n}{R^2 - 1}, \frac{R(n-1)}{(R^2 - 1)})$ consist of starlike univalent functions if $R \geq n$, but since $q'(R/n) = 0$, q is clearly not univalent on D if $R < n$.

(2) It might seem that the above theorem could be generalized to allow $\{z_k\}, k = 1, \ldots, n-1$ to be in any disc D_z not intersecting D and $\frac{zp'(z)}{p(z)}$ to lie inside any disc D_N containing $z = 1$. But the nature of the proof does not allow this: $\frac{zp'(z)}{p(z)} \in D_N$ would be true if and only if the corresponding $F_x(z)$ had no zeroes in D, with the zeroes again taking the form $w_k \in D_z$ or $\left[\frac{B^*(x) - 1}{B^*(x) - n}\right]w_k$, where $B^*(x)$ traces out the boundary of D_N. The first case causes no problem, but since $\left[\frac{B^*(x) - 1}{B^*(x) - n}\right]$ loops around the origin as x traces the boundary of D, $\left[\frac{B^*(x) - 1}{B^*(x) - n}\right]w$ missing D depends only on the magnitude of w. Thus the hypothesis must be relaxed to the form $|z_k| \geq R$ for some R.

It is now easy to relate Theorem 2 to various classes to which the polynomials may belong.

Corollary 2. *Let $n > 1$ be an integer, $R \geq n$, $0 \leq \lambda \leq \frac{R-n}{R-1}$. Let $p \in P_n$ with $|z_k| > R$, $k = 1, \ldots, n-1$. Then $p \in SP(\alpha, \lambda) \bigcap SP(-\alpha, \lambda)$, where $\lambda = (\frac{R^2-n}{R^2-1})\cos(\alpha) - \frac{R(n-1)}{R^2-1}$. In particular, $p \in St(\frac{R-n}{R-1})$, and $p \in SP(\alpha) \bigcap SP(-\alpha)$ for $\alpha = \cos^{-1}(\frac{R(n-1)}{R^2-n})$. The result is best possible in the following sense: for any fixed value of α, no larger value of λ will suffice, and for any fixed value of λ, no larger value of α will suffice.*

Proof: This follows by an elementary geometric consideration of how $D_{R,n}$ is rotated rigidly around the origin while still remaining in $\{z : \Re(z) > \lambda\}$. The examples $p(z) = z(1 - z/Re^{i\theta})^{(n-1)}$ show that the inclusions are best possible. □

Note: It follows from a renormalization and some calculation that these results imply the "if" portion of Theorem 2 in [5]. Our method could be adapted to achieve "if" portion the case $\beta = n$ of Theorem 1 of that paper, but we have chosen to omit consideration of this extra parameter. Our method gives a stronger result in the "if" directions, however.

We close this section by relating the above to other function classes.

Corollary 3. *If n is an integer, $n > 1$, then $p_n(z) = z(1 - z/n)^{n-1} \in S(\frac{n}{n+1}, \frac{n}{n+1})$.*

Proof: Take $R = n$ in Theorem 2. □

The class $S(a, b)$, where $|1 - a| < b$, was introduced in [8]. In that paper the authors also defined $R(a, b)$ as the set of f analytic in D for which $|f'(z) - a| < b$ for all $z \in D$ and $T(a, b)$ for $a \geq b > 0$ as the set of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ analytic in D for which $\sum_{n=2}^{\infty}(n - a + b)|a_n| \leq b - |1 - a|$. Inclusions among the three classes are discussed. As a result of Corollary 2 we can add to their results the following.

Theorem 3. *Let $p_n(z) = z(1 - z/n)^{n-1}$ for n integer, $n > 1$. Then $p_n \in S(\frac{n}{n+1}, \frac{n}{n+1})$ but $p_n \notin R(\frac{n}{n+1}, \frac{n}{n+1})$ and $p_n \notin T(n/(n+1), n/(n+1))$. Furthermore, $p_n \in S(1, 1)$, but $p_n \notin T(1, 1)$ and $p_n \notin R(1, 1)$.*

Proof: The first statement is Corollary 3. The second statement follows from the fact that $|p_n'(z) - \frac{n}{n+1}| < \frac{n}{n+1}$ fails for $z \in D$ close to $z = -1$. The third statement follows immediately from the definition and the value of the first term of the defining sum. The fourth statement follows since $S(\frac{n}{n+1}, \frac{n}{n+1}) \subset S(1, 1)$. The fifth statement follows as the third, and the last as the fourth. □

Note: In [8] the authors show that $P_3 \bigcap R(1, 1) = P_3 \bigcap T(1, 1) \neq P_3 \bigcap S(1, 1)$. Our p_3 provides another example of a polynomial p of any degree greater than one with $p \in S(1, 1)\backslash R(1, 1)$ and $p \in S(1, 1)\backslash T(1, 1)$. The authors also state that, as a result of examining the maximum of the second coefficient for functions in $R(a, b)$, $S(a, b)$, and $T(a, b)$ that the inclusions $S(a, b) \subset R(a, b)$ and $S(a, b) \subset T(a, b)$ fail for all a and b. Our p_n shows this failure for the polynomial case if $a = b = n/(n + 1)$, as well as addressing the first result for cases $a = b \neq 1$, and any $n > 1$. Similar examples can be constructed using $q(z) = z(1 - z/R)^{n-1}$ for $R > n$.

4. A Physical Interpretation of $SP(\alpha)$

All of the work done so far has produced polynomials in $SP(\alpha)$ which are strongly starlike of order α. It is not clear that there exist polynomials in $SP(\alpha)$ which fail this extra condition. We shall construct such an example. Our example is heuristically based on an interpretation of spiral-like polynomials which we now give.

Let $p_n(z) = z\prod_{k=1}^{n-1}(1 - z/z_k)$, then $z\dfrac{p_n'(z)}{p_n(z)} = 1 + \sum_{k=1}^{n}\dfrac{z}{z - z_k}$. Let $z = re^{i\theta}$, and $|\alpha| < \pi/2$. Then the condition $p_n \in SP(\alpha)$ holds if and only if $\Re\left[e^{-i\alpha}\left(\dfrac{r}{r} + \sum_{k=1}^{n-1}\dfrac{re^{i\theta}}{re^{i\theta} - z_k}\right)\right] > 0$ for all $re^{i\theta} \in D$. But this holds if and only if $\Re\left[e^{i\alpha}\overline{\left(\dfrac{r}{r} + \sum_{k=1}^{n-1}\dfrac{re^{i\theta}}{re^{i\theta} - z_k}\right)}\right] > 0$ for all $re^{i\theta} \in D$, which in turn occurs if and only if $-\pi/2 < \arg\left[e^{i\alpha}\left(\dfrac{1}{r} + \sum_{k=1}^{n-1}\dfrac{1}{\overline{re^{i\theta}} - \overline{z_k}}e^{-i\theta}\right)\right] < \pi/2$. Thus we have: If $p_n(z) = z\prod_{k=1}^{n-1}(1 - z/z_k)$, then $p_n(z) \in SP(\alpha)$ if and only if $(\theta - \alpha) - \pi/2 < \arg\left[\dfrac{1}{\overline{re^{i\theta}} - \overline{0}} + \sum_{k=1}^{n-1}\dfrac{1}{\overline{re^{i\theta}} - \overline{z_k}}\right] < (\theta - \alpha) + \pi/2$ for $|z| < 1$.

The quantity $\dfrac{1}{\overline{re^{i\theta}} - \overline{0}} + \sum_{k=1}^{n-1}\dfrac{1}{\overline{re^{i\theta}} - \overline{z_k}}$ has a physical interpretation credited to Gauss: the vector $\dfrac{1}{\overline{re^{i\theta}} - \overline{z_k}}$ represents the force at $re^{i\theta}$ due to a particle of unit mass at z_k which repels with a force whose magnitude is equal to its mass divided by the distance. *Thus if unit masses are placed at each* $|z_k| > 1$ *and at zero itself with a repulsive force as above, then starlikeness of* p_n *is equivalent to the condition that the sum of the repulsive forces directed at any point* $e^{i\theta} \in \partial D$ *points in a direction outward from the boundary. Spiral-likeness tilts this outward orientation at* $e^{i\theta}$ *by an angle* α, $|\alpha| < \pi/2$. We may add that this interpretation is valid for $R(z) = z + a_2z^2 + \cdots$ rational in D, where the forces may be regarded as repulsive or attractive depending on whether they are zeroes or poles of R.

The field $\overrightarrow{F}(z) = 1/\overline{z} + \sum_{k=1}^{n-1} 1/(\overline{z} - \overline{z}_k)$ is sourceless (aside from the obvious poles at z_k) and irrotational, according to Pólya's interpretation of the Cauchy-Riemann equations for the rational function $1/z + \sum_{k=1}^{n-1} 1/(z - z_k)$ (see [11]), and $\nabla(\frac{1}{2}\ln|p_n(z)|^2) = \overrightarrow{F}(z)$. Thus the streamlines of $\overrightarrow{F}$, the lines following the direction of $\overrightarrow{F}$ in the domain of p_n, get mapped by p_n to the lines of constant argument. *Then* $p_n \in St$ *if and only if* D *is contained in the set of streamlines emanating from the origin, exiting* ∂D *(to provide the exit angle for* $\overrightarrow{F}$ *in the outward direction), and not returning to* D *(this disallows an* $\overrightarrow{F}(e^{i\theta})$ *pointing inward to* ∂D*). If* $p_n \in SP(\alpha)$*, then the streamlines emanating from the origin exit* D *at* $e^{i\theta}$ *with an angle between* $-\alpha - \pi/2$ *and* $-\alpha + \pi/2$ *off the normal, and a streamline may exit and re-enter. In this case some streamlines in* D *may also originate from a zero other than the origin.*

We note finally that for any $p_n \in St$ a streamline emanating from the origin does so with increasing magnitude, a fact which can easily be proved for arbitrary $f \in St$ as well:

Theorem 4. *For any* $f \in St$, $\theta \in \mathbb{R}$ *fixed, we have* $|f^{-1}(re^{i\theta})|$ *is an increasing function of* r.

Proof: Let $f : D \to D$ be univalent starlike, and fix $0 \le \theta < 2\pi$ and let $R =$ the ray from the origin in D with angle θ from the axis of positive reals. Then if $\mathcal{L} = f^{-1}(R)$ emanates from the origin we have $\dfrac{\partial}{\partial r}\log|f^{-1}(re^{i\theta})| = \dfrac{\partial}{\partial r}\mathrm{Re}\left[\log f^{-1}(re^{i\theta})\right] = \mathrm{Re}\left[\dfrac{\partial}{\partial r}\log(f^{-1}(re^{i\theta}))\right] =$

$\mathrm{Re}\left[\dfrac{d}{dw}\log f^{-1}(re^{i\theta})\dfrac{\partial w}{\partial r}\right]$ (where $w = re^{i\theta}$) $= \mathrm{Re}\left[\dfrac{1}{f^{-1}(re^{i\theta})}\left(\dfrac{d}{dw}f^{-1}(re^{i\theta})\right)e^{i\theta}\right]$ $= \mathrm{Re}\left[\dfrac{1}{f^{-1}(re^{i\theta})}\dfrac{1}{f'(z)}e^{i\theta}\right]$ (where $w = f(z)$) $= \mathrm{Re}\left[\dfrac{1}{z}\dfrac{(f(z)/|f(z)|)}{f'(z)}\right] > 0$. Thus $\log|f^{-1}(re^{i\theta})|$ is an increasing function of r, for fixed θ, and the result follows upon exponentiation. □

We may also note for completeness that a version of the "exiting vectorfield" condition holds for arbitary $f \in St$. Assume for simplicity that $f \in \mathcal{A}$ is starlike univalent in a neighborhood of $\overline{D}$, then $\mathrm{Re}\left[\dfrac{zf'(z)}{f(z)}\right] > 0$ on ∂D is equivalent to $-\pi/2 < \arg\{zf'(z)/f(z)\} < \pi/2$, which in turn is equivalent to $-\pi/2 < \arg\{zf'(z)\} - \arg\{f(z)\} < \pi/2$ for $z \in \partial D$. Thus the angle between the outward normal to $f(\partial D)$ at z and the ray $\mathcal{R}$ from the origin through $f(z)$ is between $-\pi/2$ and $\pi/2$. But by conformality of f at z this means that the angle between the outward normal at z and the arc $f^{-1}(\mathcal{R})$ from 0 to z also lies between $-\pi/2$ and $\pi/2$. In dealing with polynomials one can control the field and streamlines by adjusting the zeroes; this observation is the basis for our example.

5. An Example

The above interpretation can be used as a tool to construct many examples of starlike and spiral-like polynomials. We note, for example, that it follows immediately from this interpretation that setting $|z_k| \geq n, k = 1 \cdots n-1$ for $p \in P_n$ implies that the resulting polynomial is starlike univalent in D. This follows since locating sources outside or on $\{z : |z| = n\}$ assures the net force contributed inward to D at a point $e^{i\theta} \in \partial D$ does not exceed 1, the force emanating from the origin; thus $\overrightarrow{F}$ always points outward on ∂D. This result was obtained in [1] by a different method.

Less trivial examples can be produced. We will conclude by indicating how one can use the interpretation heuristically to suggest how to proceed, then construct our example. Placing a zero at the origin and $z_1 = c, 1 < c < 2$ will give a polynomial with a critical point $c_1, 0 < c_1 < 1$. By the Gauss-Lucas Theorem placing $z_3 = di$ for $d > 1$ will lift c_1 to the interior of the triangle formed by the origin, z_1, and z_2. It also introduces a second critical point c_2 in the same triangle. One needs to adjust c and d so that $|c_1|, |c_2| > 1$ and c_1 is close enough to the boundary of D that certain behavior occurs. Specifically, we want a streamline from the origin which exits D near c_1, is deflected back into the interior of D below the exit point by the force stream from z_1, then exits D once more below the re-entry point. We also want a streamline which exits D above the first one at an angle almost π greater than the re-entry angle of the first streamline.

By choosing $z_1 = 1.85553$ and $z_2 = 2.7i$, we obtain a cubic $q_3(z) = z(1 - z/z_1)(1 - z/z_2)$. Using the computer algebra system MAPLE IX one can verify that:

1. $q_3 \in SP(0.4)$ (Since $zq_3'(z)/q_3(z)$ is harmonic on $\overline{D}$, we need only test on the boundary of D.
2. If $g(\theta) = \text{argument}[1 + \dfrac{e^{-i\theta}}{e^{-i\theta} - 1.85553} + \dfrac{e^{-i\theta}}{e^{-i\theta} - (-2.7i)}]$, then $g(.146) = -1.969 < -\pi/2$ and $g(.152) = 1.155\ldots$.
3. $c_1 = .988911\cdots + (0.14857\ldots)i$, and $|c_1| = 1.000009722\ldots$.

It follows from (2) that $q_3 \notin St$. Furthermore, with $\alpha = 0.4$ we have $-\alpha - \pi/2 = -1.9707\cdots < g(0.146)$ and $g(0.152) < -\alpha + \pi/2 = 1.17079\ldots$, so the angle spread for the exiting vectorfield is nearly π. Thus $zq_3'(z)/q_3(z)$ lies nearly tangentially in the upper half plane formed by rotating the imaginary axis by 0.4 radians counterclockwise. One can compare this with Corollary 2, which requires $|z_k| > R = (1 + \sqrt{(1 + \cos^2(0.4)}/\cos(0.4) = 3.1299045\ldots$ for membership in the strongly starlike class of order $\alpha = 0.4$.

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Mathematical Sciences, Villanova University, Villanova, PA, USA, 19085, Phone:610-519-7334, FAX:610-519-6322

E-mail address: `frederick.hartmann@villanova.edu, alan.gluchoff@villanova.edu`