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APPROXIMATE AND GENERALIZED CONFIDENCE BANDS FOR SOME PARAMETRIC FUNCTIONS OF THE LOGNORMAL DIFFUSION PROCESS WITH EXOGENOUS FACTORS

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Abstract. Approximate and generalized confidence bands for some parametric functions of the univariate lognormal diffusion process with exogenous factors are obtained. The procedures to obtain these bands are developed from the suitable adaptation of the available methods for building confidence intervals for the mean of the lognormal distribution. The obtained bands are similar to those calculated for the homogeneous lognormal diffusion process but, in this case it is not possible a general comparative study in terms of coverage errors, by means of simulation studies, because of the dependence on the exogenous factors of each particular model. Therefore, in each case a particular study is necessary. In this sense, in this paper two models are considered modelling the *gross national product* of Spain and the *global manmade methane emissions*, respectively.

1 Introduction. The lognormal distribution and the lognormal diffusion process have been used frequently as probabilistic models in several scientific fields when the variable under consideration shows an exponential trend. For example, to determine even-time distributions (Lawrence [22]), in ecology as population growth model (Capocelli and Ricciardi [5]; Ricciardi [28]), in geology, etc. Possibly, economic and financial fields are the areas of application where the lognormal diffusion process has been more considered in order to model dynamic variables. Important contributions have been made in this direction by Cox and Ross [6], Merton [26] and Markus and Shaked [25], showing the theoretical and practical importance of this process in this environment. For example, these processes appear associated with the Black and Scholes model (Black and Scholes [4]) and later extensions, for example terminal swap-rate models (Lamberton and Lapeyre [17]; Hunt and Kennedy [15]).

The motivation that leads to include exogenous factors in the model is to introduce an explanation about the behavior of the studied variable by the diffusion (endogenous variable) in terms of a set of external variables. Their time behavior is assumed as known and they must contribute to the description of the evolution of the process as well as its external control with forecasting aims.

Usually, the way of introducing the external variables in the model is by means of a time function h which must be continuous in the time interval where the process is observed. The possibility of being several external influences to the endogenous variable of the process makes usual to consider $h(t) = \beta_0 + \sum_{j=1}^{q} \beta_j F_j(t)$, with $\beta_j \in \mathbb{R}$ and F_j time-continuous functions $j = 1, \ldots, q$.

This case has been widely studied related to some aspects about the inference as well as first passage times (Torres [33]; Gutiérrez *et al.* [7], [8], [9]) and applied for modelling time-variables in several fields. For example, Gutiérrez *et al.* ([10], [12]) built a non-homogeneous

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lognormal diffusion process to fit the *gross national product* in Spain by considering the *consumer spending* and the *gross domestic fixed capital formation* as exogenous variables. This application is an example where the endogenous variable itself shows who the exogenous factors are. Nevertheless, there are situations in that external variables to the process having an influence on the system are not available or their functional expressions are unknown. In such case Gutiérrez *et al.* [13] suggest to approach the exogenous factors by means of polynomial functions, that is, in this case $h(t) = \sum_{j=0}^{k} \beta_j^{k)} P_j^{k)}(t)$, where $P_j^{k)}$ is a k-degree polynomial ($P_0^{k)} = 1$) and $\beta_j^{k)}$ are real fixed parameters, $j = 1, \ldots, k$.

The possibility of controlling the endogenous variable by means of the exogenous factors makes very useful this process in forecasting. In this sense, some of its characteristics, as the mean and mode functions, could be used for predictions. Therefore, the inference of these two functions has been the object of considerable study, both from the point of view of point estimation and of that of estimation by confidence intervals.

With respect to the former, in Gutiérrez *et al.* [11] a more general study was made to obtain maximum likelihood estimators (MLE), uniformly minimum variance unbiased estimators (UMVUE) and expressions for the relative efficiency of MLE with respect to UMVUE for more general parametric functions (which include the mean and mode functions, together with their conditional versions, as special cases).

Concerning estimation by confidence bands, Gutiérrez *et al.* [12] extended the results obtained by Land ([18], [21]) on exact confidence intervals for the mean of a lognormal distribution, thus obtaining confidence bands for the mean and mode functions of the lognormal process with exogenous factors, expressing these functions in a more general form. The calculation of these bands runs into the same problems as does that of the exact confidence intervals obtained by Land, namely that they are based on conditional pivot statistics, the calculation of which is fairly complex, as it involves determining quantiles on the basis of integrals that must be resolved numerically. Therefore, it is necessary to employ tables of quantiles [20] with the consequent restrictions of available values, or computer programs, such as that proposed by Lyon and Land in [24] based on numerical algorithms that are unstable for certain ranges of values of the sample mean and quasi-variance of the lognormal random variable being examined (or the corresponding values in the case of the former process). Moreover, Singh *et al.* [30] suggested that upper confidence limits values based on Land's method are too high and lead to incorrect conclusions.

With these considerations in mind, for the case of the mean of a lognormal distribution, various authors have developed approximate confidence intervals, and studies have been devoted to obtain these and to compare them in terms of the coverage probability, the average length, etc., by means of simulation studies; see, for example, Zhou and Gao [35] and Lefante and Shah [23]. On the other hand, in 2003 Krishnamoorthy and Mathew [16] obtained a generalized confidence interval for the mean of a lognormal distribution based on the concepts of the generalized pivotal quantity and the generalized confidence interval, following Weerahandi [34].

The goal of this paper is to obtain approximate and generalized confidence bands for some parametric functions associated to the lognormal diffusion process with exogenous factors. These bands will be obtained by the suitable adaptation of the corresponding methods available for the mean of a lognormal distribution; we also propose an alternative method, referred to as the proposed method. The bands are similar to the obtained for the homogeneous lognormal diffusion process (Rico [29]; Gutiérrez *et al.* [14]) but, in this case it is not possible a general comparative study, by means of simulation studies, in terms of coverage errors because of the dependence on the exogenous factors of each particular model.

2 The lognormal diffusion process with exogenous factors. The lognormal diffusion process with exogenous factors is defined by a diffusion $\{X(t); t_0 \leq t \leq T\}$, taking values on $\mathbb{R}^+$ and with infinitesimal moments

$$A_1(x,t) = h(t)x \qquad (1)$$
$$A_2(x,t) = \sigma^2 x^2$$

where h is a continuous function in $[t_0, T]$ and $\sigma > 0$. We have to note this definition generalizes that of the homogeneous version of this process, in which case the function h is $h(t) = m$, with $m \in \mathbb{R}$.

This process can be studied from the point of view of the partial differential equations. The starting point for this is the forward (or Fokker-Planck) equation

$$\frac{\partial f(x,t|y,s)}{\partial t} = -h(t)\frac{\partial[xf(x,t|y,s)]}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2[x^2 f(x,t|y,s)]}{\partial x^2}$$

and the backward (or Kolmogorov) equation

$$\frac{\partial f(x,t|y,s)}{\partial s} + h(s)y\frac{\partial f(x,t|y,s)}{\partial y} + \frac{\sigma^2}{2}y^2\frac{\partial^2 f(x,t|y,s)}{\partial y^2} = 0. \qquad (2)$$

These equations verify the conditions for the existence and uniqueness of the solution, with respective initial conditions $\lim_{t\downarrow s} f(x,t|y,s) = \delta(x-y)$ and $\lim_{s\uparrow t} f(x,t|y,s) = \delta(x-y)$, where $\delta(\cdot)$ is the Dirac's delta function.

Alternatively, we can consider the Ito stochastic differential equation given by

$$dX(t) = h(t)X(t)dt + \sigma X(t)dW(t) \qquad (3)$$
$$X(t_0) = c$$

where $\{W(t); t \in [t_0, T]\}$ is a one-dimensional standard Wiener process and c is a positive random variable. Considering the analytical properties of h, it follows (see for example Arnold [3]) that the equation (3) has an unique solution that will be the $(0, +\infty)$-valued diffusion process with initial distribution c and infinitesimal moments given by (1).

3 Distribution of the process. In this section we obtain the distribution of the process considering the two approaches mentioned above. In any case, the finite dimensional distributions will be calculated.

3.1 Distribution of the process from stochastic differential equations. Let us consider the transformation $Y(t) = \ln(X(t))$. By virtue of Ito's lemma, equation (3) turns into

$$dY(t) = \left(h(t) - \frac{\sigma^2}{2}\right)dt + \sigma dW(t)$$
$$Y(t_0) = \ln(c)$$

which solution is

$$Y(t) = \ln(c) + \int_{t_0}^{t} h(\lambda)d\lambda - \frac{\sigma^2}{2}(t-t_0) + \sigma(W(t) - W(t_0)). \qquad (4)$$

Furthermore, following Arnold [3], $Y(t)$ is a gaussian process if and only if $\ln(c)$ is constant or normally distributed. In such a case, the mean and covariance function of $Y(t)$ is given by

$$m(t) = E[\ln(c)] + \int_{t_0}^{t} h(\lambda)d\lambda - \frac{\sigma^2}{2}(t-t_0)$$

and

$$R(t,s) = Var[\ln(c)] + \sigma^2(t \wedge s - t_0),$$

respectively, where $t \wedge s = \min(t,s)$. Hence, the finite dimensional distributions of the process $Y(t)$ are normal, that is,

$$(Y(t_1), Y(t_2), \ldots, Y(t_n))' \sim \mathcal{N}_n(\mu; \Sigma)$$

where the i-th component of the vector μ is $m(t_i)$, $i = 1, \ldots, n$, whereas Σ is a definite positive matrix whose components are $R(t_i, t_j)$, $i, j = 1, \ldots, n$.

From (4), the solution of (3) is

$$X(t) = \exp(Y(t)) = c\exp\left(\int_{t_0}^{t} h(\lambda)d\lambda - \frac{\sigma^2}{2}(t-t_0) + \sigma(W(t) - W(t_0))\right)$$

and, therefore, the finite dimensional distributions of the process $X(t)$ are lognormal $\Lambda_n(\mu; \Sigma)$. Obviously, from the two-dimensional distributions one can calculate the transition probability density function (p.d.f.).

3.2 Distribution of the process from partial differential equations. The transition p.d.f. of the process can be obtained by looking for a transformation

$$\begin{aligned} t' &= \phi(t) \\ x' &= \psi(x,t) \end{aligned}$$

that changes its Kolmogorov equation (2) into that of the standard Wiener process. Indeed, the infinitesimal moments (1) verify the conditions of the theorem 1 in Ricciardi [27], so such transformation exists. Concretely

$$\begin{aligned} \psi(x,t) &= \frac{(k_1)^{1/2}}{\sigma}\left(\ln(x/z) - \int_{t_2}^{t} h(\lambda)d\lambda + \frac{\sigma^2}{2}(t-t_2)\right) + k_2 \\ \phi(t) &= k_1(t-t_1) + k_3 \end{aligned}$$

where $z \in \mathbb{R}^+$, $t_i > 0$ and the k_i's are arbitrary constants with $k_1 > 0$. This transformation allows to obtain the transition p.d.f. for the considered process, resulting

$$f(x,t|y,s) = \frac{1}{x\sqrt{2\pi\sigma^2(t-s)}} \exp\left(-\frac{\left[\ln(x/y) - \int_s^t h(\lambda)d\lambda + \frac{\sigma^2}{2}(t-s)\right]^2}{2\sigma^2(t-s)}\right), \; t > s, \tag{5}$$

that corresponds with the density function of a lognormal variable, i.e.

$$X(t)|X(s) = y \sim \Lambda_1\left(\ln(y) + \int_s^t h(\lambda)d\lambda - \frac{\sigma^2}{2}(t-s);\; \sigma^2(t-s)\right), \; t > s. \tag{6}$$

Since the process being considered is a markovian process, the obtaining of the finite-dimensional distributions depends on the initial one and the transition p.d.f. In our case, the transition is lognormal given by (6), so it only remains to choose the initial distribution. Accordingly, two are the distributions here considered: a degenerate distribution in $x_0 > 0$, that is, $P[X(t_0) = x_0] = 1$, and a lognormal distribution $X(t_0) \sim \Lambda_1(\mu_0; \sigma_0^2)$, these choices ensuring that the finite dimensional distributions are lognormal (in accordance with that established in the above approach to the distribution of the process). We have to note that the former choice can be seen as a particular case of the second considering $\sigma_0 = 0$ (that implies $\mu_0 = \ln(x_0)$). Moreover, the degenerate initial distribution is the real situation when only a sample path is available whereas the lognormal case requires several trajectories.

Particularly, as $X(t)|X(t_0) = x_0 \sim \Lambda_1\left(\ln(x_0) + \int_{t_0}^t h(\lambda)d\lambda - \frac{\sigma^2}{2}(t-t_0);\ \sigma^2(t-t_0)\right)$, taking $X(t_0) \sim \Lambda_1(\mu_0; \sigma_0^2)$, one can calculate the joint distribution of $(X(t_0), X(t))'$, resulting a two-dimensional lognormal distribution $\Lambda_2(\mu; \Sigma)$ with

$$\mu = \begin{pmatrix} \mu_0 \\ \mu_0 + \int_{t_0}^t h(\lambda)d\lambda - \frac{\sigma^2}{2}(t-t_0) \end{pmatrix} \quad \text{and} \quad \Sigma = \sigma_0^2\mathbf{I}_2 + \sigma^2\begin{pmatrix} 0 & 0 \\ 0 & t-t_0 \end{pmatrix},$$

where $\mathbf{I}_2$ denotes the two-order identity matrix. Therefore, $X(t)$ is distributed as a random lognormal variable $\Lambda_1(\gamma; \sigma^2(t-t_0))$ with $\gamma = \mu_0 + \int_{t_0}^t h(\lambda)d\lambda - \frac{\sigma^2}{2}(t-t_0)$, $t > t_0$.

A similar development to the one-dimensional case leads to the obtaining of the bivariate distributions. In this case we can check that $(X(s), X(t))' \sim \Lambda_2(\mu; \Sigma)$, being now

$$\mu = \begin{pmatrix} \mu_0 + \int_{t_0}^s h(\lambda)d\lambda - \frac{\sigma^2}{2}(s-t_0) \\ \\ \mu_0 + \int_{t_0}^t h(\lambda)d\lambda - \frac{\sigma^2}{2}(t-t_0) \end{pmatrix} \quad \text{and} \quad \Sigma = \sigma_0^2\mathbf{I}_2 + \sigma^2\begin{pmatrix} s-t_0 & s\wedge t - t_0 \\ s\wedge t - t_0 & t-t_0 \end{pmatrix}.$$

Obviously, by virtue of the markovian property of the process, it is possible to obtain any finite-dimensional distribution, being lognormal in all cases.

3.3 Some characteristics. Once the finite dimensional distributions have been calculated, the main characteristics of the process can be obtained. We now describe some of them, focussing particularly on two of the most commonly employed in practice, especially for forecasting. These characteristics are the mean and the mode functions (as well as their conditional versions), which expressions can be formulated jointly for the two initial distributions considered. Expressions for other characteristics (covariance function, quantile function, etc...) have not been included here because there will be not considered in the remainder of the paper.

- Mean function

$$m(t) = E[X(t)] = E[X(t_0)]\exp\left(\int_{t_0}^t h(\lambda)d\lambda\right),\ t \geq t_0.$$

- Conditional mean function. Given s and x_s,

$$m(t|s) = E[X(t)|X(s) = x_s] = x_s\exp\left(\int_s^t h(\lambda)d\lambda\right),\ t > s \geq t_0.$$

- Mode function

$$Mo(t) = \text{Mode}[X(t)] = \text{Mode}[X(t_0)]\exp\left(\int_{t_0}^t h(\lambda)d\lambda - (t-t_0)\frac{3\sigma^2}{2}\right),\ t \geq t_0.$$

- Conditional mode function. Given s and x_s,

$$Mo(t|s) = \text{Mode}[X(t)|X(s) = x_s] = x_s\exp\left(\int_s^t h(\lambda)d\lambda - (t-s)\frac{3\sigma^2}{2}\right),\ t > s \geq t_0.$$

3.4 Particular case: lognormal diffusion process with h a linear function. In the previous sections we have done a brief overview about the lognormal process with exogenous factors without taking an explicit expression for the external variables being included in the model. In the following we will consider a special case for the h function: a linear combination of partially known functions, that is, $h(t) = \beta_0 + \sum_{j=1}^{q} \beta_j F_j(t)$, with $\beta_j \in \mathbb{R}$ and F_j time-continuous functions in $[t_0, T]$, $j = 1, \dots, q$. It is important to note that for inferential purposes about this process, it is not necessary to know the functional form of the exogenous factors but the value of their integrals between two time values included in the interval $[t_0, T]$. In any case, the exogenous factors must be independent of unknown parameters.

With the choice of the h function and denoting

- $a_0 = \beta_0 - \frac{\sigma^2}{2}$ and $a_j = \beta_j$, $j = 1 \dots, q$;
- $\mathbf{a} = (a_0, a_1, \dots, a_q)'$;
- $\bar{\mathbf{u}}(t,s) = \left(t - s, \int_s^t F_1(\tau) d\tau, \dots, \int_s^t F_q(\tau) d\tau \right)'$,

the transition p.d.f. (5) can be expressed as

$$f(x,t|y,s) = \frac{1}{x\sqrt{2\pi\sigma^2(t-s)}} \exp\left(-\frac{[\ln(x/y) - \bar{\mathbf{u}}(t,s)'\mathbf{a}]^2}{2\sigma^2(t-s)} \right), \tag{7}$$

whereas from (7), and considering the initial distribution $\mathrm{P}[X(t_0) = x_0] = 1$, the aforementioned parametric functions that represent the mean and mode functions of the process are expressed as

- $m(t) = x_0 \exp\left(\bar{\mathbf{u}}(t)'\mathbf{a} + \frac{1}{2}\sigma^2(t - t_0) \right)$, $t \geq t_0$
- $m(t|s) = x_s \exp\left(\bar{\mathbf{u}}(t,s)'\mathbf{a} + \frac{1}{2}\sigma^2(t - s) \right)$, $t > s \geq t_0$
- $Mo(t) = x_0 \exp\left(\bar{\mathbf{u}}(t)'\mathbf{a} - (t - t_0)\sigma^2 \right)$, $t \geq t_0$
- $Mo(t|s) = x_s \exp\left(\bar{\mathbf{u}}(t,s)'\mathbf{a} - (t - s)\sigma^2 \right)$, $t > s \geq t_0$

where $\bar{\mathbf{u}}(t) = \bar{\mathbf{u}}(t, t_0)$. Note that we have chosen the degenerate initial distribution because it is the situation that will appear in the subsequent examples.

The above expressions can be summarized in a single formula, concretely

$$\exp\left(\mu(t,s) + \lambda\sigma^2(t,s) \right) \tag{8}$$

where the values of $\mu(t,s)$, λ and $\sigma^2(t,s)$ are given in table 1.

In the following, we will consider this version of the lognormal process, and because the no conditional versions of the mean and mode functions are particular cases of the conditional, with $s = t_0$ and $x_s = x_0$ since we have considered $\mathrm{P}[X(t_0) = x_0] = 1$, we will refer to these last.

4 Maximum likelihood estimators of the parameters and parametric functions. Let us consider a discrete sampling of the process, that is, for fixed times $t_1, \dots, t_n$, $(n > q+2)$, we observe the variables $X(t_1), \dots, X(t_n)$ whose values will be the basic sample from which we carry out the inferential process. Furthermore, we suppose that $\mathrm{P}[X(t_1) = x_1] = 1$. Let $x_1, \dots, x_n$ the observed values of the sampling. Now we transform these values by means

Table 1: Values of $\mu(t,s)$, λ and $\sigma^2(t,s)$ to obtain the mean and mode functions from $\exp\left(\mu(t,s)+\lambda\sigma^2(t,s)\right)$.

Function	$\mu(t,s)$	λ	$\sigma^2(t,s)$
$m(t)$	$\ln(x_0)+\bar{\mathbf{u}}(t)'\mathbf{a}$	$1/2$	$(t-t_0)\sigma^2$
$m(t\|s)$	$\ln(x_s)+\bar{\mathbf{u}}(t,s)'\mathbf{a}$	$1/2$	$(t-s)\sigma^2$
$Mo(t)$	$\ln(x_0)+\bar{\mathbf{u}}(t)'\mathbf{a}$	-1	$(t-t_0)\sigma^2$
$Mo(t\|s)$	$\ln(x_s)+\bar{\mathbf{u}}(t,s)'\mathbf{a}$	-1	$(t-s)\sigma^2$

of $v_1=x_1$ and $v_i=(t_i-t_{i-1})^{-1/2}\ln(x_i/x_{i-1})$, $i=2,\ldots,n$. From (7), the likelihood function for the transformed sample is

$$L_{v_2,\ldots,v_n}(\mathbf{a},\sigma^2)=\frac{1}{(2\pi)^{(n-1)/2}(\sigma^2)^{(n-1)/2}}\exp\left(-\frac{1}{2\sigma^2}(\mathbf{v}-\mathbf{U}'\mathbf{a})'(\mathbf{v}-\mathbf{U}'\mathbf{a})\right) \tag{9}$$

where $\mathbf{v}=(v_2,\ldots,v_n)'$ and $\mathbf{U}$ is the $(q+1)\times(n-1)$ matrix given by $\mathbf{U}=(\mathbf{u}_2,\ldots,\mathbf{u}_n)$ with $\mathbf{u}_i=(t_i-t_{i-1})^{-1/2}\bar{\mathbf{u}}(t_i,t_{i-1})$. By supposing that $\mathrm{rg}(\mathbf{U})=q+1$, the maximum likelihood estimators of $\mathbf{a}$ and σ^2 are

$$\widehat{\mathbf{a}}=(\mathbf{U}\mathbf{U}')^{-1}\mathbf{U}\mathbf{v}\quad\text{and}\quad\widehat{\sigma}^2=\frac{1}{n-1}\mathbf{v}'[\mathbf{I}_{n-1}-\mathbf{U}'(\mathbf{U}\mathbf{U}')^{-1}\mathbf{U}]\mathbf{v}.$$

These estimators are independent and jointly sufficient and complete for $(\mathbf{a},\sigma^2)$. Furthermore,

$$\widehat{\mathbf{a}}\sim\mathcal{N}_{q+1}\left(\mathbf{a};\sigma^2(\mathbf{U}\mathbf{U}')^{-1}\right)\quad\text{and}\quad\frac{(n-1)\widehat{\sigma}^2}{\sigma^2}\sim\chi^2(n-q-2).$$

Taking into account these estimators, for fixed t and s, those corresponding to $\mu(t,s)$ and $\sigma^2(t,s)$ are immediately obtained. In this sense, $B(t,s)=\ln(x_s)+(t-s)\hat{a}$ is the MLE of $\mu(t,s)$, whereas for $\sigma^2(t,s)$ we consider the unbiased estimator $S^2(t,s)=(t-s)S^2$, where $S^2=\frac{(n-1)\hat{\sigma}^2}{n-2}$. These estimators are also independent, jointly sufficient and complete for $(\mu(t,s),\sigma^2(t,s))$ and verify

$$B(t,s)\sim\mathcal{N}\left(\mu(t,s);C(t,s)\sigma^2(t,s)\right)$$

and

$$\frac{(n-q-2)S^2(t,s)}{\sigma^2(t,s)}\sim\chi^2(n-q-2)$$

where $C(t,s)=\dfrac{t-s}{t_n-t_1}\cdot$

5 Confidence bands. In order to obtain approximate and generalized confidence bands for the parametric functions aforementioned, and because these functions represent some characteristics associated to lognormal distributions, it seems obvious to consider known similar results in this context. Thus, let us cite the results summarized in Zhou and Gao [35] which are related to the building of approximate and generalized confidence intervals for the mean of the lognormal distribution. In a first approach, one might think that the confidence bands could be calculated in a direct form from the distribution results, simply by obtaining confidence intervals for fixed values of t and s and, subsequently, varying t

to get the bands. This procedure is valid if one has n independent sample paths because, in such a case, and for fixed t and s, the same sample information as for the inference on the lognormal distribution is available, that is, we have a simple random sample of n observations for the distribution being considered. Nevertheless, if only one trajectory is observed (which is the usual situation in certain real applications such as those proposed in this paper), these results are not directly applicable in this way. However, the consideration of the likelihood function of the sample path, by virtue of the Markov structure of the process considered, allows us to apply the point estimation of the parameters involved, and thus the adaptation and extension of the results for the lognormal distribution is possible.

In this section we give the expressions of the approximate and generalized confidence bands for the $\exp\left(\mu(t,s)+\lambda\sigma^2(t,s)\right)$ functions in the case of the lognormal diffusion process with h a linear function, which obtaining is similar to that developed for the homogenous version of the process (see Rico [29]; Gutiérrez *et al.* [14]). However, since the estimations of the parameters are different (more complicated in this case), the obtained expressions show differences regarding the associated distributions that appear, so another approach in their calculation is needed. Moreover, in contrast to the homogeneous case, where the coverage probabilities associated to each one of the intervals that constitute the confidence band remain constant, in the case of the non homogenous process these probabilities change through the time, being this dependence motivated by the inclusion of the exogenous factors. For this reason, in this last case it is not possible a general comparative study, in terms of coverage errors, as the realized in the last mentioned works.

In the next, we give a brief summary including the expressions of the approximate and generalized confidence bands for the parametric functions, including a new proposal for this purpose, as well as some comments about their obtaining.

5.1 Approximate confidence bands.

5.1.1 Transformation methods. The adaptation of these methods to the process being considered leads to calculate, for each t and s, a confidence interval for $\mu(t,s)$ (that can be obtained from the distribution of its estimator $B(t,s)$) and then take the appropriate transformation in order to obtain the desired confidence interval.

The naive method considers the exponential transformation, resulting the following confidence intervals at level $1-\alpha$

$$\exp\left(B(t,s) \pm t_{n-q-2;1-\alpha/2}\sqrt{C(t,s)}S(t,s)\right),$$

whereas the adaptation of Patterson's transformation leads to add, before taking the exponential transformation, the term $\lambda S^2(t,s)$. Hence the intervals, from which the confidence band is obtained, are

$$\exp\left(B(t,s) + \lambda S^2(t,s) \pm t_{n-q-2;1-\alpha/2}\sqrt{C(t,s)}S(t,s)\right),$$

where $t_{n;\alpha}$ is the αth quantile of a Student's t distribution with n degrees of freedom.

5.1.2 Direct methods. These methods are based on estimators of the characteristics for which we want to build the confidence intervals or of some function of them. With these methods it is assumed that the estimators are normally distributed, with a known or estimated variance, from which the approximate confidence intervals can be calculated.

In our case the interest characteristics are $\mu(t,s)+\lambda\sigma^2(t,s)$. For these functions, the adaptation of the Cox's method [6] considers their UMVUEs, $B(t,s)+\lambda S^2(t,s)$, as estimators, whose variances are also estimated by the corresponding UMVUEs, that is

$$\begin{aligned}\text{UMVUE}\left[\text{Var}\left(B(t,s)+\lambda S^2(t,s)\right)\right] &= \text{UMVUE}\left[C(t,s)\sigma^2(t,s)+2\lambda^2\frac{\sigma^4(t,s)}{n-q-2}\right]\\ &= C(t,s)S^2(t,s)+2\lambda^2\frac{S^4(t,s)}{n-q}\cdot\end{aligned}$$

Finally, from the above considerations, the confidence band takes the form

$$\exp\left(B(t,s)+\lambda S^2(t,s)\pm z_{1-\alpha/2}\sqrt{C(t,s)S^2(t,s)+2\lambda^2\frac{S^4(t,s)}{n-q}}\right)$$

where z_α is the αth quantile of a normal standard distribution.

The method considered here is an adaptation of one that appears in [19]. However, other versions of this method have also been proposed. For example, following Zhou and Gao [35] or Lefante and Shah [23], we can estimate Var $\left(B(t,s)+\lambda S^2(t,s)\right)$ simply by replacing $\sigma^2(t,s)$ by $S^2(t,s)$.

5.1.3 Methods based on pivot statistics.

Angus' conservative method.

The adaptation of the method proposed by Angus [1] leads us to consider, for $\mu(t,s)+\lambda\sigma^2(t,s)$, the following pivot statistic

$$\frac{B(t,s)+\lambda S^2(t,s)-\left(\mu(t,s)+\lambda\sigma^2(t,s)\right)}{\sqrt{S^2(t,s)C(t,s)+\frac{2\lambda^2}{n-q-2}S^4(t,s)}}\cdot$$

This statistic is asymptotically equivalent to the likelihood ratio statistic for testing hypothesis about $\mu(t,s)+\lambda\sigma^2(t,s)$ and its cumulative distribution function is monotone increasing on $\sigma(t,s)$. From this statistic, and taking into account the asymptotic distributions associated when $\sigma(t,s)$ tends to 0 and to infinity (see [29] for details), we calculate the confidence band as follows

$$\left(\exp\left(B(t,s)+\lambda S^2(t,s)-t_{n-q-2;1-\alpha/2}\sqrt{C(t,s)S^2(t,s)+\frac{2\lambda^2S^4(t,s)}{n-q-2}}\right),\right.$$

$$\left.\exp\left(B(t,s)+\lambda S^2(t,s)+\sqrt{\frac{n-q-2}{2}}\left(\frac{n-q-2}{\chi^2_{n-q-2;\alpha/2}}-1\right)\sqrt{C(t,s)S^2(t,s)+\frac{2\lambda^2S^4(t,s)}{n-q-2}}\right)\right).$$

In this case $\chi^2_{n;\alpha}$ denotes the αth quantile of a chi-squared distribution with n degrees of freedom.

Parametric bootstrap method.

In this method, the confidence band is made by following the next Monte Carlo algorithm, similar to the one proposed in [2]. This algorithm is used to avoid the numerical problems derived by applying the t-percentile method to the approximate pivotal statistic considered in the previous method.

- Generate k values N_i^* of a normal standard distribution $\mathcal{N}(0;1)$ and k values $\chi_i^{2^*}$ of a chi-squared distribution with $n-q-2$ degrees of freedom $\chi^2(n-q-2)$ independently.

- Calculate, from the simulated values

$$T_i^*(t,s) = \frac{N_i^* + \frac{\lambda S(t,s)}{\sqrt{C(t,s)}}\left(\frac{\chi_i^{2*}}{n-q-2} - 1\right)}{\sqrt{\frac{\chi_i^{2*}}{n-q-2}\left(1 + 2\lambda^2 \frac{S^2(t,s)}{C(t,s)} \frac{\chi_i^{2*}}{(n-q-2)^2}\right)}}.$$

- Sort out the values $T_i^*(t,s)$ in $T_{(1)}^*(t,s) < T_{(2)}^*(t,s) < \ldots < T_{(k)}^*(t,s)$.
- Calculate $k_1(t,s)$ and $k_2(t,s)$ as

$$k_1^{boot}(t,s) = T^*_{[(1-\alpha/2)k]}(t,s) \text{ and } -k_2^{boot}(t,s) = T^*_{[\alpha/2k]}(t,s)$$

where $[a]$ denotes the integer part of a.

- Construct the bootstrap confidence band for $\exp\big(\mu(t,s) + \lambda\sigma^2(t,s)\big)$ as

$$\left(\exp\left(B(t,s) + \lambda S^2(t,s) - k_1^{boot}(t,s)\sqrt{C(t,s)S^2(t,s) + \frac{2\lambda^2}{n-q-2}S^4(t,s)}\right),\right.$$

$$\left.\exp\left(B(t,s) + \lambda S^2(t,s) + k_2^{boot}(t,s)\sqrt{C(t,s)S^2(t,s) + \frac{2\lambda^2}{n-q-2}S^4(t,s)}\right)\right).$$

5.1.4 Proposed method. We construct the confidence band by combining, for each (t,s), the limits of the corresponding optimal confidence intervals for $\mu(t,s)$ and for $\sigma^2(t,s)$. With this procedure both the variability in the estimation of $\mu(t,s)$ and in the estimation of $\sigma^2(t,s)$ are considered whereas the previous procedures only take into account the former.

The result is

$$\left(\exp\left(B(t,s) - t_{n-q-2;1-\alpha/2}S(t,s)\sqrt{C(t,s)} + \lambda\frac{(n-q-2)S^2(t,s)}{\chi^2_{n-q-2;1-\alpha/2}}\right),\right.$$

$$\left.\exp\left(B(t,s) + t_{n-q-2;1-\alpha/2}S(t,s)\sqrt{C(t,s)} + \lambda\frac{(n-q-2)S^2(t,s)}{\chi^2_{n-q-2;\alpha/2}}\right)\right).$$

5.2 Generalized confidence band. Following the guidelines of Krishnamoorthy and Mathew [16], we consider the generalized pivotal quantity for $\mu(t,s) + \lambda\sigma^2(t,s)$

$$R(t,s) = b(t,s) - \frac{B(t,s) - \mu(t,s)}{\sqrt{C(t,s)}S(t,s)}\sqrt{C(t,s)}s(t,s) + \lambda\frac{\sigma^2(t,s)}{S^2(t,s)}s^2(t,s)$$

which has the same distribution as

$$b(t,s) - \frac{Z}{\frac{U}{\sqrt{n-q-2}}}\sqrt{C(t,s)}s(t,s) + \lambda\frac{s^2(t,s)}{\frac{U^2}{n-q-2}}$$

where $Z \sim \mathcal{N}(0;1)$ and $U^2 \sim \chi^2(n-q-2)$ are independent, and where the observed values of each random variable that appears are denoted by lower case print.

Since $r(t,s) = \mu(t,s) + \lambda\sigma^2(t,s)$, it is sufficient to obtain the corresponding quantiles in order to construct the desired confidence band. Hence we use the following algorithm:

- Obtain, from a sample observed path of the process, the values of $b(t,s)$ and $s^2(t,s)$.
- Generate $Z_i \sim \mathcal{N}(0;1)$ and $U_i^2 \sim \chi^2(n-q-2)$, $i=1,\dots,k$ independently.
- Calculate

$$R_i(t,s) = b(t,s) - \frac{Z_i}{\frac{U_i}{\sqrt{n-q-2}}}\sqrt{C(t,s)}s(t,s) + \lambda \frac{s^2(t,s)}{\frac{U_i^2}{n-q-2}}, \quad i=1,\dots,k$$

  and, from these values, calculate the $100(\alpha/2)$th and the $100(1-\alpha/2)$th quantiles, denoted by $R_{(t,s)}(\alpha/2)$ and $R_{(t,s)}(1-\alpha/2)$ respectively.
- Construct the generalized confidence band, at the $1-\alpha$ confidence level, for $\exp(\mu(t,s)+\lambda\sigma^2(t,s))$ as

$$\left(\exp(R_{(t,s)}(\alpha/2)), \exp(R_{(t,s)}(1-\alpha/2))\right).$$

Note that this confidence band fits a Monte Carlo procedure for calculating the approximate confidence band obtained by the proposed method.

6 Comparative studies. Once the different confidence bands have been obtained, the logical next step to do would be a comparative study of the confidence bands in the same line of those carried out for the lognormal distribution by Zhou and Gao [35] and for the homogeneous lognormal diffusion process by Rico [29] and Gutiérrez *et al.* [14]. In these studies the confidence bands are compared in terms of coverage probabilities, coverage errors and average lengths, by means of simulation procedures.

Nevertheless, in this context, the process being considered in this paper has some particularities that make it be different from the homogeneous case.

Firstly, we show these particular features justifying that no general studies are possible, and hence each specific case must be dealt with separately. Secondly, we consider two different models for which the comparative study is realized.

6.1 Special features of the model. Observe that approximate confidence bands can be written in the general form

$$\begin{aligned}&\Big(x_s \exp\left(\overline{\mathbf{u}}(t,s)'\widehat{\mathbf{a}} + K_1\lambda(t-s)S^2 - K_2\sqrt{\overline{\mathbf{u}}(t,s)'(\mathbf{U}\mathbf{U}')^{-1}\overline{\mathbf{u}}(t,s)S^2 + K_3\lambda^2(t-s)^2S^4}\right),\\ &\;x_s \exp\left(\overline{\mathbf{u}}(t,s)'\widehat{\mathbf{a}} + K_1\lambda(t-s)S^2 - K_2^*\sqrt{\overline{\mathbf{u}}(t,s)'(\mathbf{U}\mathbf{U}')^{-1}\overline{\mathbf{u}}(t,s)S^2 + K_3\lambda^2(t-s)^2S^4}\right)\Big)\end{aligned} \tag{10}$$

taking the values of K_1, K_2, K_2^* and K_3 that are related in table 2.

From this general form, and remembering that the generalized band fits a Monte Carlo procedure for calculating the approximate confidence band obtained by the proposed method, we remark the following considerations:

- The results are independent on the parameters β_j, $j=0,1,\dots,q$. For each fixed value of (t,s), a change in the estimation of the coefficients β_j, $j=0,1,\dots,q$, will only affect the estimation of the vector $\mathbf{a}$ and the corresponding confidence interval for $\exp\left(\mu(t,s)+\lambda\sigma^2(t,s)\right)$ by a scale change, since n and S^2 are unchanged, given that the vector $\overline{\mathbf{u}}(t,s)$ and the quadratic form $\overline{\mathbf{u}}(t,s)'(\mathbf{U}\mathbf{U}')^{-1}\overline{\mathbf{u}}(t,s)$ depend on the exogenous factors but they are independent on unknown parameters.

Table 2: Values of K_1, K_2, K_2^* and K_3.

Method	K_1	K_2	K_2^*	K_3
Naive	0	$t_{n-q-2;1-\alpha/2}$	$-t_{n-q-2;1-\alpha/2}$	0
Patterson	1	$t_{n-q-2;1-\alpha/2}$	$-t_{n-q-2;1-\alpha/2}$	0
Cox (Land)	1	$z_{1-\alpha/2}$	$-z_{1-\alpha/2}$	$2/(n-q)$
Angus	1	$t_{n-q-2;1-\alpha/2}$	$\sqrt{\frac{n-q-2}{2}}\left(\frac{n-q-2}{\chi^2_{n-q-2;\alpha/2}}-1\right)$	$2/(n-q-2)$
Bootstrap	1	$k_1^{boot}(t,s)$	$-k_2^{boot}(t,s)$	$2/(n-q-2)$
Proposed	$\frac{n-q-2}{\chi^2_{n-q-2;1-\alpha/2}}$	$t_{n-q-2;1-\alpha/2}$	$-t_{n-q-2;1-\alpha/2}$	0

- The coverage probability and the length of the intervals change through the time. From (10) we observe that both of them change through the time according to the form of the vector $\overline{\mathbf{u}}(t,s)$ and the quadratic form $\overline{\mathbf{u}}(t,s)'(\mathbf{UU}')^{-1}\overline{\mathbf{u}}(t,s)$, which also depend on the exogenous factors within the model.

  Therefore, the comparative study of the approximate and generalized confidence bands must be done for a concretely model, that is, for previously fixed functions F_i (i.e., for each choice of the exogenous factors).

6.2 Gross National Product (G.N.P.) in Spain. This case has been widely studied in [10] and [12], where the exogenous factors are built from the knowledge of another related variables like *consumer spending* and *gross domestic fixed capital formation* during the same time period.

6.2.1 The model. Remarks about the exogenous factors. Gutiérrez *et al.* [10], proposed a model that fits the behavior of the G.N.P. in Spain by a lognormal diffusion process with exogenous factors. For building the model, this study contemplated two stages: the first was to decide which was the external information that must be considered in the model, and that constitutes the exogenous factors, and the second how this information is included. Taking into account that the G.N.P. depends mainly on the *national demand*, the search of the exogenous factors was focused on its components. About how the information is included, in this kind of studies (see Tintner and Sengupta [32]) normally is considered that the exogenous variables remain constant between two consecutive observed times (usually equally spaced). This supposition is not according with the continuity hypothesis established in the definition of the process and can be discussed because of economic variables, essentially, evolve continuously and not by jumps.

Given these two questions, the following procedure was proposed in order to solve them:

1. For the selection of the exogenous factors, a stepwise regression study for the *national demand* on its components was realized. The variables selected were *consumer spending* and *gross domestic fixed capital formation*.

2. Given the selected factors, a function of them was built by polygonal functions such that the integrals between two consecutive times coincide with the observed value of the exogenous factors. That is, the exogenous factors are really functions not directly observable but such that their influence on the process is given by the observed values of the considered variables. Therefore, the chosen exogenous factors were the

polygonal functions F_1 and F_2 such that

$$\int_{t_{i-1}}^{t_i} F_1(\tau)d\tau = Y_1(t_i), \quad \int_{t_{i-1}}^{t_i} F_2(\tau)d\tau = Y_2(t_i), \ i = 2, \ldots, n, \quad Y_1(t_1) = Y_2(t_1) = 1.$$

where

(a) Y_1: *Increases of the private consumption.*

(b) Y_2: *Increases of the gross fixed capital formation.*

Once the external information and the procedure to include it in the model are established, the estimation of the parameters is possible. In this sense, and considering the annual observed values of the endogenous and exogenous variables in the time interval [1970, 2002], the estimations of the parameters of the model are

$$\begin{aligned} \widehat{\beta}_0 &= -0.7146756702074915 \\ \widehat{\beta}_1 &= 0.6297030524705661 \\ \widehat{\beta}_2 &= 0.09370917397490128 \\ \widehat{\sigma}^2 &= 0.00003799426796598253 \end{aligned}$$

6.2.2 Study of the approximate and generalized confidence bands. In order to decide which confidence band is the optimal one with respect to the considered model, we have simulated 1000 random sample paths of the diffusion process with infinitesimal moments

$$\begin{aligned} A_1(x,t) &= \left(\widehat{\beta}_0 + \widehat{\beta}_1 F_1(t) + \widehat{\beta}_2 F_2(t)\right) x \\ A_2(x,t) &= \widehat{\sigma}^2 x^2. \end{aligned}$$

Each sample path consists of 33 data in times $1970 + i, \ i = 0, \ldots, 32$ with initial value $x_1 = 213032$.

From these paths we have calculated the approximate and generalized confidence bands, at level 0.9, for the particular case of the mean function, as well as the average and the range of variation through the time of the coverage probabilities, coverage errors and the range of variation of the average lengths in the observation times.

The results are shown in table 3 and allow the comparison of the confidence bands in a general form.

According with these results, the generalized confidence band shows the least variation range of the coverage errors, containing the imposed confidence level. Because of this confidence band fits a Monte Carlo scheme for the proposed method, and the observed differences between the two methods can be motivated by the number of random variables used for the calculations, we can select any of them.

In this case, the naive confidence band is similar, in coverage error, to the proposed one but with less range for the average length (the optimality in this case is given by a small value of the estimation of σ^2).

If our interest is focused on the length of the confidence bands, with an acceptable coverage error, we will choose the confidence band given by Cox's method.

The conservative confidence band presents the biggest coverage error and the biggest length.

Table 3: Range of variation of the coverage probability, average of the coverage probability, range of variation of the coverage error, average of the coverage error and range of variation of the average lengths for the approximate and generalized confidence bands, with confidence level 0.9, for the mean function of the G.N.P. in the case of *manmade global methane emissions.*

Method	Variation cov. prob.	Average cov. prob.	Variation cov. error	Average cov. error	Variation average length
Naive	0.882-0.909	0.89178125	0-0.018	0.00946875	2417.9491-65302.5648
Patterson	0.881-0.907	0.8913125	0-0.019	0.0096875	2417.9968-65343.8726
Cox	0.87 -0.9	0.88125	0-0.03	0.01875	2340.7633-63254.8897
Conservativ	0.938-0.955	0.94453125	0.017-0.055	0.04378125	3248.8546-89484.6992
Bootstrap	0.872-0.952	0.8881875	0.001-0.052	0.016	2421.0044-65539.2623
Proposed	0.882-0.908	0.89209375	0-0.018	0.00909375	2422.0445-65675.1914
Generalized	0.884-0.911	0.89646875	0-0.016	0.00540625	2489.6651-67243.7583

6.3 Global methane emissions. We now consider the case of the *manmade global methane emissions*. This is an example where there is not additional information available over another related variables that can be useful for the construction of the exogenous factors. For this reason, in [29] and [13] an iterative procedure is proposed in order to approach the unknown exogenous factors by means of polynomial functions.

6.3.1 The model. Remarks about the exogenous factor. In 1998 Stern and Kaufmann, [31], published a study about global manmade methane emissions since 1860 to 1994. In this study the authors gave the estimation, in the mentioned period of time, for the total emissions considering each one of the seven components which constitute it. The *global methane emission* is the addition of those components, where each one is estimated from other variables like population or coal production. The target of this study was to obtain an approximation to the actually value of the methane emissions and other fossil combustibles such that it was compatible with the estimations of the *Intergovernmental Panel on Climate Change*.

Gutiérrez *et al.* [13] studied the fit of the observed data by means of a lognormal diffusion process owing to the exponential trend followed by the data. Nevertheless, when a homogeneous lognormal diffusion process is fitted, the estimated trend shows deviations to the observed data. For this reason one can think in the existence of some external influences that the homogeneous process is not considering. These influences must be time dependent variables affecting the trend but, however, unknown. For this reason, an approach to this unknown factors is taken into account by considering polynomial exogenous factors. In this sense, in that paper, the authors developed an iterative procedure to estimate the model including this kind of exogenous factors. This method includes the recursive estimation of the models, resulting from the successive addition of a polynomial function, and the criterion for selecting the optimum one, which is based on the forecasting capacity of the model. The model being chosen was that with infinitesimal moments (see the aforementioned references for details):

$$\begin{aligned} A_1(x,t) &= \left(0.0109222 - 0.000292911t + 6.982539013 \times 10^{-6}t^2 - 3.579618802 \times 10^{-8}t^3\right) x \\ A_2(x,t) &= 0.00007457051282638727x^2. \end{aligned}$$

6.3.2 Study of the approximate and generalized confidence bands. With the selected model, and with the objective to decide which confidence band is optimum, we have realized a similar study than the one established in the previous section.

We have simulated 1000 sample paths of the previously fitted lognormal diffusion process, with 135 values on each one, beginning at the year $t_1 = 1860$ with $x_1 = 79.3$.

From these paths we have calculated the approximated and generalized confidence bands, at level 0.9, in the particular case of the mean function, and the average and range of variation through the time of their coverage probabilities, coverage errors and the range of variation of the average length in the observation times.

The results are showed in table 4. They allow the comparison of all methods in a global form.

Table 4: Range of variation of the coverage probability, average of the coverage probability, range of variation of the coverage error, average of the coverage error and range of variation of the average lengths for the approximate and generalized confidence bands, with confidence level 0.9, for the mean function in the case of *manmade global methane emissions*.

Method	Variation cov. prob.	Average cov. prob.	Variation cov. error	Average cov. error	Variation average length
Naive	0.872-0.906	0.89191791	0-0.028	0.00891791	0.7691-124.0001
Patterson	0.87-0.904	0.891768657	0-0.03	0.008723881	0.7692-124.6266
Cox	0.867-0.901	0.889134328	0-0.033	0.010880597	0.7637-123.7328
Conservative	0.904-0.938	0.928156716	0.004-0.0379	0.028152985	0.892-148.5159
Bootstrap	0.86-0.897	0.880970149	0.003-0.04	0.019029851	0.7636-124.3261
Proposed	0.871-0.906	0.893492537	0-0.029	0.008059701	0.7704-125.4415
Generalized	0.867-0.905	0.88541791	0-0.033	0.015149254	0.7715-124.4133

The coverage errors obtained are very similar between all confidence bands, being the greatest the corresponding to the conservative and bootstrap bands and showing the latter the biggest length. Excepting these two confidence bands, all of the other will be valid.

Cox's method is the optimum in terms of average length, but if the interest is focused on the minimal coverage error, the proposed method must be selected.

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