PROPAGATING COALITIONS IN NETWORKS OF NONLINEAR OSCILLATORS

Frank Hoppensteadt and Kevin Lin

Received April 28, 2006

Abstract. A brain can perform pattern recognition tasks that are not yet possible for an electronic computer. We continue here our investigation of electronic circuits that are inspired by knowledge of structures in a brain. These circuits are oscillators that are motivated by principles of neuroscience, but yet are constructible as micro circuits, and possibly as nano-circuits. Populations of such oscillators can exhibit patterns in their output frequencies. This may be in response to an external signal being applied across the population or in response to internal waves that propagate through the population. We have investigated the former phenomenon where coalitions of oscillators, classified by their output frequency, form in response to a common driving signal. The resulting patterns have been used to characterize the input signal and they serve as a basis for comparison of signals and other pattern recognition tasks. In this paper, we investigate pattern formation that results from the propagation of synchronized activity waves within the population of oscillators. The novelty here is in the model: We derive a nonlinear wave equation to describe networks of oscillators, and simulate solutions to demonstrate some basic results.

We use knowledge of how a brain works to design networks of electronic circuits that can perform pattern recognition tasks. The circuits are inspired by electronic circuits that have been derived from Hodgkin's and Huxley's seminal work [Hodgkin, 1948, 1952; Huxley, 1952]. Their work shows that (1) there are mechanisms to maintain a membrane potential in the absence of stimulation, (2) there is negative differential resistance partly due to the opening and closing of ionic channels, (3) there is a homeostatic mechanism that returns the membrane potential to rest, and (4) charge accumulates on both sides of the membrane. A useful model of a neuron focuses on an Action Potential Generator (APG) region that can be described in terms of the membrane potential in that region [Hoppensteadt, 2004]. There is substantial experimental evidence that the electrical dynamics of an APG is similar to that of a mechanical pendulum [Hodgkin, 1948; Hoppensteadt, 1997; Izhikevich, 2004]. For reference here, an oscillator is a device, possibly at rest, that can sustain regular oscillations when stimulated with constant input. For example, we refer to a pendulum, even at rest, as being an oscillator since it can perform regular oscillations when a constant torque is applied to the support point. Its frequencies might not be apparent in the absence of external forcing. The state of a pendulum is naturally described in terms of an angle variable.

The novelty of this work is in the derivation from neuroscience principles of a nonlinear wave equation (4) that supports isolated waves of synchronization through a population of oscillators, and we provide an illustrative simulation of its solution structure in a two dimensional array.

Pendulum equations appear in many investigations ranging from neuroscience to quantum mechanics, and the angle variable that they govern are related to integral of a voltage. For example, in quantum mechanics the dynamics of a Josephson junction [Feynman, *et*

2000 *Mathematics Subject Classification.* 34C15, 35L70, 92B20, 94C99.
Key words and phrases. Nonlinear oscillators, neural networks, circuits.

al.,1965, III.21-15 ff.] is described by a pendulum equation for a phase variable $\theta(t)$ where $\dot\theta = qV$ (electron charge times applied voltage) and the current through the device is proportional to $\sin\theta$. The close connection between frequency and voltage is used to design and study electrical systems from large dynamos and rotating electrical machinery, to microscopic electronic devices such as phase-locked loops and MEMS, and to super-conducting quantum mechanical devices.

Phase variables are natural state variables for modeling neural membrane potentials [Hoppensteadt, 1997, 2006]. Writing the membrane potential as a frequency $V = \dot\theta$, we can interpret the principles identified by Hodgkin and Huxley in terms of electronic components in the following way: The membrane voltage is $\dot\theta$, the current into the homeostatic mechanism is $a\sin\theta$, the current into the negative differential resistance mechanism is $f(\dot\theta)$ where f is an $\mathcal{N}$-shaped function, and the current into the capacitor is $C\ddot\theta$. Balancing currents leads us to the equation

$$\tau\ddot\theta + \alpha\dot\theta + f(\dot\theta) + a\sin\theta = \omega. \tag{1}$$

where τ is a time constant, α is a damping factor, f describes the negative differential resistance in the circuit, $a\sin\theta$ accounts for homeostasis, and ω summarizes currents injected into the circuit.

Because the exact shape of f changes from one application to another, we change variables if necessary, and write the nonlinearity as being a canonical cubic function

$$f(V) \approx \alpha(V^2 - \lambda)V. \tag{2}$$

where α and λ are parameters related to circuit parameters. We have completed an analysis of equation (1) on the phase-cylinder

$$\mathcal{C} = \{(\theta, v) : 0 \le \theta < 2\pi, -\infty < v < \infty\}$$

[Hoppensteadt, 2006].

1 Coalition Formation in Frequency-Gradient Aggregate Consider an array of N of these oscillators that are not connected, but that have a common periodic parametric forcing. We refer to them as forming an aggregate rather than a network. This configuration and choice of parametric structure is suggested by our work on the hippocampus [Borisyuk, 1999]. We assume that the center frequencies of these oscillators are graded, say $\omega_1 < \omega_2 < \cdots < \omega_N$. Such gradients are known in various structures in the brain, principally in the auditory pathway, but also possibly in the hippocampus.

Based on the considerations above, we describe the oscillators by the equations

$$\tau\ddot\theta_j + F(\dot\theta_j) + a\cos\mu t\sin\theta_j = \omega_j \tag{3}$$

for $j = 1, 2, 3, \ldots, N$, where μ is the common forcing frequency. We describe the output of these oscillators in terms of output frequencies

$$\rho_j = \lim_{t\to\infty}\frac{\theta_j(t)}{t}.$$

The simulation depicted in Figure 1 shows a typical output. The output frequencies resemble a staircase whose treads correspond to intervals of phase locking. A collection of oscillators, although not identical to each other, that are locked at the same output frequency form a *coalition*. This provides a classification of oscillators by their output frequency: The sets of oscillators having the same output frequency define coalitions among the population. For

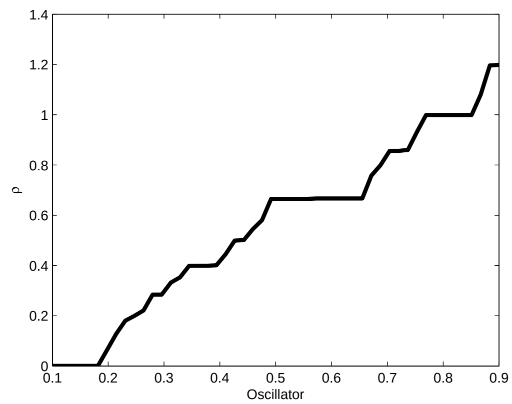

Figure 1: Output frequencies of the array in Eq. (3) with $0.1 < \omega < 0.9$ and $\mu = 2.0$.

example, the oscillators with $0.5 < \omega < 0.65$ form a coalition, and each output frequency is approximately 0.75. A different input frequency μ creates another pattern, and in this way we see that various inputs create distinctive coalitions. These patterns of firing frequencies lay a basis for pattern recognition methodologies that are discussed elsewhere [Archibald, *et al.*., 2005; Borisyuk, *et al.*., 1999; Hoppensteadt, 2006; Vinogradova, 2001].

This example describes what we mean by coalitions of oscillators. These are defined by their response to particular inputs, and different inputs will result in different coalitions being formed. Coalitions can be formed by other mechanisms than gradients of center frequencies. We show next that they can also be formed by internal waves in the population.

2 Coalition Propagation in Inductively Coupled Networks: Nonlinear Wave Propagation A next step in investigating pattern formation in arrays of oscillators is to consider an array of them, say indexed by a spatial variable $x \in E^1$ or $x \in E^2$. We write the state variable at position x as being $\theta(x,t)$. We suppose here that the circuits are as in equation (1), but now coupled through inductors. Recall that if V is the voltage across an inductor and I is the current through it, then $V = L\dot{I}$ where L is a physical parameter called the inductance. In terms of the state variable, we have $\dot{\theta} = L\dot{I}$, so the current between sites is proportional to θ. A short calculation shows that the model is

$$\ddot{\theta} + F(\dot{\theta}) + p(x,t)\sin\theta = \omega(x) + \sigma\nabla^2\theta \tag{4}$$

where $\nabla^2\theta$ is the Laplacian in 1D or 2D and where σ is related to neighboring connection strength. This is a nonlinear wave equation which generalizes work on cellular nonlinear networks [Archibald, *et al.*., 2005; Yang, *et al.*., 2001].

Consider this equation on a line segment, or a square, respectively, with periodic boundary conditions. When $F \equiv 0$ and p and ω are constants, this equation is known as the sine-Gordon equation. It is known to support solitons in 1D [McLaughlin, 1978]. We show for the full model (4), which includes damping and external inputs, that there is an elaborate nonlinear wave structure of frequency synchronization. In particular, useful interference patterns can emerge among its solutions. In fact, the following simulation demonstrates an emergent pattern of waves and the eventual complexity of the pattern.

Some insight to this wave propagation is gained from the 1D case. By introducing the steady progressing wave coordinates $x \pm \sqrt{\sigma}t$ and writing $\theta(x,t) = \Theta(x \pm \sqrt{\sigma}t)$, the equation is reduced to a pendulum equation

$$\sigma\Theta'' + F(\Theta') + p\sin\Theta = \omega.$$

[See Hoppensteadt, 2006]. Depending the choices of the forcing constants p and ω, the pendulum can respond to an initial perturbation by returning to rest, or by executing one or a few full circle rotations, corresponding to isolated waves propagating out from the initial perturbation, or by executing self-sustained oscillations, corresponding to a wave train.

The following simulation is done for the equation

$$\ddot{\theta} + 0.1\,(\dot{\theta}^3 - \dot{\theta}) + \sin\theta = 1.0 + 0.001\,\nabla^2\theta \tag{5}$$

for $0 \leq x_j \leq 2\pi$, for $j = 1, 2$. Initial conditions are

$$\theta(x_1, x_2, 0) = \cos x_1 \cos x_2,$$

and the boundary conditions are periodic

$$\theta(x_1, x_2, t) \equiv \theta(x_1 + 2\pi, x_2, t) \equiv \theta(x_1, x_2 + 2\pi, t)$$

for all $t > 0$ and all $(x_1, x_2) \in E^2$.

The problem is solved on a square of side 2π, but the results are presented in Figure 2 in four identical contiguous squares.

The simulation in Figure 2 begins with three circular waves propagating out from a point. The wave fronts in four identical contiguous patches are shown in the top figure at $t = 60$. By time $t = 200$ a complex pattern of wave fronts has emerged. Note that since the voltage is constant along each wave front, the frequencies are identical along each wave front. As a result, the wave fronts represent waves of synchronized oscillator activity. The gray levels in these simulations indicate the size of the voltage, hence the firing frequency. The pattern in the bottom of Figure 2 continues to evolve in time - it is not a static form. In the terminology of the preceding section, these represent coalitions that propagate across the population of oscillators.

3 Summary We have demonstrated several important facts in this report. First, we derived a model of an APG in neurobiology from the principles of neuroscience, but in terms that are consistent with the fabrication of electronic circuits. Second, we demonstrated that useful pattern formation is possible in terms of voltages in totally disconnected aggregates of circuits that are driven by a common signal and that are aligned with a gradient of center frequencies (ω), which explains what coalitions are and provides one mechanism for their formation. Next, we derived an equation for nearest-neighbor coupled oscillators of the form in equation (1). We showed that when coupling is through inductors, the result is a nonlinear wave equation. Mathematical analysis of this proceeds by demonstrating the existence of stable steady progressing wave solutions of this equation. However, the waves can be isolated and the eventual wave patterns in bounded domains with periodic boundary conditions can become quite complex. The simulations of this nonlinear wave equation presented here demonstrate some aspects of this complexity and its sources.

We are ultimately interested in computational architectures that 1) are intrinsically parallel and usually asynchronous, 2) that can potentially implement algorithms like associate memory in natural and efficient ways, and 3) that can potentially be implemented on structureless systems, with coalitions forming dynamically. Here we addressed two mechanisms for coalition formation by patterns of phase locking, and we presented two interesting examples of them.

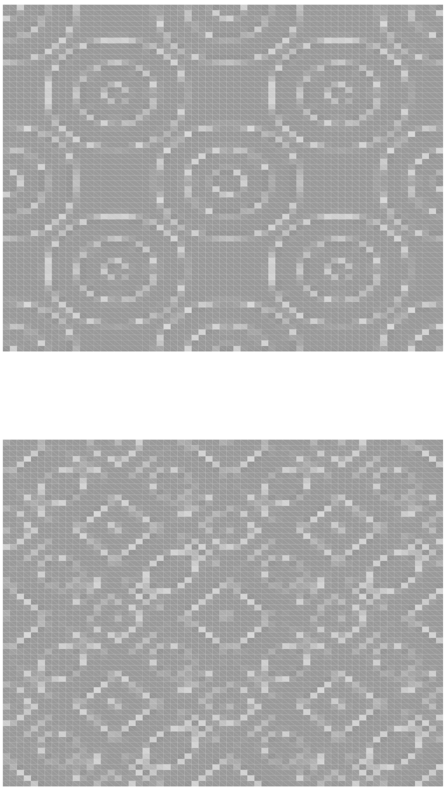

Figure 2: Output voltages of the array in (5). Top: $t = 60$. Three circular rings are wave fronts that are propagating outward from their centers. Bottom: $t = 200$. A snapshot of the emerging pattern synchronization.

Acknowledgement Supported in part by MARCO grant #2003-NT-1107. Presented to the Conference on Challenges in Mathematical Applications and Analysis at the University of Heidelberg, October 28, 2005.

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Frank Hoppensteadt, Kevin Lin
Courant Institute of Mathematical Sciences, New York University
70 Washington Square South, New York, NY 10012
E-mail: `frank.hoppensteadt@nyu.edu, klin@cims.nyu.edu`