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LATENT TRAITS OF STANDARD NOISES AND THEIR APPLICATIONS

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Abstract. There are many random phenomena such that their probability distributions are not Gaussian but other particular distributions with fat tail. They are the so-called fractional power distributions. We can see that their mathematical models can, in some favorable cases, be embedded in stable stochastic processes, which are expressed as superpositions of Poisson processes with various magnitudes of jump. Thus, our mathematical theory, which characterizes latent traits of Poisson noise, would effectively be applied to the random phenomena in question, in order to describe their biological characteristics.

1 Introduction We shall study effect of fluctuation or noise involved in biological phenomena, and more generally, in random complex systems.

If we are allowed to speak in an intuitive level, we may say as follows. Behaviors that we can observe in natural phenomena used to be controlled by some principles like symmetry, optimality and/or some other principles, in addition they are fluctuating. The nature is waiting for our approach so as those principles to be discovered, although they are often latent.

Probability theory and mathematical analysis can help us to carry on such trials of discovering. We are going to present a method which would be useful to study actual random phenomena in this line, noting that they are evolutional complex systems.

The goal of this paper is to investigate mathematical methods to discover latent traits of biological phenomena, which are random and have the probability distributions with fat tails.

The recipe of our approach is as follows:

1. Obtain characteristic properties of Poisson noise which will be useful to biological study. Those properties are revisited and need to be rediscovered.

2. Then, we can easily come to the investigation of stable process which can be expressed as a superposition of Poisson processes with various magnitudes of jump. There the way of superposing is determined by the given data from the phenomena in question.

3. Let those mathematical properties obtrained so far correspond to the biological (maybe latent) phenomena, respectively.

Regarding the study of Poisson noise, there were two motivations ; one was in quantum optics, where Poisson distribution appears in photon emissions and the other, to be more important, was in the interest in particular probability distributions with fat tail, which

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often appear in biological data (see e.g. [1]). We were therefore led to the study of characteristics of Poisson noise and compound Poisson noise that serve as basic random driving force, in addition to the study of the roles of Gaussian noise. Thus, we have a question on how to discover characteristics of Poisson noise so that we can give some interpretation to those random phenomena from the viewpoint of probability theory. For this purpose it is necessary to remind harmonic analysis arising from groups such as rotation group and symmetric group, in particular unitary representation of the groups. Representation of a rotation group is familiar to us in stochastic analysis, while we need some background for the theory of symmetric group.

It is noted that many formulas in statistics are based on Gaussian distribution. There are, however, many good applications which can be described in terms of stable distributions with exponent $\alpha < 2$. Appendix may help us when actual data are given.

2 Unitary representation of rotation group In this section we recall the unitary representations of rotation groups referring to ([3]).

Let μ be a white noise measure on the space E^* of generalized functions, and let $L^2(E^*, \mu) \equiv (L^2)$ be a complex Hilbert space. The space E is taken to be a real nuclear space and E^* is the dual space of E. Then, we have a Gel'fand triple :

$$(S) \subset (L^2) \subset (S)^*,$$

where (S) and (S^*) are spaces of test functionals and generalized white noise functionals, respectiveely.

It is well known that (Gaussian) white noise measure μ is invariant under the infinite dimensional rotation group $O^*(E^*) = g^*; g \in O(E)$, where $O(E)$ is the group of all rotations of a nuclear space E.

The Hilbert space (L^2) has a Fock space representation such that

$$(L^2) = \bigoplus_n \mathcal{H}_n.$$

For any $\varphi \in (L^2)$ and for $g \in O(E)$ define U_g by

$$(U_g \varphi)(x) = \varphi(g * x).$$

Then U_g is unitary on (L^2), and the collection $\mathbf{U} = \{U_g, g \in O(E)\}$ forms a group which is isomorphic to $O(E)$.

Theorem 1 *i)* $\{U_g, g \in O(E); (L^2)\}$ *is a unitary representation of the infinite dimensional rotation group* $O(E)$.

ii) The $\{\mathcal{H}_n\}$ *is an irreducible unitary representation.*

Theorem 1 suggests us to consider finite dimensional approximations (S^n, σ_n) to (E^*, μ), where S^n is the n-dimensional sphere and σ_n is the uniform probability measure on S^n. The σ_n is invariant under the rotation group $SO(n+1)$. Analogous approximation can be seen to a Poisson noise, but the structure is different.

3 Unitary representation of a symmetric group Let $S(n)$ be the symmetric group of order n. Unitary representations of the symmetric group $S(n)$ may be discussed in connection with Poisson noise.

Define $U_\pi, \pi \in S(n)$ on R^n by

$$U_\pi x = (x_{i_1}, x_{i_2}, \cdots, x_{i_n}), \tag{3.1}$$

where

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}, \tag{3.2}$$

and $x = (x_1, x_2, \cdots, x_n) \in R^n$. Since $\|x\| = \sqrt{\sum x_i^2} = \|U_\pi x\|$ and U_π is linear, U_π is a unitary representation of $S(n)$ on R^n.

Let $R^n = R^1 \oplus R^{n-1}$, where $R^1 = \{(x, x, \cdots, x)\}$. Then U_π is identity on R^1, that is, a trivial unitary representation, which is irreducible, although it looks very particular.

The representation can be lifted up to $L^2(R^n_+)$, where $R^n_+ = \{(x_1, \cdots, x_n); x_i \geq 0\}$. In fact, we may take a Sobolev space $H^{(n+1)/2}(R^n_+)$ of order n, instead of $L^2(R^n_+)$.

Define an operator V_π such that

$$(V_\pi f)(x) = f(x_{i_1}, x_{i_2}, \cdots, x_{i_n}), \tag{3.3}$$

where f is in $H^{(n+1)/2}(R^n_+)$.

Then V_π defines a linear isomorphism of $H^{(n+1)/2}(R^n_+)$ and keeps $H^{(n+1)/2}(R^n_+)$-norm invariant.

Consider a subspace H of $H^{(n+1)/2}(R^n_+)$ such that for any f in $H^{(n+1)/2}(R^n_+)$ there exists a function f^d on R_1^+ satisfying

$$f(x_1 + x_2 + \cdots + x_n) = f^d(x_1 + x_2 + \cdots + x_n).$$

Then, we can easily prove

Theorem 1 The V_π is restricted to H and the pair (H, V_π) is an irreducible rpresentation of $S(n)$.

4 An irreducible representation of infinite symmetric group The infinite symmetric group $S(\infty)$ is usually defined to be the inductive limit of $S(n)$. We shall, however, consider the inductive limit of the pair $S(n)$ and its irreducible representation. Hence, our proposal is as follows:

1. The Hilbert space on which the representation of $S(n)$ is a subspace of $L^2(R^n)$, where the Lebesgue measure is involved. For the infinite dimensional case, it is natural to replace the Lebesgue measure with the (Gaussian) white noise measure, denoted by μ. In other words, we use a Brownian motion $B(t)$ or the white noise $\dot{B}(t)$. The Hilbert space, where a representation is given, has to be a suitable subspace of (L^2).

2. With the choice of (L^2) we reformulate the irreducible representation of $S(n)$. In Section 3, the basic space is R_n^+, so that we have to take positive variables. The simplest choice is a family $\{(\Delta_k B)^2\}$, where $\{\Delta_k\}$ is a partition of $[0,1]$ with $|\Delta_k| = \frac{1}{n}$.

3. To fix the idea to take the inductive limit, we take partitions of $[0,1]$ to be $\Delta_{n,k}, 1 \leq k \leq 2^n,\ n = 1, 2, \cdots$, where $|\Delta_{n,k}| = 2^{-n}$. Actually the partition becomes finer as n increases. A representation of $S(2^n)$ is obtained by
$$\sum_k (\Delta_{n,k} B)^2.$$
It is known that this sum tends to 1 as $n \to \infty$, almost surely, by the theorem for the second variation of Brownian paths.

4. We wish to modify the above sum in the following form in order to have correct meaning in the limit as $n \to \infty$.
$$\sum_k \frac{:(\Delta_{n,k} B)^2:}{\Delta_{n,k}^2} \times \Delta_{n,k}^2,$$
where : : denotes the Wick product.

   The limit has a formal expression
$$\int_0^1 :\dot{B}(t)^2:(dt)^2.$$

5. The integrand in the above integral has meaning as a generalized white noise functional living in $(S)^*$, but $(dt)^2$ need a plausible interpretation. Changing a viewpoint, we remind that $:\dot{B}(t)^2 := (\partial_t^*)^2 1$, where ∂_t^* is the creation operator which is the adjoint of the differential operator
$$\partial_t = \frac{\partial}{\partial \dot{B}(t)}.$$

Theorem 2 *An irreducible representation of $S(\infty)$ is given by the following operator acting on $(S)^*$:*
$$\Delta_L^* = \int_0^t (\partial_t^*)^2 (dt)^2$$

Remark The reason why we use the notation Δ_L^* is that the above integral may be considered as the adjoint of the Lévy Laplacian which can be expressed as $\int (\partial_t)^2 (dt)^2$.

Detailed discussion will appear in the authors' forthcoming paper.

5 Lévy group and Lévy Laplacian Lévy group

Let π be a permutation defined by (3.2) by setting $n = N$ and let $\xi = \sum a_n \xi_n \in E$. The the Lévy group is defined by

$$\mathcal{G} = \{g_\pi; g_\pi \xi = a_n \xi_{\pi(n)}, g_\pi \in O_\infty, d(\pi) = 0\},$$

where $d(\pi) = \limsup \frac{1}{N} \#\{\pi(n) > n; n \leq N\}$, the density of π.

Obviously the symmetric group $S(n)$ is a subgroup of the Lévy group $\mathcal{G}$. The same for the inductive limit $S(\infty)$.

Theorem 3 *The group $S(\infty)$ is a normal subgroup of $\mathcal{G}$.*

Proof. For any $S(n)$ and for $g \in \mathcal{G}$, there exists an integer $k(\geq n)$ such that

$$g^{-1} S(n) g \subset S(k).$$

Letting $n \to \infty$, (hence $k \to \infty$,) we have

$$g^{-1} S(\infty) g = S(\infty).$$

Note that any $h \in S(\infty)$, we can easily see that the average power, a.p.$(h) = 0$. While, there exists many g's in $\mathcal{G}$ such that a.p.$(g) > 0$. In this sense, $\mathcal{G}$ is essentially infinite dimensional, but $S(\infty)$ is not.

Note. The symbol a.p. means the *average power*.

Lévy Laplacian and its eigen functionals

We keep a complete orthonormal sytem $\{\xi_n\}$ in $L^2([0,1])$ as before. The Lévy Laplacian acting on function space is defined by

$$\Delta_L = \lim \frac{1}{N} \sum_1^N \frac{\partial^2}{\partial \xi_n^2},$$

where $\frac{\partial}{\partial \xi}$ is the Fréchet derivative.

The following assertion is well known.

Proposition 1 *For any $g_\pi \in \mathcal{G}$, it holds that*

$$g_\pi \Delta_L = \Delta_L g_\pi.$$

Lévy Laplacian on $(S)^*$ and eigen functionals

There is the so-called S-transform that connects the space $(S)^*$ of generalized white noise functionals and function space $\mathcal{F}$. For $\varphi \in (S)^*$ define $(S\varphi)(\xi)$ by

$$(S\varphi)(\xi) = \exp[-\frac{1}{2}\|\xi\|^2] \int_{E^*} \exp[\langle x, \xi \rangle] \varphi(x) d\mu(x).$$

The space $\mathcal{F}$ is the image of $(S)^*$ under the S-transform and is topologized so as to be isomorphic to $(S)^*$.

The Laplacian acting on $(S)^*$ of genetralized white noise functionals is

$$S^{-1}\Delta_L S,$$

which is also denoted by Δ_L, if no confusion occurs.

Here is an example

$$\varphi(x) = \int_0^1 f(t) : \dot{B}(t)^2 : dt,\ f : smooth,$$

where we understand $x = \dot{B}$, since $x \in E^*$ is viewed as a sample function of $\dot{B}$. Then we have

$$\Delta_L \varphi = 2\int_0^1 f(t)dt.$$

If f is taken to be a constant 1, then we may see an interesting contrast between the above functional and the vector appeared in the unitary representation of $S(\infty)$.

The fact that was announced in Section 4 can be rephrased as a theorem.

Theorem 4 *The vector giving an irreducible representation of $S(\infty)$ defines a quadratic form of the creation operator which is in agreement with the Lévy Laplacian Δ_L in the sense that*

$$\left(\int_0^t (\partial_t^*)^2 (dt)^2\right)^* = \int_0^t \partial_t^2 (dt)^2 = \Delta_L.$$

6 Poisson noise We have a quick review of the paper [9] by one of the authors. We explain that the symmetric group either finite or infinite can describe the symmetry of Poisson noise. In addition, we can see below that the poissson noise has maximum information so far as the number of shots is limited.

Let the time parameter space still be kept as $I = [0, 1]$. In this case, the characteristic functional $C_P^I(\xi)$ of a Poisson noise is of the form (to make the matters simple we take the intensity $\lambda = 1$)

$$C_P^I(\xi) = \exp[\int_0^1 (e^{i\xi(u)} - 1)du.$$

It has the Taylor series expansion:

$$C_P^I(\xi) = \sum_0^\infty \frac{1}{n!} C_{P,n}(\xi),$$

where

$$C_{P,n}(\xi) = (\int_0^1 e^{i\xi(u)}du)^n.$$

We now have some observation. A Poisson measure is now introduced on the space E^*.

Define $P(t, x) = \langle x, \chi_{[0,t]}\rangle, 0 \le t \le 1,\ x \in E^*$, by a stochastic bilinear form, where χ is the indicator function. Then, $P(t, x)$ is a Poisson process with parameter set $[0, 1]$.

Let A_n be the event on which there are n jump points over the time interval I. That is

$$A_n = \{x \in E^*; P(1,x) = n\}, \tag{6.1}$$

where n is any non-negative integer.

Then, the collection $\{A_n, n \geq 0\}$ is a partition of the entire space E_1^*. Namely, up to measure 0, the following relations hold:

$$A_n \bigcap A_m = \phi,\ n \neq m;\ \bigcup A_n = E^*. \tag{6.2}$$

Given A_n, the conditional probability μ_P^n is defined :

$$\mu_P^n(A) = \frac{\mu_P(A_n \cap A)}{\mu_P(A_n)},\ A \subset E_1^*.$$

For $C \subset A_k$, the probability measure μ_P^k on a probability measure space $(A_k, \mathbf{B}_k, \mu_P^k)$, is such that

$$\mu_P^k(C) = \mu_P(C|A_k) = \frac{\mu_P(C)}{\mu_P(A_k)},$$

where $\mathbf{B}_k$ is the sigma field generated by measurable subsets of A_k, determined by $P(t,x)$.

Proposition 2 *(Ref. [9]) The conditional characteristic functional is*

$$E[e^{i\langle \dot{P},\xi\rangle}|A_n] = C_{P,n}(\xi) = \left(\int_0^1 e^{i\xi(t)}dt\right)^n. \tag{6.3}$$

Concerned with the following theorem we enjoyed conversation with Prof. K. Saitô.

Theorem 5 *The $C_{P,n}(\xi)$, expressed in* **??**, *is an eigen vector of Lévy Laplacian and the correspondibng eigen value is $-n$.*

Proof. The functional derivative of $C_{P,n}(\xi)$ is

$$\frac{\delta}{\delta\xi(t)} C_{P,n}(\xi) = ine^{i\xi(t)}C_{n-1}(\xi).$$

Again taking the functional derivative of $C_{P,n}(\xi)$, we have

$$\frac{\delta^2}{\delta\xi(t)^2} C_{P,n}(\xi) = -ne^{i\xi(t)}C_{n-1}(\xi) + n(n-1)(ie^{i\xi(t)})^2 C_{n-2}(\xi).$$

Then we have

$$\int \frac{\delta^2}{\delta\xi(t)^2} C_{P,n}(\xi)(dt)^2 = -nC_{P,n}(\xi).$$

Thus the assertion is proved.

We now see an interesting intrinsic properties of Poisson noise as well as good connection with (Gaussian) white noise.

7 Applications to Biology We are now ready to discuss applications of what we have discussed how to deal with biological data. If the data can be assumed to be Gaussian, then there are a lot of formulas and methods. However, we often meet fractional power distributions which is entirely different from Gaussian. A fractional power distribution means a probability distribution over R^1, the density function of which has tail of order $|x|^{\alpha+1}$, namely a *fat tail*. Good references are found in [1] and papers refered there.

Given a fractional power distribution as a statistics of a certain biological phenomenon, we may consider that the distribution belongs to the domain of attraction of a stable distribution (see Appendix 2)). Once a stable distribution is given, it can be considered as the probability distribution os a stable process at a certain instant t, although we need to check this is fitting: Namely additive property (independent increments property) and stationarity. In other words, we should check the possibility to be embedded in a stable process with stationary increments.

Once it is known that such embedding is acceptable, we should see the *self-similarity* as a characteristic of the given statistics. Then, we can proceed to the main assertion. Appealing to the Lévy-Itô decomposition of such a process, we can , theortically, decompose the process in question into elemetal Poisson processes with various magnitudes of jump. Thus, we can list all the characteristics of Poisson noise that we theoretically know, and we try to find if there can be corresponding properties of the actual biological phenomena.

Finally, we should try to discover the latent traits from the statistics of the given phenomena. In fact, not everything, but most significant properties would be discovered, we hope.

Appendix

1) Domain of attraction of Gaussian distribution:

$$\frac{K^2 \int_{|x|>K} dF(x)}{\int_{|x|<K} dF(x)} \ \rightarrow \ 0,$$

as K tends to ∞.

2) Domain of attraction of stable distribution with exponent α:

i)

$$\frac{F(-x)}{1-F(x)} \ \rightarrow \ c, \qquad (x \rightarrow \infty),$$

ii) for every positive constant k

$$\frac{1-F(x)+F(-x)}{1-F(kx)+F(-kx} \rightarrow k^{\alpha} \qquad (x \rightarrow \infty).$$

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