ON THE CONSTRUCTION OF FIRST-PASSAGE-TIME DENSITIES FOR DIFFUSION PROCESSES.

Virginia Giorno, Amelia G. Nobile, Enrica Pirozzi and Luigi M. Ricciardi

Received April 26, 2006

Abstract. A new method for constructing first-passage-time probability density functions in the case of regular one-dimensional time homogeneous diffusion processes restricted between constant boundaries is proposed. Some diffusion processes of particular interest in neuronal modeling are considered and thoroughly discusses.

1 Introduction

The determination of the first-passage-time (FPT) probability density function (pdf) for diffusion processes is known to play a relevant role for various biological systems modeling. For instance, when modeling neurons firing by means of diffusion processes, the FPT pdf represents the mathematical counterpart of the neuron recorded firing interspike histograms (cf., for instance, [11] and references therein). In this paper we propose a new method for constructing FPT pdf's in the case of time homogeneous diffusion processes restricted between constant boundaries S_1, S_2 $(S_1 < S_2)$, with *(a)* S_1 absorbing and S_2 elastic or *(b)* S_1 elastic and S_2 absorbing. The elastic boundary is assumed to be partially transparent in the sense that its behavior is intermediate between total absorption and total reflection (cf. [6]). The degree of elasticity of S_i $(i = 1, 2)$ is determined by two nonnegative parameters, say α_i (absorption coefficient) and β_i (reflection coefficient), with $\alpha_i + \beta_i > 0$. The extreme cases of a totally reflecting boundary S_i occurs for $\alpha_i = 0$, $\beta_i = 0$ determining instead total absorption at S_i $(i = 1, 2)$.

Our approach consists of providing a direct construction of FPT pdf's for a preassigned diffusion process in terms of predefined FPT pdf's of a known diffusion process, without using transition pdfs, thus neglecting the well known space transformations of Kolmogorov equations (cf., for instance, [1], [2], [3], [10]) and refrain from implementation of symmetry properties (cf., for instance, [5], [8], [9]).

Let $\{X(t), t \geq 0\}$ be a regular one-dimensional time-homogeneous diffusion process with drift $A_1(x)$ and infinitesimal variance $A_2(x)$ defined in $I = (r_1, r_2)$, with $P\{X(0) = y\} = 1$ and let

$$h(x) = \exp\Big\{-2\int^x \frac{A_1(z)}{A_2(z)}\,dz\Big\} \qquad (x \in I) \tag{1.1}$$

be the scale function of $X(t)$. Furthermore, let S_1, S_2 denote two arbitrary constant boundaries such that $S_1, S_2 \in I$ and $S_1 < y < S_2$.

Hereafter, the necessary preliminary background and notation is provided. We consider first the case of FPT through the lower boundary and then that of FPT through the upper boundary.

(a) FPT through the lower boundary

Assume that S_1 is an absorbing boundary and S_2 an elastic boundary. The degree of

2000 *Mathematics Subject Classification.* 60J60; 60J70; 92C20.

Key words and phrases. Diffusion processes, elastic boundaries, neuronal models.

elasticity of the boundary S_2 depends on the choice of the two nonnegative parameters, α_2 and β_2, with $\alpha_2 + \beta_2 > 0$, representing the absorption and the reflection coefficients, respectively. We define

$$\mathcal{T}^- = \inf_{t \geq 0} \{ t : X(t) < S_1,\ X(\vartheta) \leq S_2, \forall \vartheta \in (0,t) \}, \qquad X(0) = y \tag{1.2}$$

and denote by

$$g^-(S_1, S_2, t \mid y) = \frac{\partial}{\partial t} P(\mathcal{T}^- < t) \tag{1.3}$$

its pdf. Hence, $P(\mathcal{T}^- < t)$ is the probability that $X(t)$ crosses for the first time S_1 at some time preceding t before crossing S_2. Let

$$g_\lambda^-(S_1, S_2 \mid y) = \int_0^{+\infty} e^{-\lambda t}\, g^-(S_1, S_2, t \mid y)\, dt \tag{1.4}$$

be the Laplace transform (LT in the sequel) of $g^-(S_1, S_2, t \mid y)$. Due to the time homogeneity of the diffusion process under consideration and to the temporal independence of the boundaries, $g_\lambda^-(S_1, S_2 \mid y)$ is the solution of

$$A_1(y) \frac{dv_\lambda(y)}{dy} + \frac{1}{2} A_2(y) \frac{d^2 v_\lambda(y)}{dy^2} = \lambda\, v_\lambda(y) \qquad (S_1 < y < S_2) \tag{1.5}$$

with the conditions:

$$\lim_{y \downarrow S_1} v_\lambda(y) = 1, \qquad \lim_{y \uparrow S_2} \left\{ \alpha_2\, v_\lambda(y) + \beta_2\, h^{-1}(y) \frac{dv_\lambda(y)}{dy} \right\} = 0. \tag{1.6}$$

Even though the inverse LT of the function $g_\lambda^-(S_1, S_2 \mid y)$, obtained via Eq. (1.5) with conditions (1.6), cannot be explicitly obtained, nevertheless it provides useful information on the probability of ultimate FPT through the lower boundary. Indeed, by setting $\lambda = 0$ in (1.5) and (1.6), one obtains:

$$P^-(S_1, S_2 \mid y) := \int_0^{+\infty} g^-(S_1, S_2, t \mid y)\, dt = \frac{\beta_2 + \alpha_2 \displaystyle\int_y^{S_2} h(u)\, du}{\beta_2 + \alpha_2 \displaystyle\int_{S_1}^{S_2} h(u)\, du}. \tag{1.7}$$

Note that if r_2 is an inaccessible boundary, by setting $\beta_2 = 0$ and taking the limit as $S_2 \to r_2$ in (1.3) and in (1.7) one is led to the FPT pdf $g(S_1, t \mid y) := \partial P(T_1 < t)/\partial t$ and to the FPT probability $P(S_1 \mid y) := P(T_1 < +\infty)$, where

$$T_1 = \inf_{t \geq 0} \{ t : X(t) < S_1 \}, \qquad X(0) = y > S_1.$$

(b) FPT through the upper boundary
Assume now that S_1 is an elastic boundary and S_2 an absorbing boundary. The degree of elasticity of the boundary S_2 depends on the choice of the two nonnegative parameters, α_1 (absorbing coefficient) and β_1 (reflecting coefficient), with $\alpha_1 + \beta_1 > 0$. Denote

$$\mathcal{T}^+ = \inf_{t \geq 0} \{ t : X(t) > S_2,\ X(\vartheta) \geq S_1, \forall \vartheta \in (0,t) \}, \qquad X(0) = y \tag{1.8}$$

and

$$g^+(S_1,S_2,t \mid y)=\frac{\partial}{\partial t}P(\mathcal{T}^+<t) \tag{1.9}$$

its pdf. Hence, $P(\mathcal{T}^+<t)$ is the probability that $X(t)$ crosses for the first time S_2 at some time preceding t before crossing S_1. The LT

$$g^+_\lambda(S_1,S_2 \mid y)=\int_0^{+\infty} e^{-\lambda\,t}\,g^+(S_1,S_2,t \mid y)\,dt \tag{1.10}$$

of $g^+(S_1,S_2,t \mid y)$ can obtained as solution of (1.5) with the conditions

$$\lim_{y\downarrow S_1}\Big\{\alpha_1\,v_\lambda(y)-\beta_1\,h^{-1}(y)\,\frac{dv_\lambda(y)}{dy}\Big\}=0, \qquad \lim_{y\uparrow S_2} v_\lambda(y)=1. \tag{1.11}$$

Setting $\lambda=0$ in (1.5) and (1.11), the probability of ultimate FPT through the upper boundary is then given by:

$$P^+(S_1,S_2 \mid y):=\int_0^{+\infty} g^+(S_1,S_2,t \mid y)\,dt=\frac{\beta_1+\alpha_1\displaystyle\int_{S_1}^{y}h(u)\,du}{\beta_1+\alpha_1\displaystyle\int_{S_1}^{S_2}h(u)\,du}\,. \tag{1.12}$$

Note that if r_1 is inaccessible, setting $\beta_1=0$ and taking the limit as $S_1\to r_1$ in (1.9) and in (1.12), the FPT pdf $g(S_2,t \mid y):=\partial P(T_2<t)/\partial t$ and the FPT probability $P(S_2 \mid y):=P(T_2<+\infty)$ can be obtained, where T_2 denotes the random variable

$$T_2=\inf_{t\ge 0}\big\{t: X(t)>S_2\big\}, \qquad X(0)=y<S_2.$$

Example 1.1 Let $\{X(t),t\ge 0\}$ be a Wiener diffusion process with drift and infinitesimal variance:

$$A_1=\xi, \quad A_2=\omega^2 \qquad (\xi\in\mathbb{R},\ \omega>0) \tag{1.13}$$

respectively, defined in $\mathbb{R}$, with $P\{X(0)=y\}=1$. The scale function of $X(t)$ is:

$$h(x)=B\,\exp\Big\{-\frac{2\,\xi\,x}{\omega^2}\Big\} \qquad (B>0,\ x\in\mathbb{R}). \tag{1.14}$$

The general solution of (1.5) is

$$v_\lambda(y)=C_1\,\exp\Big\{y\,\vartheta_1(\lambda)\Big\}+C_2\,\exp\Big\{y\,\vartheta_2(\lambda)\Big\}, \tag{1.15}$$

with C_1,C_2 arbitrary real constants, and where $\vartheta_1(\lambda)$ and $\vartheta_2(\lambda)$ are the roots of the characteristic equation:

$$\omega^2\,\vartheta^2(\lambda)+2\,\xi\,\vartheta(\lambda)-2\,\lambda=0. \tag{1.16}$$

By imposing that (1.15) satisfies the boundaries conditions (1.6), the LT of the FPT pdf through S_1 in the presence of an elastic boundary in S_2 can be determined:

$$g^-_\lambda(S_1,S_2 \mid y)=\frac{e^{S_2\,\vartheta_2(\lambda)+y\,\vartheta_1(\lambda)}\,K_{2,2}(\lambda)-e^{S_2\,\vartheta_1(\lambda)+y\,\vartheta_2(\lambda)}\,K_{2,1}(\lambda)}{e^{S_2\,\vartheta_2(\lambda)+S_1\,\vartheta_1(\lambda)}\,K_{2,2}(\lambda)-e^{S_2\,\vartheta_1(\lambda)+S_1\,\vartheta_2(\lambda)}\,K_{2,1}(\lambda)}, \tag{1.17}$$

where

$$K_{i,j}(\lambda) = \alpha_i\, h(S_i) + \beta_i\, \vartheta_j(\lambda) \qquad (i,j = 1,2). \tag{1.18}$$

Furthermore, setting $\lambda = 0$ in (1.17), the probability of ultimate FPT through S_1 in the presence of an elastic boundary in S_2 is:

$$P^{-}(S_1, S_2 \mid y) = \begin{cases} \dfrac{\beta_2 + \alpha_2\, B\,(S_2 - y)}{\beta_2 + \alpha_2\, B\,(S_2 - S_1)}, & \xi = 0 \\[2ex] \dfrac{\beta_2 - \alpha_2\, \dfrac{\omega^2}{2\,\xi}\Big[h(S_2) - h(y)\Big]}{\beta_2 - \alpha_2\, \dfrac{\omega^2}{2\,\xi}\Big[h(S_2) - h(S_1)\Big]}, & \xi \neq 0. \end{cases} \tag{1.19}$$

In particular, if $\beta_2 = 0$ from (1.17) one has:

$$g_\lambda^{-}(S_1, S_2 \mid y) = \frac{e^{S_2\,\vartheta_2(\lambda) + y\,\vartheta_1(\lambda)} - e^{S_2\,\vartheta_1(\lambda) + y\,\vartheta_2(\lambda)}}{e^{S_2\,\vartheta_2(\lambda) + S_1\,\vartheta_1(\lambda)} - e^{S_2\,\vartheta_1(\lambda) + S_1\,\vartheta_2(\lambda)}}, \tag{1.20}$$

that identifies with the LT of the FPT pdf through S_1 in the presence of an absorbing boundary in S_2 (see, for instance, [4]). Instead, if $\alpha_2 = 0$, from (1.17) one obtains:

$$g_\lambda^{-}(S_1, S_2 \mid y) = \frac{\vartheta_2(\lambda)\, e^{S_2\,\vartheta_2(\lambda) + y\,\vartheta_1(\lambda)} - \vartheta_1(\lambda)\, e^{S_2\,\vartheta_1(\lambda) + y\,\vartheta_2(\lambda)}}{\vartheta_2(\lambda)\, e^{S_2\,\vartheta_2(\lambda) + S_1\,\vartheta_1(\lambda)} - \vartheta_1(\lambda)\, e^{S_2\,\vartheta_1(\lambda) + S_1\,\vartheta_2(\lambda)}}, \tag{1.21}$$

that identifies with the LT of the FPT pdf through S_1 in the presence of a reflecting boundary in S_2 (see, for instance, [4]).

Furthermore, by imposing that (1.15) satisfies (1.11), the LT of the FPT pdf through S_2 in the presence of an elastic boundary in S_1 is given by:

$$g_\lambda^{+}(S_1, S_2 \mid y) = \frac{e^{S_1\,\vartheta_1(\lambda) + y\,\vartheta_2(\lambda)}\, H_{1,1}(\lambda) - e^{S_1\,\vartheta_2(\lambda) + y\,\vartheta_1(\lambda)}\, H_{1,2}(\lambda)}{e^{S_1\,\vartheta_1(\lambda) + S_2\,\vartheta_2(\lambda)}\, H_{1,1}(\lambda) - e^{S_1\,\vartheta_2(\lambda) + S_2\,\vartheta_1(\lambda)}\, H_{1,2}(\lambda)}, \tag{1.22}$$

where

$$H_{i,j}(\lambda) = \alpha_i\, h(S_i) - \beta_i\, \vartheta_j(\lambda) \qquad (i,j = 1,2).$$

The probability of ultimate FPT through S_2 in the presence of an elastic boundary in S_1 is then obtained by setting $\lambda = 0$ in (1.22):

$$P^{+}(S_1, S_2 \mid y) = \begin{cases} \dfrac{\beta_1 + \alpha_1\, B\,(y - S_1)}{\beta_1 + \alpha_1\, B\,(S_2 - S_1)}, & \xi = 0 \\[2ex] \dfrac{\beta_1 - \alpha_1\, \dfrac{\omega^2}{2\,\xi}\Big[h(y) - h(S_1)\Big]}{\beta_1 - \alpha_1\, \dfrac{\omega^2}{2\,\xi}\Big[h(S_2) - h(S_1)\Big]}, & \xi \neq 0. \end{cases} \tag{1.23}$$

In particular, if $\beta_1 = 0$ from (1.22) one has:

$$g_\lambda^{+}(S_1, S_2 \mid y) = \frac{e^{S_1\,\vartheta_1(\lambda) + y\,\vartheta_2(\lambda)} - e^{S_1\,\vartheta_2(\lambda) + y\,\vartheta_1(\lambda)}}{e^{S_1\,\vartheta_1(\lambda) + S_2\,\vartheta_2(\lambda)} - e^{S_1\,\vartheta_2(\lambda) + S_2\,\vartheta_1(\lambda)}}, \tag{1.24}$$

that identifies with the LT of the FPT pdf through S_2 in the presence of an absorbing boundary in S_1 (see, for instance, [4]). Instead, if $\alpha_1 = 0$, from (1.22) one obtains:

$$g_\lambda^+(S_1, S_2 \mid y) = \frac{\vartheta_1(\lambda)\, e^{S_1\,\vartheta_1(\lambda)+y\,\vartheta_2(\lambda)} - \vartheta_2(\lambda)\, e^{S_1\,\vartheta_2(\lambda)+y\,\vartheta_1(\lambda)}}{\vartheta_1(\lambda)\, e^{S_1\,\vartheta_1(\lambda)+S_2\,\vartheta_1(\lambda)} - \vartheta_2(\lambda)\, e^{S_1\,\vartheta_2(\lambda)+S_2\,\vartheta_1(\lambda)}}, \tag{1.25}$$

that identifies with the LT of the FPT pdf through S_2 in the presence of a reflecting boundary in S_1.

Finally, taking the limit as $S_2 \to +\infty$ in (1.20) or taking the limit as $S_1 \to -\infty$ in (1.24) one obtains the well-known result:

$$g_\lambda(S \mid y) = \exp\Big\{\frac{\xi}{\omega^2}\,(S-y) - \frac{|S-y|}{\omega^2}\,\sqrt{\xi^2 - 2\,\sigma^2\,\lambda}\Big\} \qquad (S \neq y), \tag{1.26}$$

from which it follows:

$$g(S, t \mid y) = \frac{|S-y|}{\omega\,\sqrt{2\,\pi\,t^3}}\,\exp\Big\{-\frac{(S-y-\xi\,t)^2}{\omega^2\,t}\Big\} \qquad (S \neq y). \tag{1.27}$$

In Section 2 we will make use of the equation (1.5) with conditions (1.6) or (1.11) to provide a direct construction of FPT pdf's for a preassigned diffusion process in terms of predefined FPT pdf's of a known diffusion process. Finally, in Section 3 some diffusion processes of particular interest for neurons activity modeling are considered and thoroughly analyzed.

2 The construction of FPT pdf's

Let $\{X(t), t \geq 0\}$ be a regular one-dimensional time-homogeneous diffusion process with drift $A_1(x)$ and infinitesimal variance $A_2(x)$ defined in $I = (r_1, r_2)$, with $P\{X(0) = y\} = 1$ and let S_1, S_2 denote arbitrary constant boundaries such that $S_1, S_2 \in I$ and $S_1 < y < S_2$. The scale function of $X(t)$ is given in (1.1). As in Section 1, we denote by $g^-(S_1, S_2, t \mid y)$ and $P^-(S_1, S_2 \mid y)$ the FPT pdf and the probability of ultimate FPT through S_1 in the presence of the elastic boundary S_2 and by $g^+(S_1, S_2, t \mid y)$ and $P^+(S_1, S_2 \mid y)$ the FPT pdf and the probability of ultimate FPT through S_2 in the presence of the elastic boundary S_1.

Furthermore, let $\{\widehat{X}(t), t \geq 0\}$ be a regular one-dimensional time-homogeneous diffusion process with drift $\widehat{A}_1(x)$ and infinitesimal variance $\widehat{A}_2(x)$ defined in $\widehat{I} = (\widehat{r}_1, \widehat{r}_2)$, with $P\{\widehat{X}(0) = \widehat{y}\} = 1$, and let $\widehat{S}_1, \widehat{S}_2$ denote arbitrary constant boundaries such that $\widehat{S}_1, \widehat{S}_2 \in \widehat{I}$. The scale function of $\widehat{X}(t)$ is then:

$$\widehat{h}(x) = \exp\Big\{-2\int^x \frac{\widehat{A}_1(z)}{\widehat{A}_2(z)}\,dz\Big\} \qquad (x \in \widehat{I}). \tag{2.1}$$

We now denote by $\gamma^-(\widehat{S}_1, \widehat{S}_2, t \mid \widehat{y})$ and $Q^-(\widehat{S}_1, \widehat{S}_2 \mid \widehat{y})$ the FPT pdf and the probability of ultimate FPT through $\widehat{S}_1$ in the presence of the elastic boundary $\widehat{S}_2$, respectively, and by $\gamma^+(\widehat{S}_1, \widehat{S}_2, t \mid \widehat{y})$ and $Q^+(\widehat{S}_1, \widehat{S}_2 \mid \widehat{y})$ the FPT pdf and the probability of ultimate FPT through $\widehat{S}_2$ in the presence of the elastic boundary $\widehat{S}_1$, respectively.

As is well-known, if $X(t)$ and $\widehat{X}(t)$ are obtainable from one another by means of a strictly monotonic transformation, then the infinitesimal moments of the processes are mutually related, as shown in following Remark.

Remark 2.1 *Let $\widehat{X}(t)$ be a regular diffusion process defined in $\widehat{I}$ with drift $\widehat{A}_1(x)$ and infinitesimal variance $\widehat{A}_2(x)$ and let $\psi : I \to \widehat{I}$ be a strictly monotonic function such that*

$\psi(y) \in C^2(I)$. Then, $X(t) = \psi^{-1}[\widehat{X}(t)]$ defines a regular diffusion process defined in I with infinitesimal moments:

$$
\begin{aligned}
A_1(y) &= \Big(\frac{d\psi(y)}{dy}\Big)^{-1} \widehat{A}_1[\psi(y)] - \frac{1}{2}\Big(\frac{d\psi(y)}{dy}\Big)^{-3} \frac{d^2\psi(y)}{dy^2}\, \widehat{A}_2[\psi(y)] \\
A_2(y) &= \Big(\frac{d\psi(y)}{dy}\Big)^{-2} \widehat{A}_2[\psi(y)].
\end{aligned}
\tag{2.2}
$$

□

In Theorem 2.1 and Theorem 2.2 we shall consider a generalization of (2.2) that involves the probabilities of ultimate FPT of two regular diffusion processes $\widehat{X}(t)$ and $X(t)$, and determine some special functional relations among the FPT pdf's of such processes. We emphasize that this is accomplished without making use of the transition pdf's of the considered processes.

Theorem 2.1 *Let $\widehat{X}(t)$ and $X(t)$ be regular diffusion processes with infinitesimal moments $\widehat{A}_i(x)$ $(i = 1, 2)$ and $A_i(x)$ $(i = 1, 2)$ defined in $\widehat{I}$ and I, respectively. Furthermore, let $\psi : I \to \widehat{I}$ be a strictly increasing function such that $\psi(y) \in C^2(I)$.*

(i) If

$$
\begin{aligned}
A_1(y) &= \Big(\frac{d\psi(y)}{dy}\Big)^{-1} \widehat{A}_1[\psi(y)] - \Big(\frac{d\psi(y)}{dy}\Big)^{-2} \widehat{A}_2[\psi(y)] \left[\frac{1}{\varphi(y)}\frac{d\varphi(y)}{dy} + \frac{1}{2}\Big(\frac{d\psi(y)}{dy}\Big)^{-1} \frac{d^2\psi(y)}{dy^2}\right] \\
A_2(y) &= \Big(\frac{d\psi(y)}{dy}\Big)^{-2} \widehat{A}_2[\psi(y)],
\end{aligned}
\tag{2.3}
$$

with

$$
\varphi(y) = \frac{P^-(S_1, S_2 \mid y)}{Q^-[\psi(S_1), \psi(S_2) \mid \psi(y)]}\,, \tag{2.4}
$$

then

$$
g^-(S_1, S_2, t \mid y) = \varphi(y)\, \gamma^-[\psi(S_1), \psi(S_2), t \mid \psi(y)] \qquad (S_1 < y < S_2). \tag{2.5}
$$

(ii) If relations (2.3) hold with

$$
\varphi(y) = \frac{P^+(S_1, S_2 \mid y)}{Q^+[\psi(S_1), \psi(S_2) \mid \psi(y)]}\,, \tag{2.6}
$$

then

$$
g^+(S_1, S_2, t \mid y) = \varphi(y)\, \gamma^+[\psi(S_1), \psi(S_2), t \mid \psi(y)] \qquad (S_1 < y < S_2). \tag{2.7}
$$

Proof. We set $\widehat{S}_1 = \psi(S_1)$, $\widehat{S}_2 = \psi(S_2)$ and $\widehat{y} = \psi(y)$. Since $S_1 < y < S_2$ and $\psi(y)$ is a strictly increasing function, one has $\widehat{S}_1 < \widehat{y} < \widehat{S}_2$.

We now prove separately the cases *(i)* and *(ii)*.

Case (i) For simplicity, we set

$$
\begin{aligned}
u_\lambda(\widehat{y}) &:= \gamma_\lambda^-(\widehat{S}_1, \widehat{S}_2 \mid \widehat{y}), \qquad v_\lambda(y) := g_\lambda^-(S_1, S_2 \mid y) \\
u_0(\widehat{y}) &:= Q^-(\widehat{S}_1, \widehat{S}_2 \mid \widehat{y}), \qquad v_0(y) := P^-(S_1, S_2 \mid y).
\end{aligned}
\tag{2.8}
$$

Note that $v_\lambda(y)$ is solution of (1.5) and (1.6) while $v_0(y)$ is given in (1.7). Instead, the LT $u_\lambda(\widehat{y})$ is solution of the differential equation

$$\widehat{A}_1(\widehat{y})\,\frac{d}{d\widehat{y}}u_\lambda(\widehat{y})+\frac{1}{2}\,\widehat{A}_2(\widehat{y})\,\frac{d^2}{d\widehat{y}^2}u_\lambda(\widehat{y})=\lambda\,u_\lambda(\widehat{y}) \tag{2.9}$$

with the conditions:

$$\lim_{\widehat{y}\downarrow\widehat{S}_1}u_\lambda(\widehat{y})=1,\qquad \lim_{\widehat{y}\uparrow\widehat{S}_2}\left\{\alpha_2\,u_\lambda(\widehat{y})+\beta_2\,\widehat{h}^{-1}(\widehat{y})\,\frac{d}{d\widehat{y}}u_\lambda(\widehat{y})\right\}=0, \tag{2.10}$$

where $\widehat{h}(x)$ is given in (2.1). Furthermore, one has

$$u_0(\widehat{y})=\frac{\beta_2+\alpha_2\displaystyle\int_{\widehat{y}}^{\widehat{S}_2}\widehat{h}(u)\,du}{\beta_2+\alpha_2\displaystyle\int_{\widehat{S}_1}^{\widehat{S}_2}\widehat{h}(u)\,du}\;\cdot \tag{2.11}$$

We shall now prove that if the drifts and the infinitesimal variances of $X(t)$ and $\widehat{X}(t)$ satisfy (2.3) with

$$\varphi(y):=\frac{v_0(y)}{u_0(\widehat{y})}=\frac{P^-(S_1,S_2\mid y)}{Q^-(\widehat{S}_1,\widehat{S}_2\mid\widehat{y})}\,, \tag{2.12}$$

then $v^*_\lambda(y):=\varphi(y)\;u_\lambda(\widehat{y})$, i.e. the right-hand-side of (2.5), is solution of Eq. (1.5) with the conditions (1.6). Since

$$\begin{aligned}
\frac{dv^*_\lambda(y)}{dy}&=\frac{d\varphi(y)}{dy}\,u_\lambda(\widehat{y})+\varphi(y)\,\frac{du_\lambda(\widehat{y})}{d\widehat{y}}\,\frac{d\psi(y)}{dy}\,,\\
\frac{d^2v^*_\lambda(y)}{dy^2}&=\frac{d^2\varphi(y)}{dy^2}\,u_\lambda(\widehat{y})+2\,\frac{d\varphi(y)}{dy}\,\frac{du_\lambda(\widehat{y})}{d\widehat{y}}\,\frac{d\psi(y)}{dy}+\varphi(y)\,\frac{d^2u_\lambda(\widehat{y})}{d\widehat{y}^2}\,\Big(\frac{d\psi(y)}{dy}\Big)^2\\
&\quad+\varphi(y)\,\frac{du_\lambda(\widehat{y})}{d\widehat{y}}\,\frac{d^2\psi(y)}{dy^2}\,,
\end{aligned}$$

one obtains:

$$\begin{aligned}
&A_1(y)\,\frac{d}{dy}v^*_\lambda(y)+\frac{1}{2}\,A_2(y)\,\frac{d^2}{dy^2}v^*_\lambda(y)-\lambda\,v^*_\lambda(y)\\
&\quad=\varphi(y)\bigg\{\Big[A_1(y)\,\frac{1}{\varphi(y)}\,\frac{d\varphi(y)}{dy}+\frac{1}{2}\,A_2(y)\,\frac{1}{\varphi(y)}\,\frac{d^2\varphi(y)}{dy^2}-\lambda\Big]\,u_\lambda(\widehat{y})\\
&\qquad+\Big[A_1(y)\,\frac{d\psi(y)}{dy}+A_2(y)\,\frac{1}{\varphi(y)}\,\frac{d\varphi(y)}{dy}\,\frac{d\psi(y)}{dy}+\frac{1}{2}\,A_2(y)\,\frac{d^2\psi(y)}{dy^2}\Big]\,\frac{du_\lambda(\widehat{y})}{d\widehat{y}}\\
&\qquad+\frac{1}{2}\,A_2(y)\,\Big(\frac{d\psi(y)}{dy}\Big)^2\,\frac{d^2u_\lambda(\widehat{y})}{d\widehat{y}^2}\bigg\}.
\end{aligned} \tag{2.13}$$

Making use of (1.7) and (2.11) one can prove that (2.12) is solution of

$$A_1(y)\,\frac{d\varphi(y)}{dy}+\frac{1}{2}\,A_2(y)\,\frac{d^2\varphi(y)}{dy^2}=0. \tag{2.14}$$

Furthermore, recalling (2.3) one has:

$$
\begin{aligned}
&A_2(y)\Big(\frac{d\psi(y)}{dy}\Big)^2 = \widehat{A}_2(\widehat{y}),\\
&A_1(y)\,\frac{d\psi(y)}{dy} + A_2(y)\,\frac{1}{\varphi(y)}\,\frac{d\varphi(y)}{dy}\,\frac{d\psi(y)}{dy} + \frac{1}{2}\,A_2(y)\,\frac{d^2\psi(y)}{dy^2} = \widehat{A}_1(\widehat{y}).
\end{aligned}
\tag{2.15}
$$

By virtue of (2.14) and (2.15), and due (2.9), Eq. (2.13) yields:

$$
\begin{aligned}
&A_1(y)\,\frac{d}{dy}v_\lambda^*(y) + \frac{1}{2}\,A_2(y)\,\frac{d^2}{dy^2}v_\lambda^*(y) - \lambda\, v_\lambda^*(y)\\
&\qquad = \varphi(y)\bigg\{\widehat{A}_1(\widehat{y})\,\frac{du_\lambda(\widehat{y})}{d\widehat{y}} + \frac{1}{2}\,\widehat{A}_2(\widehat{y})\,\frac{d^2u_\lambda(\widehat{y})}{d\widehat{y}^2} - \lambda\, u_\lambda(\widehat{y})\bigg\} = 0.
\end{aligned}
\tag{2.16}
$$

Hence, $v_\lambda^*(y)$ is solution of (1.5).
We now prove that $v_\lambda^*(y)$ satisfies the boundaries conditions (1.6). To this purpose, recalling (1.7), (2.8) and (2.11), we have

$$
\begin{aligned}
&\lim_{y\downarrow S_1} v_0(y) = 1, \qquad \lim_{y\uparrow S_2} v_0(y) = \frac{\beta_2}{\beta_2 + \alpha_2\displaystyle\int_{S_1}^{S_2} h(z)\,dz},\\
&\lim_{y\downarrow S_1} u_0(\widehat{y}) = 1 \qquad \lim_{y\uparrow S_2} u_0(\widehat{y}) = \frac{\beta_2}{\beta_2 + \alpha_2\displaystyle\int_{\widehat{S}_1}^{\widehat{S}_2} \widehat{h}(z)\,dz},
\end{aligned}
\tag{2.17}
$$

so that from (2.12) it follows:

$$
\lim_{y\downarrow S_1} \varphi(y) = 1, \qquad \lim_{y\uparrow S_2} \varphi(y) = \frac{\beta_2 + \alpha_2\displaystyle\int_{\widehat{S}_1}^{\widehat{S}_2} \widehat{h}(z)\,dz}{\beta_2 + \alpha_2\displaystyle\int_{S_1}^{S_2} h(z)\,dz}\,\cdot
\tag{2.18}
$$

Hence, recalling the first of (2.10), $v_\lambda^*(y)$ and (2.18), one has:

$$
\lim_{y\downarrow S_1} v_\lambda^*(y) \equiv \lim_{y\downarrow S_1}\Big[\varphi(y)\,u_\lambda(\widehat{y})\Big] = \lim_{\widehat{y}\downarrow \widehat{S}_1} u_\lambda(\widehat{y}) = 1,
\tag{2.19}
$$

i.e. $v_\lambda^*(y)$ satisfies the first of (1.6). To prove that $v_\lambda^*(y)$ satisfies the second of (1.6), we consider the following two cases: *(1)* $\beta_2 = 0$ and *(2)* $\beta_2 \neq 0$.
(1) If $\beta_2 = 0$, i.e. S_2 and $\widehat{S}_2$ are absorbing boundaries, then

$$
\lim_{y\uparrow S_2}\Big\{\alpha_2\, v_\lambda^*(y) + \beta_2\, h^{-1}(y)\,\frac{d}{dy}v_\lambda^*(y)\Big\} = \alpha_2\,\varphi(S_2)\,\lim_{\widehat{y}\uparrow \widehat{S}_2} u_\lambda(\widehat{y}) = 0
\tag{2.20}
$$

where the last equality follows by virtue of the second of (2.10).
(2) If $\beta_2 \neq 0$, i.e. S_2 and $\widehat{S}_2$ are reflecting or full elastic boundaries, so that

$$
\begin{aligned}
&\lim_{y\uparrow S_2}\Big\{\alpha_2\, v_\lambda^*(y) + \beta_2\, h^{-1}(y)\,\frac{d}{dy}v_\lambda^*(y)\Big\}\\
&\qquad = \varphi(S_2)\,\frac{\widehat{h}(\widehat{S}_2)}{h(S_2)}\,\frac{d\psi(y)}{dy}\bigg|_{y=S_2}\,\lim_{\widehat{y}\uparrow \widehat{S}_2}\bigg\{\alpha_2\, u_\lambda(\widehat{y}) + \beta_2\,\widehat{h}^{-1}(\widehat{y})\,\frac{du_\lambda(\widehat{y})}{d\widehat{y}}\bigg\} = 0
\end{aligned}
$$

the vanishing of the right hand side being a consequence of the second of (2.10). Hence, $v^*_\lambda(y) \equiv g^-_\lambda(S_1, S_2 \mid y)$ and thus (2.5) holds.

Case (ii) We now set

$$
\begin{aligned}
&u_\lambda(\widehat{y}) = \gamma^+_\lambda(\widehat{S}_1, \widehat{S}_2 \mid \widehat{y}), \qquad v_\lambda(y) = g^+_\lambda(S_1, S_2 \mid y)\\
&u_0(\widehat{y}) = Q^+(\widehat{S}_1, \widehat{S}_2 \mid \widehat{y}), \qquad v_0(y) = P^+(S_1, S_2 \mid y).
\end{aligned}
\tag{2.21}
$$

We recall that the LT $v_\lambda(y)$ is solution of (1.5) with the conditions (1.11) and that $v_0(y)$ is given in (1.12). Instead, the LT $u_\lambda(\widehat{y})$ is solution of the differential equation (2.9) with the conditions:

$$
\lim_{\widehat{y}\downarrow \widehat{S}_1} \left\{ \alpha_1\, u_\lambda(\widehat{y}) - \beta_1\, \widehat{h}^{-1}(\widehat{y})\, \frac{d}{d\widehat{y}} u_\lambda(\widehat{y}) \right\} = 0, \qquad \lim_{\widehat{y}\uparrow \widehat{S}_2} u_\lambda(\widehat{y}) = 1,
\tag{2.22}
$$

with $\widehat{h}(x)$ given in (2.1). Furthermore, one has:

$$
u_0(\widehat{y}) = \frac{\beta_1 + \alpha_1 \displaystyle\int_{\widehat{S}_1}^{\widehat{y}} \widehat{h}(u)\, du}{\beta_1 + \alpha_1 \displaystyle\int_{\widehat{S}_1}^{\widehat{S}_2} \widehat{h}(u)\, du}\,.
\tag{2.23}
$$

We shall now prove that if the drifts and the infinitesimal variances of $X(t)$ and $\widehat{X}(t)$ satisfy (2.3) with

$$
\varphi(y) := \frac{v_0(y)}{u_0(\widehat{y})} = \frac{P^+(S_1, S_2 \mid y)}{Q^+(\widehat{S}_1, \widehat{S}_2 \mid \widehat{y})},
\tag{2.24}
$$

then $v^*_\lambda(y) := \varphi(y)\ u_\lambda(\widehat{y})$, i.e. the right-hand-side of (2.7), is solution of Eq. (1.5) with the conditions (1.11). Similarly to case *(i)*, it is easily seen that $v^*_\lambda(y)$ satisfies Eq. (1.5). We now prove that conditions (1.11) hold for $v^*_\lambda(y)$. To this purpose, we recall (1.12), (2.21) and (2.23), to note that

$$
\begin{aligned}
&\lim_{y\downarrow S_1} v_0(y) = \frac{\beta_1}{\beta_1 + \alpha_1 \displaystyle\int_{S_1}^{S_2} h(z)\, dz}\,, \qquad \lim_{y\uparrow S_2} v_0(y) = 1,\\
&\lim_{y\downarrow S_1} u_0(\widehat{y}) = \frac{\beta_1}{\beta_1 + \alpha_1 \displaystyle\int_{\widehat{S}_1}^{\widehat{S}_2} \widehat{h}(z)\, dz}\,, \qquad \lim_{y\uparrow S_2} u_0(\widehat{y}) = 1,
\end{aligned}
\tag{2.25}
$$

so that from (2.24) it follows:

$$
\lim_{y\downarrow S_1} \varphi(y) = \frac{\beta_1 + \alpha_1 \displaystyle\int_{\widehat{S}_1}^{\widehat{S}_2} \widehat{h}(z)\, dz}{\beta_1 + \alpha_1 \displaystyle\int_{S_1}^{S_2} h(z)\, dz}\,, \qquad \lim_{y\uparrow S_2} \varphi(y) = 1.
\tag{2.26}
$$

To prove that $v^*_\lambda(y)$ satisfies the first of (1.11), we consider the following two cases: *(1)* $\beta_1 = 0$ and *(2)* $\beta_1 \neq 0$.
(1) If $\beta_1 = 0$, i.e. S_1 and $\widehat{S}_1$ are absorbing boundaries, then

$$\lim_{y \downarrow S_1} \Big\{ \alpha_1\, v^*_\lambda(y) - \beta_1\, h^{-1}(y)\, \frac{d}{dy} v^*_\lambda(y) \Big\} = \alpha_1\, \varphi(S_1) \lim_{\widehat{y} \downarrow \widehat{S}_1} u_\lambda(\widehat{y}) = 0 \tag{2.27}$$

where the last equality follows by virtue of the first of (2.22).
(2) If $\beta_1 \neq 0$, i.e. S_1 and $\widehat{S}_1$ are reflecting or full elastic boundaries, then

$$\lim_{y \downarrow S_1} \Big\{ \alpha_1\, v^*_\lambda(y) - \beta_1\, h^{-1}(y)\, \frac{d}{dy} v^*_\lambda(y) \Big\}$$
$$= \varphi(S_1)\, \frac{\widehat{h}(\widehat{S}_1)}{h(S_1)}\, \frac{d\psi(y)}{dy}\bigg|_{y=S_1} \lim_{\widehat{y} \downarrow \widehat{S}_1} \Big\{ \alpha_1\, u_\lambda(\widehat{y}) - \beta_1\, \widehat{h}^{-1}(\widehat{y})\, \frac{du_\lambda(\widehat{y})}{d\widehat{y}} \Big\}$$

that vanishes by virtue of the first of (2.22), so that $v^*_\lambda(y)$ satisfies the first of (1.11). Finally, recalling the second of (2.22) and (2.26), one has:

$$\lim_{y \uparrow S_2} v^*_\lambda(y) \equiv \lim_{y \uparrow S_2} \Big[\varphi(y)\, u_\lambda(\widehat{y}) \Big] = \lim_{\widehat{y} \uparrow \widehat{S}_2} u_\lambda(\widehat{y}) = 1, \tag{2.28}$$

i.e. $v^*_\lambda(y)$ also satisfies the second of (1.11). In conclusion, $v^*_\lambda(y) \equiv g^+_\lambda(S_1, S_2 \mid y)$ and thus (2.7) holds. □

Theorem 2.2 *Let $\widehat{X}(t)$ and $X(t)$ be regular diffusion processes with infinitesimal moments $\widehat{A}_i(x)$ $(i = 1, 2)$ and $A_i(x)$ $(i = 1, 2)$ defined in $\widehat{I}$ and I, respectively. Furthermore, let $\psi : I \to \widehat{I}$ be a strictly decreasing function such that $\psi(y) \in C^2(I)$.*

(i) If relations (2.3) hold with

$$\varphi(y) = \frac{P^-(S_1, S_2 \mid y)}{Q^+[\psi(S_2), \psi(S_1) \mid \psi(y)]}, \tag{2.29}$$

then

$$g^-(S_1, S_2, t \mid y) = \varphi(y)\, \gamma^+[\psi(S_2), \psi(S_1), t \mid \psi(y)] \qquad (S_1 < y < S_2). \tag{2.30}$$

(ii) If relations (2.3) hold with

$$\varphi(y) = \frac{P^+(S_1, S_2 \mid y)}{Q^-[\psi(S_2), \psi(S_1) \mid \psi(y)]}, \tag{2.31}$$

then

$$g^+(S_1, S_2, t \mid y) = \varphi(y)\, \gamma^-[\psi(S_2), \psi(S_1), t \mid \psi(y)] \qquad (S_1 < y < S_2). \tag{2.32}$$

Proof. We set $\widehat{S}_1 = \psi(S_1)$, $\widehat{S}_2 = \psi(S_2)$ and $\widehat{y} = \psi(y)$. Since $S_1 < y < S_2$ and $\psi(y)$ is a strictly decreasing function, one has $\widehat{S}_2 < \widehat{y} < \widehat{S}_1$.

We consider separately the cases *(i)* and *(ii)*.

Case (i) For simplicity, we set

$$\begin{aligned} &u_\lambda(\widehat{y}) = \gamma^+_\lambda(\widehat{S}_2, \widehat{S}_1 \mid \widehat{y}), \qquad v_\lambda(y) = g^-_\lambda(S_1, S_2 \mid y) \\ &u_0(\widehat{y}) = Q^+(\widehat{S}_2, \widehat{S}_1 \mid \widehat{y}), \qquad v_0(y) = P^-(S_1, S_2 \mid y). \end{aligned} \tag{2.33}$$

We note that $v_\lambda(y)$ is solution of (1.5) with the conditions (1.6) and that $v_0(y)$ is given in (1.7). Instead, $u_\lambda(\widehat{y})$ is solution of (2.9) with the conditions:

$$\lim_{\widehat{y}\downarrow \widehat{S}_2}\Big\{\alpha_2\, u_\lambda(\widehat{y})-\beta_2\,\widehat{h}^{-1}(\widehat{y})\,\frac{d}{d\widehat{y}}u_\lambda(\widehat{y})\Big\}=0,\qquad \lim_{\widehat{y}\uparrow \widehat{S}_1} u_\lambda(\widehat{y})=1, \tag{2.34}$$

where $\widehat{h}(x)$ is given in (2.1). Furthermore, one has:

$$u_0(\widehat{y})=\frac{\beta_2+\alpha_2\displaystyle\int_{\widehat{S}_2}^{\widehat{y}}\widehat{h}(u)\,du}{\beta_2+\alpha_2\displaystyle\int_{\widehat{S}_2}^{\widehat{S}_1}\widehat{h}(u)\,du}\,. \tag{2.35}$$

We shall now prove that if the drifts and the infinitesimal variances of $X(t)$ and $\widehat{X}(t)$ satisfy (2.3) and (2.29), with

$$\varphi(y):=\frac{v_0(y)}{u_0(\widehat{y})}=\frac{P^-(S_1,S_2\mid y)}{Q^+(\widehat{S}_2,\widehat{S}_1\mid \widehat{y})} \tag{2.36}$$

then $v^*_\lambda(y):=\varphi(y)\;u_\lambda(\widehat{y})$, i.e. the right-hand-side of (2.30), is solution of Eq. (1.5) with the conditions (1.6). Proceeding as in the proof of Theorem 2.1 one can see that $v^*_\lambda(y)$ satisfies Eq. (1.5). We now prove that conditions (1.6) hold for $v^*_\lambda(y)$. To this purpose, recalling (1.7), (2.33) and (2.35), we note that

$$\begin{aligned}&\lim_{y\downarrow S_1} v_0(y)=1,\qquad \lim_{y\uparrow S_2} v_0(y)=\frac{\beta_2}{\beta_2+\alpha_2\displaystyle\int_{S_1}^{S_2}h(z)\,dz}\,,\\ &\lim_{y\downarrow S_1} u_0(\widehat{y})=1,\qquad \lim_{y\uparrow S_2} u_0(\widehat{y})=\frac{\beta_2}{\beta_2+\alpha_2\displaystyle\int_{\widehat{S}_2}^{\widehat{S}_1}\widehat{h}(z)\,dz}\,.\end{aligned} \tag{2.37}$$

Hence, from (2.36) it follows:

$$\lim_{y\downarrow S_1}\varphi(y)=1,\qquad \lim_{y\uparrow S_2}\varphi(y)=\frac{\beta_2+\alpha_2\displaystyle\int_{\widehat{S}_2}^{\widehat{S}_1}\widehat{h}(z)\,dz}{\beta_2+\alpha_2\displaystyle\int_{S_1}^{S_2}h(z)\,dz}\,. \tag{2.38}$$

Hence, recalling the second of (2.34) and (2.38), one has:

$$\lim_{y\downarrow S_1} v^*_\lambda(y)\equiv\lim_{y\downarrow S_1}\Big[\varphi(y)\,u_\lambda(\widehat{y})\Big]=\lim_{\widehat{y}\uparrow\widehat{S}_1}u_\lambda(\widehat{y})=1, \tag{2.39}$$

so that $v^*_\lambda(y)$ satisfies the first of (1.6). To prove that the second of (1.6) holds, we consider the following two cases: *(1)* $\beta_2=0$ and *(2)* $\beta_2\neq 0$.

(1) If $\beta_2=0$, i.e. S_2 and $\widehat{S}_2$ are absorbing boundaries, then,

$$\lim_{y\uparrow S_2}\Big\{\alpha_2\,v^*_\lambda(y)+\beta_2\,h^{-1}(y)\,\frac{d}{dy}v^*_\lambda(y)\Big\}=\alpha_2\,\varphi(S_2)\,\lim_{\widehat{y}\downarrow\widehat{S}_2}u_\lambda(\widehat{y})=0 \tag{2.40}$$

where the last equality follows by virtue of the first of (2.34).
(2) If $\beta_2 \neq 0$, i.e. S_2 and $\widehat{S}_2$ are reflecting or full elastic boundaries, then,

$$\lim_{y\uparrow S_2}\Big\{\alpha_2\, v^*_\lambda(y)+\beta_2\, h^{-1}(y)\,\frac{d}{dy}v^*_\lambda(y)\Big\}$$
$$=-\varphi(S_2)\,\frac{\widehat{h}(\widehat{S}_2)}{h(S_2)}\,\frac{d\psi(y)}{dy}\bigg|_{y=S_2}\lim_{\widehat{y}\downarrow\widehat{S}_2}\Big\{\alpha_2\, u_\lambda(\widehat{y})-\beta_2\,\widehat{h}^{-1}(\widehat{y})\,\frac{du_\lambda(\widehat{y})}{d\widehat{y}}\Big\}$$

that vanishes by virtue of the first of (2.34), i.e. $v^*_\lambda(y)$ satisfies the second of (1.6). Hence, $v^*_\lambda(y)\equiv g^-_\lambda(S_1,S_2\mid y)$ and thus (2.30) holds.

Case (ii) We now set

$$\begin{aligned} &u_\lambda(\widehat{y})=\gamma^-_\lambda(\widehat{S}_2,\widehat{S}_1\mid\widehat{y}), \qquad v_\lambda(y)=g^+_\lambda(S_1,S_2\mid y)\\ &u_0(\widehat{y})=Q^-(\widehat{S}_2,\widehat{S}_1\mid\widehat{y}), \qquad v_0(y)=P^+(S_1,S_2\mid y). \end{aligned} \tag{2.41}$$

We recall that the LT $v_\lambda(y)$ is solution of the differential equation (1.5) with the conditions (1.11) and that $v_0(y)$ is given in (1.12). Instead, the LT $u_\lambda(\widehat{y})$ is solution of the differential equation (2.9) with the conditions:

$$\lim_{\widehat{y}\downarrow\widehat{S}_2}u_\lambda(\widehat{y})=1, \qquad \lim_{\widehat{y}\uparrow\widehat{S}_1}\Big\{\alpha_1\, u_\lambda(\widehat{y})+\beta_1\,\widehat{h}^{-1}(\widehat{y})\,\frac{d}{d\widehat{y}}u_\lambda(\widehat{y})\Big\}=0, \tag{2.42}$$

with $\widehat{h}(x)$ given in (2.1). Furthermore, one has:

$$u_0(\widehat{y})=\frac{\beta_1+\alpha_1\displaystyle\int_{\widehat{y}}^{\widehat{S}_1}\widehat{h}(u)\,du}{\beta_1+\alpha_1\displaystyle\int_{\widehat{S}_2}^{\widehat{S}_1}\widehat{h}(u)\,du}\,. \tag{2.43}$$

We shall now prove that if the drifts and the infinitesimal variances of $X(t)$ and $\widehat{X}(t)$ satisfy (2.3) and (2.31), with

$$\varphi(y):=\frac{v_0(y)}{u_0(\widehat{y})}=\frac{P^+(S_1,S_2\mid y)}{Q^-(\widehat{S}_2,\widehat{S}_1\mid\widehat{y})} \tag{2.44}$$

then $v^*_\lambda(y):=\varphi(y)\;u_\lambda(\widehat{y})$, i.e. the right-hand-side of (2.32), is solution of Eq. (1.5) with the conditions (1.11). Proceeding similarly to the proof of Theorem 2.1 one can see that $v^*_\lambda(y)$ satisfies Eq. (1.5). We now prove that conditions (1.11) hold for $v^*_\lambda(y)$. To this purpose, due to (1.12), (2.41) and (2.43), we have

$$\begin{aligned} &\lim_{y\downarrow S_1}v_0(y)=\frac{\beta_1}{\beta_1+\alpha_1\displaystyle\int_{S_1}^{S_2}h(z)\,dz}\,, \qquad \lim_{y\uparrow S_2}v_0(y)=1,\\ &\lim_{y\downarrow S_1}u_0(\widehat{y})=\frac{\beta_1}{\beta_1+\alpha_1\displaystyle\int_{\widehat{S}_2}^{\widehat{S}_1}\widehat{h}(z)\,dz}\,, \qquad \lim_{y\uparrow S_2}u_0(\widehat{y})=1, \end{aligned} \tag{2.45}$$

so that from (2.44) it follows:

$$\lim_{y\downarrow S_1}\varphi(y)=\frac{\beta_1+\alpha_1\displaystyle\int_{\widehat{S}_2}^{\widehat{S}_1}\widehat{h}(z)\,dz}{\beta_1+\alpha_1\displaystyle\int_{S_1}^{S_2}h(z)\,dz},\qquad \lim_{y\uparrow S_2}\varphi(y)=1. \tag{2.46}$$

To prove that $v^*_\lambda(y)$ satisfied the first of (1.11), we consider the following two cases: *(1)* $\beta_1=0$ and *(2)* $\beta_1\neq 0$.
(1) If $\beta_1=0$, i.e. S_1 and $\widehat{S}_1$ are absorbing boundaries, then

$$\lim_{y\downarrow S_1}\Big\{\alpha_1\,v^*_\lambda(y)-\beta_1\,h^{-1}(y)\,\frac{d}{dy}v^*_\lambda(y)\Big\}=\alpha_1\,\varphi(S_1)\lim_{\widehat{y}\uparrow\widehat{S}_1}u_\lambda(\widehat{y})=0 \tag{2.47}$$

where the last equality follows by virtue of the first of (2.42).
(2) If $\beta_1\neq 0$, i.e. S_1 and $\widehat{S}_1$ are reflecting or full elastic boundaries, then

$$\begin{aligned}&\lim_{y\downarrow S_1}\Big\{\alpha_1\,v^*_\lambda(y)-\beta_1\,h^{-1}(y)\,\frac{d}{dy}v^*_\lambda(y)\Big\}\\&\qquad=-\varphi(S_1)\,\frac{\widehat{h}(\widehat{S}_1)}{h(S_1)}\,\frac{d\psi(y)}{dy}\bigg|_{y=S_1}\lim_{\widehat{y}\uparrow\widehat{S}_1}\Big\{\alpha_1\,u_\lambda(\widehat{y})+\beta_1\,\widehat{h}^{-1}(\widehat{y})\,\frac{du_\lambda(\widehat{y})}{d\widehat{y}}\Big\}\end{aligned}$$

that vanishes by virtue of the second of (2.42), i.e. the first of (1.11) holds for $v^*_\lambda(y)$. Furthermore, recalling the first of (2.42) and (2.46), one has:

$$\lim_{y\uparrow S_2}v^*_\lambda(y)\equiv\lim_{y\uparrow S_2}\Big[\varphi(y)\,u_\lambda(\widehat{y})\Big]=\lim_{\widehat{y}\downarrow\widehat{S}_2}u_\lambda(\widehat{y})=1, \tag{2.48}$$

so that $v^*_\lambda(y)$ satisfies the second of (1.11). In conclusion, $v^*_\lambda(y)\equiv g^+_\lambda(S_1,S_2\mid y)$ and thus (2.32) holds. □

3 Wiener process

In this Section we shall assume that $\{\widehat{X}(t),t\geq 0\}$ is the Wiener process with drift and infinitesimal variance

$$\widehat{A}_1=\mu,\qquad \widehat{A}_2=\sigma^2\qquad(\mu\in\mathbb{R},\,\sigma>0), \tag{3.1}$$

respectively, defined in $\mathbb{R}$. The scale function is known to be

$$\widehat{h}(x)=\widehat{B}\,\exp\Big\{-\frac{2\,\mu\,x}{\sigma^2}\Big\}\qquad(\widehat{B}>0,\;x\in\mathbb{R}). \tag{3.2}$$

Starting from $\widehat{X}(t)$, we shall provide a direct construction of FPT pdf's for a new diffusion process $\{X(t),t\geq 0\}$ in terms of the FPT pdf's of the diffusion process (3.1). We shall separately consider the cases when $X(t)$ is the Wiener process, the Feller process and the hyperbolic process.

3.1 Wiener to Wiener processes

Let $\{X(t),t\geq 0\}$ be the Wiener process with drift and infinitesimal variance

$$A_1=\xi,\quad A_2=\omega^2\qquad(\xi\in\mathbb{R},\;\omega>0), \tag{3.3}$$

Wiener processes $X(t)$ **and** $\widehat{X}(t)$ $\psi(z)=\dfrac{\sigma}{\omega}\,z+c \quad (c\in\mathbb{R}),\quad \widehat{S}_1=\psi(S_1),\quad \widehat{S}_2=\psi(S_2),\quad \widehat{y}=\psi(y)$			
Boundaries			
$S_1,\widehat{S}_1$	$S_2,\widehat{S}_2$	Conditions	Relations
Absorbing $(\beta_1=0)$	Reflecting $(\alpha_2=0)$	$\mu/\sigma=\xi/\omega$	$g^-(S_1,S_2,t\mid y)$ $=\gamma^-(\widehat{S}_1,\widehat{S}_2,t\mid\widehat{y})$
Absorbing $(\beta_1=0)$	Absorbing $(\beta_2=0)$	$\mu/\sigma=\xi/\omega$	$g^-(S_1,S_2,t\mid y)$ $=\gamma^-(\widehat{S}_1,\widehat{S}_2,t\mid\widehat{y})$ $g^+(S_1,S_2,t\mid y)$ $=\gamma^+(\widehat{S}_1,\widehat{S}_2,t\mid\widehat{y})$
Reflecting $(\alpha_1=0)$	Absorbing $(\beta_2=0)$	$\mu/\sigma=\xi/\omega$	$g^+(S_1,S_2,t\mid y)$ $=\gamma^+(\widehat{S}_1,\widehat{S}_2,t\mid\widehat{y})$
Absorbing $(\beta_1=0)$	Elastic $(\alpha_2>0,\beta_2>0)$	$\mu/\sigma=\xi/\omega$ $B=\widehat{B}\,\dfrac{\sigma}{\omega}\,e^{-2\,\mu\,c/\sigma^2}$	$g^-(S_1,S_2,t\mid y)$ $=\gamma^-(\widehat{S}_1,\widehat{S}_2,t\mid\widehat{y})$
Elastic $(\alpha_1>0,\beta_1>0)$	Absorbing $(\beta_2=0)$	$\mu/\sigma=\xi/\omega$ $B=\widehat{B}\,\dfrac{\sigma}{\omega}\,e^{-2\,\mu\,c/\sigma^2}$	$g^+(S_1,S_2,t\mid y)$ $=\gamma^+(\widehat{S}_1,\widehat{S}_2,t\mid\widehat{y})$

Table 1: FPT pdf's of Wiener processes $\widehat{X}(t)$ and $X(t)$, defined in (3.1) and (3.3) respectively. Here $\psi(z)$ is strictly increasing.

respectively, defined in $\mathbb{R}$ with scale function (1.14), $r_1=-\infty$ and $r_2=+\infty$ being natural boundaries.

If $\psi:\mathbb{R}\to\mathbb{R}$ is a strictly increasing function such that $\psi(y)\in C^2(\mathbb{R})$, making use of (3.1) and (3.3) in (2.3), one obtains:

$$\begin{aligned} &\psi(y)=\frac{\sigma}{\omega}\,y+c \qquad (c\in\mathbb{R})\,,\\ &\xi-\frac{\omega}{\sigma}\,\mu=-\,\omega^2\,\frac{1}{\varphi(y)}\,\frac{d\varphi(y)}{dy}\cdot \end{aligned} \tag{3.4}$$

Table 3.1 lists the boundaries (columns 1 and 2) and conditions on parameters ξ,ω,μ,σ (column 3) of $X(t)$ and $\widehat{X}(t)$ such that (3.4) are satisfied with $\varphi(y)$ given in (2.4) or in (2.6). The last column shows the relations (2.5) and (2.7) implied by Theorem 2.1.

Note that if $\mu/\sigma=\xi/\omega$, by removing either S_1 or S_2, when both are absorbing boundaries, one has:

$$g(S,t\mid y)=\gamma\Big(\frac{\sigma}{\omega}\,S+c,t\Big|\frac{\sigma}{\omega}\,y+c\Big) \qquad (S\neq y,\,c\in\mathbb{R}),$$

with $\gamma(x,t\mid y)$ and $g(x,t\mid y)$ denoting the FPT pdf's to x starting from y for the Wiener processes (3.1) and (3.3), respectively.

If $\psi:\mathbb{R}\to\mathbb{R}$ is a strictly decreasing function such that $\psi(y)\in C^2(\mathbb{R})$, making use of

Wiener processes $X(t)$ **and** $\widehat{X}(t)$ $\psi(z) = -\frac{\sigma}{\omega}\, z + c \quad (c \in \mathbb{R}), \quad \widehat{S}_1 = \psi(S_1), \quad \widehat{S}_2 = \psi(S_2), \quad \widehat{y} = \psi(y)$			
Boundaries			
$S_1, \widehat{S}_1$	$S_2, \widehat{S}_2$	Conditions	Relations
Absorbing ($\beta_1 = 0$)	Reflecting ($\alpha_2 = 0$)	$\mu/\sigma = -\xi/\omega$	$g^-(S_1, S_2, t \mid y) = \gamma^+(\widehat{S}_2, \widehat{S}_1, t \mid \widehat{y})$
Absorbing ($\beta_1 = 0$)	Absorbing ($\beta_2 = 0$)	$\mu/\sigma = -\xi/\omega$	$g^-(S_1, S_2, t \mid y) = \gamma^+(\widehat{S}_2, \widehat{S}_1, t \mid \widehat{y})$ $g^+(S_1, S_2, t \mid y) = \gamma^-(\widehat{S}_2, \widehat{S}_1, t \mid \widehat{y})$
Reflecting ($\alpha_1 = 0$)	Absorbing ($\beta_2 = 0$)	$\mu/\sigma = -\xi/\omega$	$g^+(S_1, S_2, t \mid y) = \gamma^-(\widehat{S}_2, \widehat{S}_1, t \mid \widehat{y})$
Absorbing ($\beta_1 = 0$)	Elastic ($\alpha_2 > 0, \beta_2 > 0$)	$\mu/\sigma = -\xi/\omega$ $B = \widehat{B}\, \frac{\sigma}{\omega}\, e^{-2\mu c/\sigma^2}$	$g^-(S_1, S_2, t \mid y) = \gamma^+(\widehat{S}_2, \widehat{S}_1, t \mid \widehat{y})$
Elastic ($\alpha_1 > 0, \beta_1 > 0$)	Absorbing ($\beta_2 = 0$)	$\mu/\sigma = -\xi/\omega$ $B = \widehat{B}\, \frac{\sigma}{\omega}\, e^{-2\mu c/\sigma^2}$	$g^+(S_1, S_2, t \mid y) = \gamma^-(\widehat{S}_2, \widehat{S}_1, t \mid \widehat{y})$
Absorbing ($\beta_1 = 0$)	Absorbing ($\beta_2 = 0$)	$\mu = \xi,\ \sigma = \omega$	$g^-(S_1, S_2, t \mid y) = e^{-2\xi(y - S_1)/\omega^2} \times\gamma^+(\widehat{S}_2, \widehat{S}_1, t \mid \widehat{y})$ $g^+(S_1, S_2, t \mid y) = e^{2\xi(S_2 - y)/\omega^2} \times\gamma^-(\widehat{S}_2, \widehat{S}_1, t \mid \widehat{y})$

Table 2: Same as Table 3.1, but with $\psi(z)$ strictly decreasing.

(3.1) and (3.3) in (2.3) one obtains:

$$\psi(y) = -\frac{\sigma}{\omega}\, y + c \qquad (c \in \mathbb{R}),$$

(3.5)

$$\xi + \frac{\omega}{\sigma}\, \mu = -\,\omega^2\, \frac{1}{\varphi(y)}\, \frac{d\varphi(y)}{dy}\cdot$$

Table 3.1 lists the boundaries (columns 1 and 2) and conditions on parameters ξ, ω, μ, σ (column 3) of $X(t)$ and $\widehat{X}(t)$ such that (3.5) hold with $\varphi(y)$ given in (2.29) or in (2.31). In the last column the relations (2.30) and (2.32) given in Theorem 2.2 are shown.

If $\mu/\sigma = -\xi/\omega$ by removing one of the two boundaries S_1 or S_2, when both are absorbing, one obtains:

$$g(S, t \mid y) = \gamma\Big(-\frac{\sigma}{\omega}\, S + c, t \Big| -\frac{\sigma}{\omega}\, y + c\Big) \qquad (S \neq y,\ c \in \mathbb{R}).$$

Table 3.1 also shows the relations among FPT pdf's for $\mu = \xi$ and $\sigma = \omega$ (namely when $X(t)$ and $\widehat{X}(t)$ are identical) when both boundaries are absorbing:

$$g^-(S_1, S_2, t \mid y) = \exp\Big\{-\frac{2\,\xi\,(y - S_1)}{\omega^2}\Big\}\, g^+(-S_2 + c, -S_1 + c, t \mid -y + c),$$

(3.6)

$$g^+(S_1, S_2, t \mid y) = \exp\Big\{\frac{2\,\xi\,(S_2 - y)}{\omega^2}\Big\}\, g^-(-S_2 + c, -S_1 + c, t \mid -y + c),$$

with $S_1 < y < S_2$ and $c \in \mathbb{R}$. By removing one of the two boundaries S_1 or S_2, from (3.6) one has:

$$g(S,t \mid y) = \exp\Big\{\frac{2\,\xi\,(S-y)}{\omega^2}\Big\}\, g(-S+c,t \mid -y+c) \qquad (S \neq y,\ c \in \mathbb{R}). \tag{3.7}$$

Setting $c = 2\,S$ in (3.7) one obtains the well-known symmetry relation for the FPT pdf of the Wiener process (cf, for instance, [9]):

$$g(S,t \mid y) = \exp\Big\{\frac{2\,\xi\,(S-y)}{\omega^2}\Big\}\, g(S,t \mid 2\,S-y) \qquad (S \neq y).$$

3.2 Wiener to Feller processes

Let $\{X(t), t \geq 0\}$ be the Feller process with drift and infinitesimal variance

$$A_1 = p\,x + q, \quad A_2(x) = 2\,r\,(x-\nu) \qquad (p,q,\nu \in \mathbb{R},\ r>0), \tag{3.8}$$

respectively, defined in $I = (\nu, +\infty)$, with scale function:

$$h(x) = B\,(x-\nu)^{-(p\,\nu+q)/r}\, \exp\Big\{-\frac{p}{r}\,x\Big\} \qquad (B>0,\ x>\nu). \tag{3.9}$$

As is well known, for the process (3.8) $x = \nu$ is an exit boundary if $p\,\nu + q \leq 0$, a regular boundary if $0 < p\,\nu + q < r$ and an entrance boundary if $p\,\nu + q \geq r$; instead, boundary $+\infty$ is natural. We recall that (cf., for instance, [7]) if $p\,\nu + q < r$ the FPT pdf through v starting from y can be explicitly obtained:

$$g(\nu,t \mid y) = \begin{cases} \dfrac{1}{t\,\Gamma\Big(1-\dfrac{q}{r}\Big)} \Big(\dfrac{y-\nu}{r\,t}\Big)^{1-q/r} \exp\Big\{-\dfrac{y-\nu}{r\,t}\Big\} & q<r,\ p=0 \\[2ex] \dfrac{1}{\Gamma\Big(1-\dfrac{p\,\nu+q}{r}\Big)}\, \dfrac{p}{e^{p\,t}-1}\, \Big[\dfrac{p\,(y-\nu)\,e^{p\,t}}{r\,(e^{p\,t}-1)}\Big]^{1-(p\,\nu+q)/r} & \\ \qquad\qquad \times \exp\Big\{-\dfrac{p\,(y-\nu)\,e^{p\,t}}{r\,(e^{p\,t}-1)}\Big\} & p\,\nu+q<r,\ p\neq 0. \end{cases}$$

If $\psi : I \to \mathbb{R}$, with $I = (\nu, +\infty)$, is a strictly increasing function such that $\psi(y) \in C^2(I)$, making use of (3.1) and (3.8) in (2.3) one obtains:

$$\begin{aligned} &\psi(y) = c + \sqrt{\frac{2\,\sigma^2}{r}\,(y-\nu)} \qquad (c \in \mathbb{R}), \\ &p\,y+q = \frac{r}{2} + \frac{\mu}{\sigma}\sqrt{2\,r\,(y-\nu)} - 2\,r\,(y-\nu)\,\frac{1}{\varphi(y)}\,\frac{d\varphi(y)}{dy}, \end{aligned} \tag{3.10}$$

with $\varphi(y)$ given by (2.4) or (2.6). Table 3.2 summarizes the obtained results for specified choices of boundaries and parameters.

If $\psi : I \to \mathbb{R}$, with $I = (\nu, +\infty)$, is a strictly decreasing function such that $\psi(y) \in C^2(I)$, making use of (3.1) and (3.8) in (2.3) one obtains:

$$\begin{aligned} &\psi(y) = c - \sqrt{\frac{2\,\sigma^2}{r}\,(y-\nu)} \qquad (c \in \mathbb{R}), \\ &p\,y+q = \frac{r}{2} - \frac{\mu}{\sigma}\sqrt{2\,r\,(y-\nu)} - 2\,r\,(y-\nu)\,\frac{1}{\varphi(y)}\,\frac{d\varphi(y)}{dy}, \end{aligned} \tag{3.11}$$

Feller process $X(t)$ **and Wiener process** $\widehat{X}(t)$			
$\psi(z)=c+\sqrt{\frac{2\,\sigma^2}{r}}\,(z-\nu)\quad(c\in\mathbb{R}),\quad \widehat{S}_1=\psi(S_1),\quad \widehat{S}_2=\psi(S_2),\quad \widehat{y}=\psi(y)$			
Boundaries			
$S_1,\widehat{S}_1$	$S_2,\widehat{S}_2$	Conditions	Relations
Absorbing ($\beta_1=0$)	Reflecting ($\alpha_2=0$)	$p=0,\,\mu=0$ $q/r=1/2$	$g^-(S_1,S_2,t\mid y)=\gamma^-(\widehat{S}_1,\widehat{S}_2,t\mid\widehat{y})$
Absorbing ($\beta_1=0$)	Absorbing ($\beta_2=0$)	$p=0,\,\mu=0$ $q/r=1/2$	$g^-(S_1,S_2,t\mid y)=\gamma^-(\widehat{S}_1,\widehat{S}_2,t\mid\widehat{y})$ $g^+(S_1,S_2,t\mid y)=\gamma^+(\widehat{S}_1,\widehat{S}_2,t\mid\widehat{y})$
Reflecting ($\alpha_1=0$)	Absorbing ($\beta_2=0$)	$p=0,\,\mu=0$ $q/r=1/2$	$g^+(S_1,S_2,t\mid y)=\gamma^+(\widehat{S}_1,\widehat{S}_2,t\mid\widehat{y})$
Absorbing ($\beta_1=0$)	Elastic ($\alpha_2>0,\beta_2>0$)	$p=0,\,\mu=0$ $q/r=1/2$ $B=\widehat{B}\sqrt{\frac{\sigma^2}{2\,r}}$	$g^-(S_1,S_2,t\mid y)=\gamma^-(\widehat{S}_1,\widehat{S}_2,t\mid\widehat{y})$
Elastic ($\alpha_1>0,\beta_1>0$)	Absorbing ($\beta_2=0$)	$p=0,\,\mu=0$ $q/r=1/2$ $B=\widehat{B}\sqrt{\frac{\sigma^2}{2\,r}}$	$g^+(S_1,S_2,t\mid y)=\gamma^+(\widehat{S}_1,\widehat{S}_2,t\mid\widehat{y})$
Absorbing ($\beta_1=0$)	Absorbing ($\beta_2=0$)	$p=0,\,\mu=0$ $q/r=3/2$	$g^-(S_1,S_2,t\mid y)=\sqrt{\frac{S_1-\nu}{y-\nu}}\times\gamma^-(\widehat{S}_1,\widehat{S}_2,t\mid\widehat{y})$ $g^+(S_1,S_2,t\mid y)=\sqrt{\frac{S_2-\nu}{y-\nu}}\times\gamma^+(\widehat{S}_1,\widehat{S}_2,t\mid\widehat{y})$

Table 3: Relations between the FPT pdf's of the Wiener and Feller processes $\widehat{X}(t)$ and $X(t)$, defined in (3.1) and (3.8) respectively. Here $\psi(z)$ is a strictly increasing function.

with $\varphi(y)$ given by (2.29) or (2.31). Table 3.2 lists the results of interest.

Note that when $p=0$ and $q/r=1/2$ the boundary $x=\nu$ of the Feller process (3.8) is regular. Instead, $x=\nu$ is an entrance boundary when $p=0$ and $q/r=3/2$.

We consider the Feller process (3.8) with $p=0$ and $q/r=1/2$. If both boundaries S_1 and S_2 are absorbing, taking the limit as $S_2\to+\infty$, from Tables 3.2 and 3.2, for all $c\in\mathbb{R}$ one obtains:

$$
\begin{aligned}
g_\lambda(S_1\mid y)&=\gamma_\lambda\biggl(c+\sqrt{\frac{2\,\sigma^2}{r}}\,(S_1-\nu)\;\Big|\;c+\sqrt{\frac{2\,\sigma^2}{r}}\,(y-\nu)\biggr)\\
&=\gamma_\lambda\biggl(c-\sqrt{\frac{2\,\sigma^2}{r}}\,(S_1-\nu)\;\Big|\;c-\sqrt{\frac{2\,\sigma^2}{r}}\,(y-\nu)\biggr)\\
&=\exp\biggl\{-2\sqrt{\frac{\lambda}{r}\,(y-\nu)}+2\sqrt{\frac{\lambda}{r}\,(S_1-\nu)}\biggr\}\qquad(\nu<S_1<y),
\end{aligned}
$$

where $g_\lambda(S_1\mid y)$ is the LT of the FPT pdf $g(S_1,t\mid y)$ from y to S_1 for the Feller process (3.8) with $p=0$ and $q/r=1/2$, and $\gamma_\lambda(x\mid z)$ denotes the LT of the FPT pdf $\gamma(x,t\mid z)$ from z to x for the Wiener process (3.1) with $\mu=0$. Furthermore, if both boundaries S_1 and S_2

Feller process $X(t)$ **and Wiener process** $\widehat{X}(t)$			
$\psi(z)=c-\sqrt{\frac{2\,\sigma^2}{r}\,(z-\nu)}$ $(c\in\mathbb{R})$, $\widehat{S}_1=\psi(S_1)$, $\widehat{S}_2=\psi(S_2)$, $\widehat{y}=\psi(y)$			
Boundaries			
$S_1,\widehat{S}_1$	$S_2,\widehat{S}_2$	Conditions	Relations
Absorbing ($\beta_1=0$)	Reflecting ($\alpha_2=0$)	$p=0,\ \mu=0$ $q/r=1/2$	$g^-(S_1,S_2,t\mid y)$ $=\gamma^+(\widehat{S}_2,\widehat{S}_1,t\mid\widehat{y})$
Absorbing ($\beta_1=0$)	Absorbing ($\beta_2=0$)	$p=0,\ \mu=0$ $q/r=1/2$	$g^-(S_1,S_2,t\mid y)$ $=\gamma^+(\widehat{S}_2,\widehat{S}_1,t\mid\widehat{y})$ $g^+(S_1,S_2,t\mid y)$ $=\gamma^-(\widehat{S}_2,\widehat{S}_1,t\mid\widehat{y})$
Reflecting ($\alpha_1=0$)	Absorbing ($\beta_2=0$)	$p=0,\ \mu=0$ $q/r=1/2$	$g^+(S_1,S_2,t\mid y)$ $=\gamma^-(\widehat{S}_2,\widehat{S}_1,t\mid\widehat{y})$
Absorbing ($\beta_1=0$)	Elastic ($\alpha_2>0,\beta_2>0$)	$p=0,\ \mu=0$ $q/r=1/2$ $B=\widehat{B}\sqrt{\frac{\sigma^2}{2\,r}}$	$g^-(S_1,S_2,t\mid y)$ $=\gamma^+(\widehat{S}_2,\widehat{S}_1,t\mid\widehat{y})$
Elastic ($\alpha_1>0,\beta_1>0$)	Absorbing ($\beta_2=0$)	$p=0,\ \mu=0$ $q/r=1/2$ $B=\widehat{B}\sqrt{\frac{\sigma^2}{2\,r}}$	$g^+(S_1,S_2,t\mid y)$ $=\gamma^-(\widehat{S}_2,\widehat{S}_1,t\mid\widehat{y})$
Absorbing ($\beta_1=0$)	Absorbing ($\beta_2=0$)	$p=0,\ \mu=0$ $q/r=3/2$	$g^-(S_1,S_2,t\mid y)=\sqrt{\frac{S_1-\nu}{y-\nu}}$ $\times\gamma^+(\widehat{S}_2,\widehat{S}_1,t\mid\widehat{y})$ $g^+(S_1,S_2,t\mid y)=\sqrt{\frac{S_2-\nu}{y-\nu}}$ $\times\gamma^-(\widehat{S}_2,\widehat{S}_1,t\mid\widehat{y})$

Table 4: Same as Table 3.2, but with $\psi(z)$ strictly decreasing.

are absorbing, taking the limit as $S_1\to\nu$, with ν absorbing boundary, from Tables 3.2 and 3.2, for all $c\in\mathbb{R}$ one has:

$$
\begin{aligned}
g_\lambda(S_2\mid y)&=\gamma_\lambda^+\Bigl(c,c+\sqrt{\frac{2\,\sigma^2}{r}\,(S_2-\nu)}\;\Bigl|\;c+\sqrt{\frac{2\,\sigma^2}{r}\,(y-\nu)}\Bigr)\\
&=\gamma_\lambda^-\Bigl(c-\sqrt{\frac{2\,\sigma^2}{r}\,(S_2-\nu)},c\;\Bigl|\;c-\sqrt{\frac{2\,\sigma^2}{r}\,(y-\nu)}\Bigr)\\
&=\frac{\sinh\Bigl[2\sqrt{\frac{\lambda\,(y-\nu)}{r}}\Bigr]}{\sinh\Bigl[2\sqrt{\frac{\lambda\,(S_2-\nu)}{r}}\Bigr]}\qquad(\nu<y<S_2),
\end{aligned}
$$

where $g_\lambda(S_2\mid y)$ is the LT of $g(S_2,t\mid y)$ for the Feller process (3.8) with $p=0$ and $q/r=1/2$ in the presence of an absorbing boundary at ν, whereas $\gamma_\lambda^-(u,v\mid z)$ and $\gamma_\lambda^+(u,v\mid z)$ denote the LT's of the FPT pdf's $\gamma^-(u,v,t\mid z)$ and $\gamma^+(u,v,t\mid z)$ in the presence of two absorbing boundaries u and v $(u<z<v)$ for the Wiener process (3.1) with $\mu=0$. Instead, if S_1 is reflecting and S_2 is absorbing, taking the limit as $S_1\to\nu$, with ν reflecting boundary, from

Tables 3.2 and 3.2, for all $c \in \mathbb{R}$ one obtains:

$$
\begin{aligned}
g_\lambda(S_2 \mid y) &= \gamma_\lambda^+\left(c, c+\sqrt{\frac{2\,\sigma^2}{r}\,(S_2-\nu)}\;\middle|\; c+\sqrt{\frac{2\,\sigma^2}{r}\,(y-\nu)}\right)\\
&= \gamma_\lambda^-\left(c-\sqrt{\frac{2\,\sigma^2}{r}\,(S_2-\nu)}, c \;\middle|\; c-\sqrt{\frac{2\,\sigma^2}{r}\,(y-\nu)}\right)\\
&= \frac{\cosh\left[2\sqrt{\dfrac{\lambda\,(y-\nu)}{r}}\right]}{\cosh\left[2\sqrt{\dfrac{\lambda\,(S_2-\nu)}{r}}\right]} \qquad (\nu < y < S_2),
\end{aligned}
$$

where $g_\lambda(S_2 \mid y)$ is the LT of $g(S_2, t \mid y)$ for the Feller process (3.8) with $p = 0$ and $q/r = 1/2$ in the presence of a reflecting boundary at ν, and $\gamma_\lambda^-(x, c \mid z)$ and $\gamma_\lambda^+(c, x \mid z)$ denote the LT's of the FPT pdf's $\gamma^-(x, c, t \mid z)$ and $\gamma^+(c, x, t \mid z)$ in the presence of the reflecting boundary c and of the absorbing boundary x for the Wiener process (3.1) with $\mu = 0$.

We now consider the Feller process (3.8) with $p = 0$ and $q/r = 3/2$. Taking the limit as $S_2 \to +\infty$ from Tables 3.2 and 3.2, for all $c \in \mathbb{R}$ one has:

$$
\begin{aligned}
g_\lambda(S_1 \mid y) &= \sqrt{\frac{S_1-\nu}{y-\nu}}\;\gamma_\lambda\left(c+\sqrt{\frac{2\,\sigma^2}{r}\,(S_1-\nu)}\;\middle|\; c+\sqrt{\frac{2\,\sigma^2}{r}\,(y-\nu)}\right)\\
&= \sqrt{\frac{S_1-\nu}{y-\nu}}\;\gamma_\lambda\left(c-\sqrt{\frac{2\,\sigma^2}{r}\,(S_1-\nu)}\;\middle|\; c-\sqrt{\frac{2\,\sigma^2}{r}\,(y-\nu)}\right)\\
&= \sqrt{\frac{S_1-\nu}{y-\nu}}\;\exp\left\{-2\sqrt{\frac{\lambda}{r}\,(y-\nu)}+2\sqrt{\frac{\lambda}{r}\,(S_1-\nu)}\right\} \qquad (\nu < S_1 < y),
\end{aligned}
$$

where $g_\lambda(S_1 \mid y)$ is the LT of $g(S_1, t \mid y)$ for the Feller process (3.8) with $p = 0$ and $q/r = 3/2$, and $\gamma_\lambda(x \mid z)$ denotes the LT of $\gamma(x, t \mid z)$ for the Wiener process (3.1) with $\mu = 0$. Furthermore, since an entrance boundary cannot be attained from the interior of the diffusion interval, we take the limit as $S_1 \to \nu$, with ν entrance boundary. Hence, for all $c \in \mathbb{R}$, Tables 3.2 and 3.2 yield:

$$
\begin{aligned}
g_\lambda(S_2 \mid y) &= \sqrt{\frac{S_2-\nu}{y-\nu}}\;\gamma_\lambda^+\left(c, c+\sqrt{\frac{2\,\sigma^2}{r}\,(S_2-\nu)}\;\middle|\; c+\sqrt{\frac{2\,\sigma^2}{r}\,(y-\nu)}\right)\\
&= \sqrt{\frac{S_2-\nu}{y-\nu}}\;\gamma_\lambda^-\left(c-\sqrt{\frac{2\,\sigma^2}{r}\,(S_2-\nu)}, c \;\middle|\; c-\sqrt{\frac{2\,\sigma^2}{r}\,(y-\nu)}\right)\\
&= \sqrt{\frac{S_2-\nu}{y-\nu}}\;\frac{\sinh\left[2\sqrt{\dfrac{\lambda\,(y-\nu)}{r}}\right]}{\sinh\left[2\sqrt{\dfrac{\lambda\,(S_2-\nu)}{r}}\right]} \qquad (\nu < y < S_2),
\end{aligned}
\tag{3.12}
$$

where $g_\lambda(S_2 \mid y)$ is the LT of $g(S_2, t \mid y)$ for the Feller process (3.8) with $p = 0$ and $q/r = 3/2$ in the presence of an entrance boundary at ν. Here, $\gamma_\lambda^-(u, v \mid z)$ and $\gamma_\lambda^+(u, v \mid z)$ denote the LT's of the FPT pdf's $\gamma^-(u, v, t \mid z)$ and $\gamma^+(u, v, t \mid z)$ in the presence of two absorbing boundaries u and v $(u < z < v)$ for the Wiener process (3.1) with $\mu = 0$. Equation (3.12) provides an a priori unexpected functional relation between the FPT pdf for the Feller process (3.8) with $p = 0$ and $q/r = 3/2$ (ν entrance boundary) and the FPT pdf in the presence of two absorbing boundaries for the Wiener process (3.1) with $\mu = 0$.

3.3 Wiener to hyperbolic processes

Let $\{X(t), t \geq 0\}$ be the hyperbolic process with drift and infinitesimal variance

$$A_1(x) = \mu \, \frac{1 - \zeta \, \exp\Big\{-\dfrac{2\,\mu\,x}{\sigma^2}\Big\}}{1 + \zeta \, \exp\Big\{-\dfrac{2\,\mu\,x}{\sigma^2}\Big\}}, \quad A_2 = \sigma^2 \qquad (\mu \neq 0, \ \zeta > 0, \ \sigma > 0), \tag{3.13}$$

respectively, defined in $\mathbb{R}$, with scale function:

$$h(x) = B \, \exp\Big\{-\frac{2\,\mu\,x}{\sigma^2}\Big\} \left[1 + \zeta \, \exp\Big\{-\frac{2\,\mu\,x}{\sigma^2}\Big\}\right]^{-2} \qquad (B > 0, \ x \in \mathbb{R}). \tag{3.14}$$

Here $r_1 = -\infty$ and $r_2 = +\infty$ are natural boundaries.

If $\psi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function such that $\psi(y) \in C^2(\mathbb{R})$, making use of (3.1) and (3.13) in (2.3) one obtains:

$$\begin{gathered} \psi(y) = y + c \quad (c \in \mathbb{R}), \\ \frac{2\,\zeta\,\mu \, \exp\Big\{-\dfrac{2\,\mu\,y}{\sigma^2}\Big\}}{1 + \zeta \, \exp\Big\{-\dfrac{2\,\mu\,y}{\sigma^2}\Big\}} = \sigma^2 \, \frac{1}{\varphi(y)} \, \frac{d\varphi(y)}{dy}. \end{gathered} \tag{3.15}$$

In the case when S_1 and S_2 are absorbing boundaries, it is easily seen that relations (3.15) are satisfied, with $\varphi(y)$ given in (2.4) or in (2.6). Hence, from (2.5) and (2.7) of Theorem 2.1, for all $c \in \mathbb{R}$ and $S_1 < y < S_2$ one obtains:

$$\begin{aligned} g^-(S_1, S_2, t \mid y) &= \frac{1 + \zeta \, \exp\Big\{-\dfrac{2\,\mu\,S_1}{\sigma^2}\Big\}}{1 + \zeta \, \exp\Big\{-\dfrac{2\,\mu\,y}{\sigma^2}\Big\}} \, \gamma^-(S_1 + c, S_2 + c, t \mid y + c) \\ g^+(S_1, S_2, t \mid y) &= \frac{1 + \zeta \, \exp\Big\{-\dfrac{2\,\mu\,S_2}{\sigma^2}\Big\}}{1 + \zeta \, \exp\Big\{-\dfrac{2\,\mu\,y}{\sigma^2}\Big\}} \, \gamma^+(S_1 + c, S_2 + c, t \mid y + c), \end{aligned} \tag{3.16}$$

where $\gamma^-(u, v, t \mid z)$ and $\gamma^+(u, v, t \mid z)$ denote the FPT pdf's in the presence of two absorbing boundaries u and v ($u < z < v$) for the Wiener process (3.1).

If $\psi : \mathbb{R} \to \mathbb{R}$ is a strictly decreasing function such that $\psi(y) \in C^2(\mathbb{R})$, making use of (3.1) and (3.13) in (2.3), one obtains:

$$\begin{gathered} \psi(y) = -y + c \qquad (c \in \mathbb{R}), \\ \frac{2\,\mu}{1 + \zeta \, \exp\Big\{-\dfrac{2\,\mu\,x}{\sigma^2}\Big\}} = -\sigma^2 \, \frac{1}{\varphi(y)} \, \frac{d\varphi(y)}{dy}. \end{gathered} \tag{3.17}$$

If S_1 and S_2 are absorbing boundaries, one can prove that (3.17) are satisfied with $\varphi(y)$ given in (2.29) or (2.31). Hence, from (2.30) and (2.32) of Theorem 2.2, for all $c \in \mathbb{R}$ and $S_1 < y < S_2$ one has:

$$
\begin{aligned}
g^-(S_1, S_2, t \mid y) &= \frac{1 + \zeta \exp\Bigl\{-\dfrac{2\,\mu\,S_1}{\sigma^2}\Bigr\}}{1 + \zeta \exp\Bigl\{-\dfrac{2\,\mu\,y}{\sigma^2}\Bigr\}} \exp\Bigl\{-\frac{2\,\mu\,(y - S_1)}{\sigma^2}\Bigr\} \\
&\quad \times \gamma^+(-S_2 + c, -S_1 + c, t \mid -y + c), \\
g^+(S_1, S_2, t \mid y) &= \frac{1 + \zeta \exp\Bigl\{-\dfrac{2\,\mu\,S_2}{\sigma^2}\Bigr\}}{1 + \zeta \exp\Bigl\{-\dfrac{2\,\mu\,y}{\sigma^2}\Bigr\}} \exp\Bigl\{\frac{2\,\mu\,(S_2 - y)}{\sigma^2}\Bigr\} \\
&\quad \times \gamma^-(-S_2 + c, -S_1 + c, t \mid -y + c),
\end{aligned}
\tag{3.18}
$$

where $\gamma^-(u, v, t \mid z)$ and $\gamma^+(u, v, t \mid z)$ denote the FPT pdf's in the presence of two absorbing boundaries u and v $(u < z < v)$ for the Wiener process (3.1).

By removing one of the two boundaries S_1 or S_2, from (3.16) and (3.18) for all $c \in \mathbb{R}$ one obtains the following result (cf., for instance, [8]):

$$
\begin{aligned}
g(S, t \mid y) &= \frac{1 + \zeta \exp\Bigl\{-\dfrac{2\,\mu\,S}{\sigma^2}\Bigr\}}{1 + \zeta \exp\Bigl\{-\dfrac{2\,\mu\,y}{\sigma^2}\Bigr\}} \gamma(S + c, t \mid y + c) \\
&= \frac{1 + \zeta \exp\Bigl\{-\dfrac{2\,\mu\,S}{\sigma^2}\Bigr\}}{1 + \zeta \exp\Bigl\{-\dfrac{2\,\mu\,y}{\sigma^2}\Bigr\}} \exp\Bigl\{-\frac{2\,\mu\,(y - S)}{\sigma^2}\Bigr\} \gamma(-S + c, t \mid -y + c) \qquad (S \neq y),
\end{aligned}
\tag{3.19}
$$

where $g(S, t \mid y)$ is the FPT pdf of the process (3.13), whereas $\gamma(x, t \mid y)$ denotes the FPT pdf to x starting from y for the Wiener process (3.1). Hence, recalling (1.27), from (3.19) it follows:

$$
g(S, t \mid y) = \frac{1 + \zeta \exp\Bigl\{-\dfrac{2\,\mu\,S}{\sigma^2}\Bigr\}}{1 + \zeta \exp\Bigl\{-\dfrac{2\,\mu\,y}{\sigma^2}\Bigr\}} \frac{|S - y|}{\sigma\,\sqrt{2\,\pi\,t^3}} \exp\Bigl\{-\frac{(S - y - \mu\,t)^2}{\sigma^2\,t}\Bigr\} \qquad (S \neq y).
$$

4 Concluding remarks

Within certain models of neuron activity based on diffusion processes, the FPT pdf can be viewed as the neuron firing density. In particular, if S denotes the neuron firing threshold and if one assumes that the left-hand point r_1 of the diffusion interval is either a natural or an entrance or a regular boundary with a reflection condition superimposed, $g(S, t \mid y)$ $(r_1 < y < S)$ can be viewed as the firing pdf. In addition, $g^+(S_1, S, t \mid y)$ $(S_1 < y < S)$, with S_1 and S in the interior of the diffusion interval, can be viewed as the neuron firing density for models including a reversal potential if at S_1 a reflection condition is imposed.

A challenging problem in the neuronal modeling context is to make use of experimental data in the form of interspike interval histograms to arise to a diffusion process by means of which the membrane potential time course can be modeled. If the diffusion process is assumed to be known, and if the initial state, the boundaries of the available state-space as well as the appropriate absorption, reflection and, in general, elastic boundaries are assumed to be known, the FPT pdf can be uniquely determined. Different is the situation when the nature of the boundaries and the conditions imposed are not known, since in this case the FPT pdf cannot be uniquely specified.

With such background in mind, in this paper the relation among FPT pdf's have been investigated for pairs of time-homogeneous diffusion processes without resorting to space-transformations of the transition pdf's. In particular, it is shown that if drifts and infinitesimal variances of the two processes are suitably related to one another, then their FPT pdf's are also suitably related. Thus doing, not only classical well-known results are recovered, but new FPT pdf's can be obtained, so providing a noteworthy contribution to the search of candidate data fitting densities in the realm of the biological sciences and, particularly, in models of neurobiological interest.

Acknowledgments This work has been performed under partial support by MIUR (PRIN 2005), by G.N.C.S - INdAM and by Campania Region.

References

[1] Bluman G.W. On the transformation of diffusion processes into the Wiener process. SIAM J. Appl. Math. **39** (2), 238247, 1980.

[2] Capocelli R.M. and Ricciardi L.M. On the transformation of diffusion processes into the Feller process. Mathematical Biosciences **29**, 219-234., 1976.

[3] Cherkasov I.D. On the transformation of the diffusion processes to a Wiener process. Theory Probab. Its Appl. **2**, 373-377, 1957.

[4] Cox, D.R. and Miller, H.D. (1965). *The Theory of Stochastic Processes.* Chapman and Hall, London.

[5] Di Crescenzo A., Giorno V., Nobile A.G. and Ricciardi L.M. On first-passage-time and transition densities for strongly symmetric diffusion processes. Nagoya Mathematical Journal **145**, 143-161, 1997.

[6] Feller W. The parabolic differential equations and the associated semi-groups of transformations. Ann. Math. **55**, 468-518, 1952.

[7] Giorno V., Nobile A.G., Ricciardi L.M. and Sacerdote L. Some remarks on the Rayleigh process. J. Appl. Prob. **23**, 398-408, 1986.

[8] Giorno V., Nobile A.G. and Ricciardi L.M. A new approach to the construction of first-passage-time densities. In Cybernetics and Systems 1988 (Trappl R., ed.), Kluwer Academic Publishers, 375-381, 1988.

[9] Giorno V., Nobile A.G. and Ricciardi L.M. A symmetry-based constructive approach to probability densities for one dimensional diffusion processes. J. Appl. Prob. **27**, 707-721, 1989.

[10] Ricciardi L.M. On the transformation of diffusion processes into the Wiener process. J. Math. Analysis Appl. **54**, 185-199, 1976.

[11] Ricciardi L.M., Di Crescenzo A., Giorno V. and Nobile A.G. An outline of theoretical and algorithmic approaches to first passage time problems with applications to biological modeling. Math. Japonica **50** (2): 247-322, 1999.

Virginia Giorno, Amelia Giuseppina Nobile
Dipartimento di Matematica e Informatica, Università degli Studi di Salerno
Via Ponte don Melillo, I-84084 Fisciano (SA), Italy
E-mail: {giorno, nobile}@unisa.it

Enrica Pirozzi, Luigi Maria Ricciardi
Dipartimento di Matematica e Applicazioni, Università di Napoli Federico II
Via Cintia, 80126 Napoli, Italy
E-mail: {enrica.pirozzi, luigi.ricciardi}@unina.it