LAWS OF LARGE NUMBERS FOR PRODUCT OF RANDOM VARIABLES

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Abstract. We state and prove weak and strong laws of large numbers for a product of random variables. A statistical application to a problem in geometric probability is also provided.

1 Introduction and results. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and denote by $\mathbf{E}$ the expectation with respect to $\mathbf{P}$. Consider a sequence $\xi_0, \xi_1, \xi_2, ...$ of i.i.d. random variables defined on $(\Omega, \mathcal{F}, \mathbf{P})$ and suppose that $\mathbf{E}[\xi_0^4] < +\infty$. Let $\{c_n \; ; \; n \geq 1\}$ be an increasing sequence of integers such that $c_n^{-1} n^{\alpha} \to 0$, for some $\alpha > 1$. For a fixed $\lambda > 0$, denote by $m_n := \lfloor \lambda c_n \rfloor$ the integer part of λc_n. The classical laws of large numbers (see any basic book in probability theory, as [7]) state that (even under weaker assumptions):

$$(1) \qquad \lim_n \frac{1}{m_n} \sum_{i=1}^{m_n} \xi_i \; = \; \mathbf{E}(\xi_0) \quad \textit{in } \mathbf{P}-\textit{probability},$$

and

$$(2) \qquad \lim_n \frac{1}{m_n} \sum_{i=1}^{m_n} \xi_i \; = \; \mathbf{E}(\xi_0) \quad \mathbf{P}-\textit{almost surely}.$$

Being, for any $n \geq 1$:

$$\left(\prod_{i=1}^{m_n} \xi_i \right)^{\frac{1}{c_n}} \; = \; \exp\left(\frac{1}{c_n} \sum_{i=1}^{m_n} \ln(\xi_i) \right),$$

and because of the continuity of the exponential, from (1) and (2) we may derive the following laws of large numbers for product:

$$\lim_n \left(\prod_{i=1}^{m_n} \xi_i \right)^{\frac{1}{c_n}} \; = \; \exp\left(\lambda \mathbf{E}\left(\ln(\xi_0) \right) \right), \quad \textit{in } \mathbf{P}-\textit{probability},$$

and

$$\lim_n \left(\prod_{i=1}^{m_n} \xi_i \right)^{\frac{1}{c_n}} \; = \; \exp\left(\lambda \mathbf{E}\left(\ln(\xi_0) \right) \right), \quad \mathbf{P}-\textit{almost surely}.$$

However to compute the limit one should be able to compute $\mathbf{E}(\ln(\xi_0))$.

We consider different kind of product and laws of large numbers. We define, for any $n \geq 1$:

$$(3) \qquad \widehat{S}_n \; := \; \prod_{i=1}^{m_n} \left(1 - \frac{\xi_i}{c_n} \right),$$

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and

$$s_n \ := \ \left(1 - \frac{\mathbf{E}(\xi_0)}{c_n}\right)^{m_n}. \tag{4}$$

and we will prove the following Theorems.

Theorem 1 *(Weak Law of Large Numbers) The sequence $\widehat{S}_n - s_n$ converges in $\boldsymbol{P}$-probability to 0, that is:*

$$for\ any\ \varepsilon > 0: \qquad \lim_n \ \boldsymbol{P}\left(|\widehat{S}_n - s_n| > \varepsilon\right) \ = \ 0.$$

Theorem 2 *(Strong Law of Large Numbers) The sequence $\widehat{S}_n - s_n$ converges $\boldsymbol{P}$-almost surely to 0, that is:*

$$\boldsymbol{P}\left(\lim_n \left(\widehat{S}_n - s_n\right) = 0\right) \ = \ 1.$$

In the next section we give motivations for considering these results and an application of them. In section 3 we provide proofs of the Theorem 1 and Theorem 2. In section 4 we recall previous results related to dynamical models.

2 Applications and motivations. Theorem 1 and Theorem 2 may be applied to the following mathematical model.

Let's consider a convex averaging sequence $\{C_n \ ; \ n \geq 1\}$ (see [3]), that is:

1. $C_n \subset \mathbf{R}^2$ convex Borel set,
2. $C_n \subset C_{n+1}$, for any $n \geq 1$,
3. $r(C_n) := \sup\{r > 0 \ : \ C_n$ contains a ball of radius $r\} \longrightarrow +\infty$, as $n \longrightarrow +\infty$.

So, denoting by ℓ Lebesgue measure on $\mathbf{R}^2$: $c_n \ := \ \ell(C_n) \uparrow +\infty$, as $n \longrightarrow +\infty$. We also assume that $c_n^{-1} n^\alpha \longrightarrow 0$, as $n \longrightarrow +\infty$, for some $\alpha > 1$.

For a fixed $\lambda > 0$ and for any $n \geq 1$, we consider $m_n = \lfloor \lambda c_n \rfloor$ discs $D_1, ..., D_{m_n}$ of random (bounded) sizes placed at random on C_n. The disc areas, say $\xi_1, \xi_2, ..., \xi_{m_n}$, are supposed to be i.i.d. and known. The positions of the centers of the discs, say $X_1, X_2, ..., X_{m_n}$, are supposed to be unknown (not observable). We assume they are i.i.d. uniformly distributed on C_n and independent of the areas.

Let's denote by Θ_n the union of the discs and by Δ_n the area fraction of the uncovered part of C_n:

$$\Theta_n := \bigcup_{i=1}^{m_n} D_i, \qquad \Delta_n := \frac{1}{c_n}\ell\left(C_n \setminus \Theta_n\right) = \frac{1}{c_n} \int_{C_n} I\left(x \notin \Theta_n\right) \, d\ell(x).$$

Because the model is only partially observable, we are not able to compute Δ_n. So an estimation problem arises for its expectation $\mathbf{E}(\Delta_n)$. In the following example we present concrete situations in which there is an interest in the estimation of $\mathbf{E}(\Delta_n)$.

Example 1. *Bombing model with obscuring object.* Suppose that, during a bombing, m_n bombs have been dropped on a region C_n. Because of the presence of obscuring objects (clouds, hills, ...) it is not possible to observe the hitting points of the bombs. It is assumed that each bomb destroys a circular region around it proportional to its destructive power, and that the destructive power of each dropped bomb is known. It is useful an estimation of the expected not destroyed portion of the region.

Example 2. *Sources of pollution.* Suppose it is known that m_n sources of pollution entered a region C_n but the positions of them are unknown. Suppose further that the polluting power of each source is known and that each source damages a circular region around it proportional to its polluting power. It is interesting to estimate the expected not damaged portion of the region.

In the following results we shows that $\mathbf{E}(\Delta_n)$ is equal to s_n as defined in (4). In the proof we do not consider edge effect. They are negligible if, as in our case, the region area c_n is very large with respect to the discs areas.

Theorem 3 *With the previous definitions and notations it is:*

$$s_n \;=\; \boldsymbol{E}(\Delta_n)\,.$$

Proof. If $0 \in C_n$ is a fixed test point, and D is the generic disc with area distributed as ξ_0 and center X uniformly distributed on C_n and independent of ξ_0, then:

$$\mathbf{P}(0 \notin D) \;=\; \int \mathbf{P}(0 \notin D \mid \xi = x)\; f_\xi(x)\; dx \;=$$

$$= \int \mathbf{P}\left(X \notin B\left(0, \sqrt{x/\pi}\right)\right)\; f_\xi(x)\; dx \;=$$

$$= \int \left(1 - \frac{x}{c_n}\right)\; f_\xi(x)\; dx \;= 1 - \frac{\mathbf{E}(\xi_0)}{c_n},$$

where $B(0,r)$ denotes the ball with center 0 and radius r. It follows that:

$$\mathbf{E}(\Delta_n) \;=\; \frac{1}{c_n}\,\mathbf{E}\left(\int_{C_n} I(x \notin \Theta_n)\; d\ell(x)\right) \;=$$

$$= \frac{1}{c_n}\int_{C_n} \mathbf{P}(0 \notin \Theta_n)\; d\ell(x) \;=\; \mathbf{P}(0 \notin \Theta_n) \;=$$

$$= \prod_{i=1}^{m_n} \mathbf{P}(0 \notin D_i) \;=\; \left(1 - \frac{\mathbf{E}(\xi_0)}{c_n}\right)^{m_n} = s_n. \quad \square$$

So we are looking for an estimator of s_n. One may think to consider the estimator

$$\left(1 - \frac{\overline{\xi}}{c_n}\right)^{m_n},$$

or, noting that

$$\lim_n s_n \;=\; e^{-\lambda \mathbf{E}(\xi_0)}, \tag{5}$$

to consider the estimator $e^{-\lambda\overline{\xi}}$, where $\overline{\xi}$, the sample mean of $\xi_1, ..., \xi_{m_n}$, is the natural estimator of $\mathbf{E}(\xi_0)$. But both of these estimators are not even unbiased.

Instead we consider the estimator $\widehat{S}_n$ of s_n defined in (1). By using independence and standard properties of expectation, it is easy to verify that $\widehat{S}_n$ is an unbiased estimator of s_n, that is: $\mathbf{E}\left(\widehat{S}_n\right) \;=\; s_n$. Now the questions are:

Is $\widehat{S}_n$ a weakly consistent estimator of s_n and in which sense?
Is $\widehat{S}_n$ a strongly consistent estimator of s_n and in which sense?
Theorem 1 and 2 give positive answers to the first and second question, respectively.

3 Proof of the Theorems. In this section we give the proofs of Theorem 1 and Theorem 2.

Proof of Theorem 1. Being $\widehat{S}_n$ an unbiased estimator of s_n, and because of Chebychev inequality, it is enough to prove that

$$\lim_n \left(\mathbf{E}\left(\widehat{S}_n^2\right) - s_n^2\right) \;=\; 0.$$

Because of independence,

$$\mathbf{E}\left(\widehat{S}_n^2\right) \;=\; \prod_{i=1}^{m_n} \mathbf{E}\left(\left(1-\frac{\xi_i}{c_n}\right)^2\right) \;=\; \left(1-\frac{2\mathbf{E}(\xi_0)}{c_n}+\frac{\mathbf{E}(\xi_0^2)}{c_n^2}\right)^{m_n}.$$

so that

$$\lim_n \left(\mathbf{E}\left(\widehat{S}_n^2\right)\right) \;=\; e^{-2\lambda\mathbf{E}(\xi_0)},$$

and the conclusion follows from (5). □

Proof of Theorem 2. If we prove that

$$\sum_{n=1}^{\infty} \mathbf{E}\left(\left(\widehat{S}_n - s_n\right)^4\right) \;<\; \infty, \tag{6}$$

then, because of Chebishev inequality,

$$\textit{for every } \varepsilon>0: \quad \sum_{n=1}^{\infty} \mathbf{P}\left(|\widehat{S}_n - s_n| > \varepsilon\right) \;<\; \infty,$$

and then, by the first Borel-Cantelli lemma:

$$\textit{for every } \varepsilon>0: \quad \mathbf{P}\left(\limsup_n \left\{|\widehat{S}_n - s_n| > \varepsilon\right\}\right) = 0,$$

from which the conclusion of Theorem 2 follows. So we just have to prove (6). By expanding, using independence and computing expectation we have:

$$\mathbf{E}\left(\left(\widehat{S}_n - s_n\right)^4\right) =$$

$$= \left(1-\frac{4a_1}{c_n}+\frac{6a_2}{c_n^2}-\frac{4a_3}{c_n^3}+\frac{a_4}{c_n^4}\right)^{m_n} + 6\left(1-\frac{2a_1}{c_n}+\frac{a_2}{c_n^2}\right)^{m_n}\left(1-\frac{a_1}{c_n}\right)^{2m_n}$$
$$-4\left(1-\frac{3a_1}{c_n}+\frac{3a_2}{c_n^2}-\frac{a_3}{c_n^3}\right)^{m_n}\left(1-\frac{a_1}{c_n}\right)^{m_n} - 3\left(1-\frac{a_1}{c_n}\right)^{4m_n}$$

where: $a_i = \mathbf{E}\left(\xi_0^i\right)$. Note that we may write

$$\begin{aligned}\mathbf{E}\left(\left(\widehat{S}_n - s_n\right)^4\right) = \quad & f(1/c_n; A, B_1, C_1, D_1) + 6\, f(1/c_n; A, B_2, C_2, D_2) \\ & -4\, f(1/c_n; A, B_3, C_3, D_3) - 3\, f(1/c_n; A, B_4, C_4, D_4),\end{aligned} \tag{7}$$

where

$$f(x; A, B, C, D) := \begin{cases} (1+Ax+Bx^2+Cx^3+Dx^4)^{1/x}, & \textit{for } x \neq 0 \\ e^A, & \textit{for } x = 0 \end{cases}$$

$$A := -4a_1,$$
$$B_1 := 6a_2, \quad B_2 := 5a_1^2 + a_2, \quad B_3 := 3a_1^2 + 3a_2, \quad B_4 := 6a_1^2,$$
$$C_1 := -4a_3, \quad C_2 := -(2a_1^3 + 2ab), \quad C_3 := -(3a_1a_2 + a_3), \quad C_4 := -4a_1^3$$
$$D_1 := a_4, \quad D_2 := a_1^2 a_2, \quad D_3 := a_1 a_3, \quad D_4 := a_1^4.$$

The function $f(\cdot; A, B, C, D)$ is defined and smooth in a neighborhood of $x = 0$ and it admits the following Taylor expansion around 0:

(8)
$$f(x; A, B, C, D) = e^A \left(1 + \left(B - \frac{A^2}{2}\right)x + \left(\frac{A^4}{8} + \frac{A^3}{3} - \left(\frac{A^2}{2} + A\right)B + \frac{B^2}{2} + C\right)x^2 + o(x^2)\right)$$

(notice that D does not appear).

Expanding each addendum in (7) as in (8) yields

$$\mathbf{E}\left(\left(\widehat{S}_n - s_n\right)^4\right) =$$

$$= e^A \Bigg[\quad 1 + \left(B_1 - \frac{A^2}{2}\right)\frac{1}{c_n} + \left(\frac{A^4}{8} + \frac{A^3}{3} - \left(\frac{A^2}{2} + A\right)B_1 + \frac{B_1^2}{2} + C_1\right)\frac{1}{c_n^2} +$$
$$6\left[1 + \left(B_2 - \frac{A^2}{2}\right)\frac{1}{c_n} + \left(\frac{A^4}{8} + \frac{A^3}{3} - \left(\frac{A^2}{2} + A\right)B_2 + \frac{B_2^2}{2} + C_2\right)\frac{1}{c_n^2}\right] -$$
$$4\left[1 + \left(B_3 - \frac{A^2}{2}\right)\frac{1}{c_n} + \left(\frac{A^4}{8} + \frac{A^3}{3} - \left(\frac{A^2}{2} + A\right)B_3 + \frac{B_3^2}{2} + C_3\right)\frac{1}{c_n^2}\right] -$$
$$3\left[1 + \left(B_4 - \frac{A^2}{2}\right)\frac{1}{c_n} + \left(\frac{A^4}{8} + \frac{A^3}{3} - \left(\frac{A^2}{2} + A\right)B_4 + \frac{B_4^2}{2} + C_4\right)\frac{1}{c_n^2}\right] \quad \Bigg] + o\left(\frac{1}{c_n^2}\right).$$

Whence

$$\mathbf{E}\left(\left(\widehat{S}_n - s_n\right)^4\right) = \quad e^A \Bigg[(B_1 + 6B_2 - 4B_3 - 3B_4)\frac{1}{c_n} + \Bigg(-\Big(\frac{A^2}{2} + A\Big)(B_1 + 6B_2 - 4B_3 - 3B_4) +$$
$$\frac{1}{2}(B_1^2 + 6B_2^2 - 4B_3^2 - 3B_4^2) + (C_1 + 6C_2 - 4C_3 - 3C_4)\Bigg)\frac{1}{c_n^2}\Bigg] + o\Big(\frac{1}{c_n^2}\Big).$$

A direct inspection shows that

$$B_1 + 6B_2 - 4B_3 - 3B_4 = C_1 + 6C_2 - 4C_3 - 3C_4 = 0,$$

and

$$B_1^2 + 6B_2^2 - 4B_3^2 - 3B_4^2 = 6\left(a_1^2 - a_2\right)^2,$$

thus

$$\mathbf{E}\left(\left(\widehat{S}_n - s_n\right)^4\right) = 3e^{-4a_1}\left(a_1^2 - a_2\right)^2 \frac{1}{c_n^2} + o\Big(\frac{1}{c_n^2}\Big).$$

The conclusion follows. □

4 Dynamical models. In previous papers we have considered dynamical models in which discs drop on the region at the times of a point process. In this section we recall the obtained results.

In [4] it is assumed that discs drop on C_n following a Poisson process N_n with mean measure of the form

$$\mu_n((s,t]) \ := \ c_n \cdot \int_s^t \lambda(u)du, \qquad 0 < s \leq t,$$

and that disc volume is a function $\xi(t)$ of dropping time t.

By using stochastic geometry (see [8]) and thinning properties of Poisson process it is shown that the expected free area function $\mathcal{S}$ is given by

$$\mathcal{S}(t) = \exp\left(- \int_0^t \ \lambda(s) \ \mathbf{E}(\xi(s)) \ ds \right), \quad t > 0.$$

The estimator $\widehat{\mathcal{S}}_n$ of $\mathcal{S}$ is defined by:

$$\widehat{\mathcal{S}}_n(t) \ := \ \prod_{s \leq t} \left(1 - \frac{\xi(s) \ dN_n(s)}{c_n} \right), \quad t > 0,$$

where $\prod_{s \leq t}$ means *product integral* (see [1] or [6]).

The following uniform weak law of large numbers is stated and proved.

Theorem 4 *$\widehat{\mathcal{S}}_n$ is a uniformly consistent estimator of $\mathcal{S}$ in $[0,T]$, that is*

$$\sup_{t \in [0,T]} \ |\widehat{\mathcal{S}}_n(t) - \mathcal{S}(t)| \ \stackrel{P}{\longrightarrow} \ 0, \quad \textit{as } n \longrightarrow +\infty;$$

In the proof martingales theory, Lenglart's inequality and properties of product integral are used (see [1]). Martingales theory is also used to prove the following result.

Theorem 5 *The process $\mathcal{M}_n^{\mathcal{S}} = \{\mathcal{M}_n^{\mathcal{S}}(t) \ ; \ t > 0\}$, defined by*

$$\mathcal{M}_n^{\mathcal{S}}(t) \ := \ \sqrt{c_n} \ \left(\widehat{\mathcal{S}}_n(t) - \mathcal{S}(t) \right), \quad t > 0,$$

is a zero-mean square integrable martingale, and, for any $t > 0$, its predictable variation is given by:

$$\langle \mathcal{M}_n^{\mathcal{S}} \rangle(t) \ = \ (\mathcal{S}(t))^2 \ \int_0^t \ \left(\frac{\widehat{\mathcal{S}}_n(s^-)}{\mathcal{S}(s)} \right)^2 \ \boldsymbol{E}(\xi^2(s)) \ \lambda(s) \ ds, \quad t > 0.$$

About asymptotic gaussianity, the following result is stated and its proof is obtained by using the central limit theorem for martingales, Duhamel's equation and properties of product integral (see [1]).

Theorem 6 *The process $\mathcal{M}_n^{\mathcal{S}}$ converges on the Skorokhod function space $D(0,T)$ to $-\mathcal{S} \cdot \mathcal{M}$,*

$$\mathcal{M}_n^{\mathcal{S}} \ \stackrel{\mathcal{D}}{\longrightarrow} \ -\mathcal{S} \cdot \mathcal{M}, \quad \textit{as } n \longrightarrow +\infty,$$

where $\mathcal{M}$ is a Gaussian martingale with variance function $v = \{v(t) \ ; \ t > 0\}$ defined by:

$$v(t) := \int_0^t \ \boldsymbol{E}\left(\xi^2(s)\right) \ \lambda(s) \ ds, \quad t > 0.$$

About variance estimation the following result holds.

Theorem 7 *The process* $\widehat{v}_n$ *defined by*

$$\widehat{v}_n(t) \ := \ \frac{1}{c_n} \int_0^t \xi^2(s) \ dN_n(s), \quad t > 0,$$

is a uniformly consistent estimator of v *in* $[0, T]$, *that is*

$$\sup_{t \in [0,T]} |\widehat{v}_n(t) - v(t)| \ \stackrel{P}{\longrightarrow} \ 0, \quad as \ n \longrightarrow +\infty.$$

So that confidence bands may be obtained. The asymptotic $100(1-\alpha)\%$ confidence band for $\mathcal{S}$ in $[0, T]$ is

$$\widehat{\mathcal{S}}_n(t) \left(1 \mp \frac{1 + \widehat{v}_n(t)}{\sqrt{c_n)}} \ \ e_{\alpha/2}(c) \right), \quad 0 \leq t \leq T$$

where $e_{\alpha/2}(c)$ denotes the upper $(\alpha/2)$-quantile of the distribution of $\sup_{x \in [0,c]} \ \left|W^0(x)\right|$, , W^0 standard Brownian bridge.

In [5] it is assumed that the dropping times sequence is a process

$$N_n(t) \ := \ N(c_n t), \quad 0 \leq t$$

where N is a renewal process with mean interarrival time μ. Denoting by $\omega_n(t)$ the expected free area at time t and

$$\mathcal{S}_n(t) \ := \ \left(1 - \frac{\mathbf{E}(\xi)}{c_n} \right)^{N_n(t)}, \qquad \mathcal{S}(t) \ := \ \exp\left(-\frac{\mathbf{E}(\xi)}{\mu} t \right),$$

the following result holds.

Theorem 8 *a)* $\omega_n(t) = \boldsymbol{E}[\mathcal{S}_n(t)]$, *for any* $t \geq 0$;
b) $\sup_{0 \leq t \leq T} \ |\omega_n(t) - \mathcal{S}(t)| \ \longrightarrow 0$, *as* $n \to +\infty$.

The estimator $\widehat{\mathcal{S}}_n$ of ω_n is defined by

$$\widehat{\mathcal{S}}_n(t) \ := \prod_{T_{ni} \leq t} \left(1 - \frac{\xi_{ni}}{c_n} \right), \quad 0 \leq t \leq T.$$

The following uniform weak law of large numbers is proved by using Kolmogorov's inequality and product integral properties.

Theorem 9 $\widehat{\mathcal{S}}_n$ *is a Uniform Consistent estimator of* ω_n, *that is:*

$$\sup_{0 \leq t \leq T} |\widehat{\mathcal{S}}_n(t) - \omega_n(t)| \ \Longrightarrow 0, \qquad as \ \ n \to +\infty.$$

The asymptotic Gaussianity stated below is proved by using product integral continuity properties, Duhamel's equation (see [1]), weak convergence theory and Donsker theorem (see [2]).

Theorem 10 *The process $\mathcal{M}_n^{\mathcal{S}} = \{\mathcal{M}_n^{\mathcal{S}}(t) \; : \; 0 \le t \le T\}$, defined by*

$$\mathcal{M}_n^{\mathcal{S}}(t) \; := \; \sqrt{c_n} \; \left(\frac{\widehat{\mathcal{S}}_n(t) - \mathcal{S}_n(t)}{\mathcal{S}_n(t)} \right), \qquad 0 \le t \le T,$$

converges to $W(v)$:

$$\mathcal{M}_n^{\mathcal{S}} \; \Rightarrow \; W(v), \quad \text{as} \;\; n \longrightarrow +\infty,$$

where W is a Standard Brownian motion on [0,T], and $v = \{v(t) \; ; \; 0 \le t \le T\}$ is defined by

$$v(t) := Var(\xi_0(t)) \cdot \frac{t}{\mu}, \quad 0 \le t \le T.$$

The following result concern variance estimation.

Theorem 11 *The process $\widehat{v}_n$ defined by*

$$\widehat{v}_n(t) \; := \; \frac{1}{c_n} \sum_{i=1}^{N_n(t)} \left(\xi_i - \frac{1}{N_n(t)} \sum_{i=1}^{N_n(t)} \xi_i \right)^2, \quad 0 \le t \le T.$$

is a uniformly consistent estimator of the variance function v, that is:

$$\sup_{0 \le t \le T} \; |\widehat{v}_n(t) - v(t)| \; \Longrightarrow 0, \quad \text{as} \;\; n \longrightarrow +\infty.$$

The asymptotic $100(1-\alpha)\%$ confidence band for $\mathcal{S}_n$ in $[0, T]$ is:

$$\widehat{\mathcal{S}}_n(t) \left(1 \mp \frac{1 + \widehat{v}_n(t)}{\sqrt{c_n}} e_{\alpha/2}(c) \right), \quad 0 \le t \le T,$$

where $e_{\alpha/2}(c)$ is the upper $(\alpha/2)$-quantile of the distribution of $\sup_{0 \le x \le c} \; \left| W^0(x) \right|$, W^0 standard Brownian bridge.

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