ξ-CLOSED SETS IN TOPOLOGICAL SPACES AND DIGITAL PLANES

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Received October 28, 2005

Abstract. In this paper, we introduce and investigate three classes of subsets called ξ-closed sets, ξ∗-closed sets and ξ∗∗-closed sets in topological spaces. As applications we introduce two separation axioms Tξ and Tξ∗∗ of topological spaces and we construct a group of ρ-homeomorphisms which contains the group of all homeomorphisms as a subgroup, where ρ ∈ {ξ, ξ∗, ξ∗∗}. A discussion of ρ-closed sets in the digital plane concludes the paper, where ρ ∈ {ξ, ξ∗∗}. The digital plane is a Tξ-space; it is not Tξ∗∗.

1 Introduction

In 1970, Levine [15] introduced and investigated the notion of generalized closed sets in a topological space and one of T1/2-spaces. By [15, Theorem 5.3, Corollary 5.6], it was shown that the class of the T1/2-spaces is placed between the class of the T0-spaces and one of the T1-spaces. In 1977, Dunham [9, Theorem 2.5] proved that a topological space is T1/2 if and only if every singleton is open or closed. We know that the digital line is a typical example of the T1/2-spaces (eg.[8, Example 4.6], [12]). Using the concept of α-sets (= α-open sets) [21], in 1993 Balachandran, Devi and Maki defined the concept of generalized ρ-closed sets (cf. Definition 2.1(ii)-(iii)) analogous to generalized closed sets [15], where ρ ∈ {α, α∗, α∗∗}. Recently, Devi, Bhuvaneswari and Maki define a weak form of generalized ρ-closed sets and investigate their behaviours in the digital plane, where ρ ∈ {α, α∗, α∗∗}. Moreover, Veera Kumar [27] define and investigate the notion of g∗-closed sets (cf. Definition 2.1(vi)) which is placed between the class of the closed sets and one of the generalized closed sets [15].

In Section 2 of this paper, we introduce a new class of generalized closed sets which is called ξ-closed sets (cf. Definition 2.2) and investigate some basic properties of them. In Section 3, ξ∗-closed sets and ξ∗∗-closed sets are introduced. Some implications of their generalized closed sets (cf. Remark 3.4) and some properties of their behaviours to a subspace are investigated. In Section 4, new topologies induced from families of ρ-closed sets, where ρ ∈ {ξ, ξ∗, ξ∗∗} are introduced (cf. Definition 2.1). The digital plane is an example of ξ-closed sets and it is not a ξ∗∗-space (cf. Remark 3.4). In Section 5, we introduce new separation axioms Tξ and Tξ∗∗ analogous to the axiom T1/2 [15]. The digital plane is an example of ξ-closed sets; it is not a ξ∗∗-space (cf. Remark 5.6; Theorem 7.1 and Remark 7.2 in Section 7). In Section 6, using ξ-closed sets, where ρ ∈ {ξ, ξ∗, ξ∗∗}, new classes of functions and some groups are introduced (cf. Definition 6.1, Definition 6.6). Their groups are new topological invariants (cf. Corollary 6.8(ii)). In Section 7, it is proved that the digital plane is a Tξ-space (Theorem 7.1); a discussion of ξ-closed sets and ξ∗∗-closed sets in the digital plane concludes the paper.

Throughout this paper, (X, τ) and (Y, σ) represent nonempty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Throughout this paper, “a space” means a topological space. For some undefined or related concepts, the reader is referred to [11], [20] in TOPOLOGY ATLAS, URL: http://at.yorku.ca./topology/.

2000 Mathematics Subject Classification. Primary: 54A05, 54D10, 54F65, 54H99.
Key words and phrases. generalized closed sets, g∗-closed sets, ξ-closed sets, ξ∗-closed sets, ξ∗∗-closed sets, preclosed sets, digital lines, digital planes, T1/2-spaces, Tξ-spaces.
2 On $\xi$-closed sets The purpose of this section is to introduce and investigate the notion of the $\xi$-closed sets and some relationships between well known generalized closed sets. A subset $A$ is called $\alpha$-open [21] in $(X, \tau)$ if $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$ holds; the complement of an $\alpha$-open set is called $\alpha$-closed. The family of all $\alpha$-open sets in $(X, \tau)$ is denoted by $\tau^\alpha$. The $\alpha$-closure of a subset $A$ is denoted by $\tau^\alpha\cdot\text{Cl}(A) = \cap\{F \setminus X \mid F \in \tau^\alpha \text{ and } A \subseteq F\}$.

Definition 2.1 We recall the following definitions which are used in this paper. Let $A$ be a subset of $(X, \tau)$.

(i) The set $A$ is called generalized closed [15] (briefly, $g$-closed) in $(X, \tau)$, if $\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$; the complement of a $g$-closed set of $(X, \tau)$ is called $g$-open in $(X, \tau)$. (ii) $A$ is called $go$-closed [16] (resp. $go^*$-closed, $go^{**}$-closed) in $(X, \tau)$, if $\tau^\alpha\cdot\text{Cl}(A) \subseteq U$ (resp. $\tau^\alpha\cdot\text{Cl}(A) \subseteq \text{Int}(U)$, $\tau^\alpha\cdot\text{Cl}(A) \subseteq \text{Int}(\text{Cl}(U))$) whenever $A \subseteq U$ and $U$ is $\alpha$-open in $(X, \tau)$. (iii) The complement of a $go$-closed set (resp. $go^*$-closed set, $go^{**}$-closed set) of $(X, \tau)$ is called $go$-open (resp. $go^*$-open, $go^{**}$-open) in $(X, \tau)$. (iv) $A$ is called $og$-closed [17] in $(X, \tau)$, if $\tau^\alpha\cdot\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$. (v) $A$ is called $gs$-closed [3] in $(X, \tau)$, if $s\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$, where $s\text{Cl}(A) = \cap\{F \mid F$ is a semi-closed subset of $(X, \tau)$ such that $A \subseteq F\}$ is the semi-closure of $A$. A subset $F$ is called semi-closed [14] in $(X, \tau)$, if $\text{Int}(\text{Cl}(F)) \subseteq F$ holds in $(X, \tau)$. (vi) $A$ is called $g^*$-closed [27] in $(X, \tau)$, if $\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$-open in $(X, \tau)$. Every closed set is $g^*$-closed; every $g^*$-closed set is $g$-closed [27, Theorems 3.2, 3.4].

Definition 2.2 A subset $A$ is called $\xi$-closed in $(X, \tau)$ if $\tau^\alpha\cdot\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a $g$-open set of $(X, \tau)$ (cf. Definition 2.1(iii)). The complement of a $\xi$-closed set of $(X, \tau)$ is called $\xi$-open in $(X, \tau)$.

Theorem 2.3 (i) Every closed set and every $\alpha$-closed set is $\xi$-closed. (ii) Every $\xi$-closed set is $go$-closed, $go^*$-closed, $go^{**}$-closed and $gs$-closed. □

Remark 2.4 The converse of Theorem 2.3(i) (resp. (ii)) is not true in general by the following example (i) (resp. (ii)). (i) Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{c, d\}, X\}$. For a space $(X, \tau)$, a subset $\{a, b, c\}$ is $\xi$-closed; it is neither closed nor $\alpha$-closed. Indeed, $\tau^\alpha = \{\emptyset, \{c, d\}, \{b, c, d\}, \{a, c, d\}, X\}$. (ii) Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{b\}, \{a, c\}, X\}$. For a space $(X, \tau)$, a subset $\{a\}$ is $\xi$-closed, $\alpha$-closed, $gs$-closed and $g^*$-closed; it is not $\xi$-closed.

Remark 2.5 By Theorem 2.3, we obtain the following diagram of implications. Remark 2.4 shows that implications are not reversible.

\[\text{closed} \rightarrow \alpha\text{-closed} \rightarrow \xi\text{-closed} \rightarrow \text{go}\text{-closed}\]

Remark 2.6 The following examples show that the $\xi$-closedness is independent from the $go^*$-closedness, $g^{**}$-closedness and $g^*$-closedness. (i) In the same space $(X, \tau)$ of Remark 2.4(ii), a subset $\{a\}$ is $go^*$-closed; it is not $\xi$-closed. (ii) In the same space $(X, \tau)$ of Remark 2.4(i), a subset $\{a\}$ is $\xi$-closed; it is not $go^*$-closed. (iii) Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{b\}, \{a, b\}, X\}$. For a space $(X, \tau)$, a subset $\{a\}$ is $\xi$-closed; it is neither $g$-closed nor $g^*$-closed. A subset $\{b, c\}$ is $g$-closed and $g^*$-closed; it is not $\xi$-closed.

Theorem 2.7 A subset $A$ of $X$ is $\xi$-closed in $(X, \tau)$ if and only if $A$ is $g^*$-closed in a space $(X, \tau^\alpha)$.

Proof. We recall that a subset $B$ is $go$-open in $(X, \tau)$ if and only if $B$ is $g$-open in $(X, \tau^\alpha)$ ([16, Theorem 2.3], cf. Definition 2.1(iii)). Thus, Theorem 2.7 is proved by definitions. □
Remark 2.8 Let by using Theorem 2.7, we can translate theorems in [27] into ξ-closedness version. For examples, we have the following:

(i) A set A is ξ-closed in (X, τ) if and only if τα-Cl(A) \ A does not contain any non-empty go-closed set of (X, τ) ([27, Theorem 3.14]).

(ii) The union of two ξ-closed sets is ξ-closed ([27, Remark 3.12]).

We have an alternative characterization of ξ-closed sets (cf. Theorem 2.10 below). We prepare the following notations: Gαα(τ, O) := { U \ U is go-open in (X, τ)}; Gαα(X, τ) := { f | f is go-closed in (X, τ)}; Xα := { x | x ∈ Gαα(X, τ)}; Xξ := { x | x is ξ-open in (X, τ)}; Gαα(X, τ)-Ker(α) = \{ U \ U is go-open in (X, τ) and A ⊆ U \} for a subset A of X (briefly, Gαα-O-Ker(α)).

Lemma 2.9 For any space (X, τ), X = Xα ∪ Xξ holds.

**Proof.** Let x ∈ X. Suppose that { x } is not go-closed in (X, τ) (i.e. X \ { x } is not go-open). Then, X is a unique go-open set containing X \ { x }. Thus X \ { x } is ξ-closed in (X, τ) and so { x } is ξ-open. Therefore, x ∈ Xα ∪ Xξ holds.  □

Theorem 2.10 For a subset A of (X, τ), the following properties are equivalent:

1. A is ξ-closed;
2. τα-Cl(A) ⊆ Gαα-O-Ker(α) holds;
3. (i) τα-Cl(A) ∩ Xα ⊆ A and (ii) τα-Cl(A) ∩ Xξ ⊆ Gαα-O-Ker(α) hold.

**Proof.** (1) ⇒ (2) Let x ∉ Gαα-O-Ker(α). Then, there exists a set U ∈ Gαα(X, τ) such that x ∉ U and A ⊆ U. Since A is ξ-closed, τα-Cl(A) ⊆ U and so x ∉ τα-Cl(A). (2) ⇒ (3) (i) First we claim that Gαα-O-Ker(α) ∩ Xα ⊆ A. Indeed, let x ∈ Gαα-O-Ker(α) ∩ Xα and assume that x ∉ A. Since the set X \ { x } ∈ Gαα(X, τ) and A ⊆ X \ { x }, Gαα-O-Ker(α) ⊆ X \ { x }. Then, we have that x ∈ X \ { x } and so this is a contradiction. Thus, we show that τα-Cl(A) ∩ Xα ⊆ A. By using (2), τα-Cl(A) ∩ Xα ⊆ Gαα-O-Ker(α) ∩ Xα ⊆ A. (ii) It is obtained by (2). (3) ⇒ (2) By Lemma 2.9 and (3), τα-Cl(A) = τα-Cl(A) ∩ Xα \ τα-Cl(A) ∩ Xξ = (τα-Cl(A) ∩ Xα) ∪ (τα-Cl(A) ∩ Xξ) \ U ∪ Gαα-O-Ker(α) = Gαα-O-Ker(α). That is, τα-Cl(A) ⊆ Gαα-O-Ker(α) holds. (2) ⇒ (1) Let U ∈ Gαα(O, τ) such that A ⊆ U. Then, we have that Gαα-O-Ker(α) ⊆ U and so, by (2), τα-Cl(A) ⊆ U. Therefore, A is ξ-closed. □

Corollary 2.11 Let P := \{ A | τα-Cl(A) ∩ Xξ ⊆ Gαα-O-Ker(α) \}.

(i) If \∩_{i ∈ Σ} A_i ∈ P and A_i is ξ-closed set in (X, τ) for each i ∈ Σ, then \∩_{i ∈ Σ} A_i is ξ-closed in (X, τ).

(ii) If P = P(X) and A_i is ξ-closed set in (X, τ) for each i ∈ Σ, then \∩_{i ∈ Σ} A_i is ξ-closed in (X, τ).

(iii) If Xξ = ∅ and A_i is ξ-closed set in (X, τ) for each i ∈ Σ, then \∩_{i ∈ Σ} A_i is ξ-closed in (X, τ).

(iv) If τα-Cl(A_i) ∩ Xξ ⊆ A_i and A_i is a ξ-closed set in (X, τ) for each i ∈ Σ, then \∩_{i ∈ Σ} A_i is ξ-closed in (X, τ).

**Proof.** (i) By Theorem 2.10, τα-Cl(A_i) ∩ Xα ⊆ A_i for each i ∈ Σ. Then, we have that τα-Cl(\∩_{i ∈ Σ} A_i) ∩ Xα ⊆ \∩_{i ∈ Σ} A_i. Using assumption and Theorem 2.10(3), \∩_{i ∈ Σ} A_i is ξ-closed. (ii)-(iv) By (i), they are proved. □

The following theorem is concerned on a property of gα-closedness in a subspace. As a corollary, we have a property of ξ-closedness in a subspace (cf. Corollary 2.13 below).
We recall the following notations and some properties. For a space \((X, \tau)\) and a subset \(H\) of \((X, \tau)\), \(GO(X, \tau) := \{ U \mid U \text{ is g-open in } (X, \tau) \}\); \(GO(H, \tau|H) := \{ V \mid V \text{ is g-open in } (H, \tau|H) \}\). If \(U \in GO(X, \tau)\) and \(V \in GO(X, \tau)\), then \(U \cap V \in GO(X, \tau)\) ([15, Theorem 2.4]). If \(U \in GO(X, \tau)\), \(H \in \tau\) and \(X \setminus H \in \tau\), then \(U \cap H \in GO(H, \tau|H)\) ([24, Lemma 2.10(ii)]). If \(U \in GO(X, \tau)\) and \(V \in \tau\), then \(U \cup V \in GO(X, \tau)\) ([15, Corollary 2.7]).

**Theorem 2.12** Let \(B\) and \(H\) be subsets in \((X, \tau)\) such that \(B \subseteq H\).

(i) If \(B\) is \(g^*\)-closed in \((H, \tau|H)\) and \(H\) is open and closed in \((X, \tau)\), then \(B\) is \(g^*\)-closed in \((X, \tau)\).

(ii) Suppose that, for \((X, \tau)\) and \(H\),

\(\text{(i)}\) \(GO(H, \tau|H) \subseteq \{ H \cap O \mid O \in GO(X, \tau) \}\) holds.

\(\text{(ii)}\) If \(B\) is \(g^*\)-closed in \((X, \tau)\), then \(B\) is \(g^*\)-closed in \((H, \tau|H)\).

Proof. (i) Let \(O \in GO(X, \tau)\) such that \(B \subseteq O\). We have that \(H \cap O \in GO(H, \tau|H)\) and \(B \subseteq H \cap O\). Then, \(H \cap O \subseteq H \cap \tau\) holds. It is shown that \(H \subseteq O \cup (X \setminus Cl(B))\) and the subset \(O \cup (X \setminus Cl(B))\) \(\subseteq GO(X, \tau)\). Since \(H\) is \(g^*\)-closed in \((X, \tau)\), \(Cl(H) \subseteq O \cup (X \setminus Cl(B))\) and so \(Cl(B) \subseteq O \cup (X \setminus Cl(B))\). Therefore, we have that \(Cl(B) \subseteq O\) and so \(B\) is \(g^*\)-closed in \((X, \tau)\).

(ii) Let \(V \in GO(H, \tau|H)\) such that \(B \subseteq V\). Using assumption (i), there exists a subset \(O \in GO(X, \tau)\) such that \(V = H \cap O\). Then, we have that \(Cl(B) \subseteq O\) and so \((\tau|H)-Cl(B) = Cl(B) \cap H \subseteq O \cap H = V\). Therefore, \(B\) is \(g^*\)-closed in \((H, \tau|H)\).

Using Theorem 2.12 for \((X, \tau^o)\) and Theorem 2.7, we prove the following property on \(\xi\)-closedness in a subspace.

**Corollary 2.13** Let \(B\) and \(H\) be subsets of \((X, \tau)\) such that \(B \subseteq H\).

(i) If \(B\) is \(\xi\)-closed in \((H, \tau|H)\) and \(H\) is open and closed in \((X, \tau)\), then \(B\) is \(\xi\)-closed in \((X, \tau)\).

(ii) Suppose that, for \((X, \tau^o)\) and \(H\),

\(\text{(i)}\) \(GO(H, \tau^o|H) \subseteq \{ H \cap O \mid O \in GO(X, \tau^o) \}\) holds.

\(\text{(ii)}\) If \(B\) is \(\xi\)-closed in \((X, \tau)\) and \(H\) is open in \((X, \tau)\), then \(B\) is \(\xi\)-closed in \((H, \tau|H)\).

Proof (i) Using Theorem 2.7, we have that \(B\) is \(\xi\)-closed in \((H, \tau|H)\) if and only if \(B\) is \(g^*\)-closed in \((H, (\tau|H)^o)\). Then, the set \(B\) is \(g^*\)-closed in \((H, (\tau^o|H))\), because \((\tau^o|H)^o = \tau^o|H\) holds if \(H \in \tau\) (eg.[16, Lemma 2.4(ii)]). Using Theorem 2.12(i) for \((X, \tau^o)\), \(B\) is \(g^*\)-closed in \((X, \tau^o)\) because \(H\) is open and closed in \((X, \tau^o)\). Therefore, using Theorem 2.7, \(B\) is \(\xi\)-closed in \((X, \tau)\).

(ii) By Theorem 2.12(ii) for \((X, \tau^o)\), it is shown that \(B\) is \(g^*\)-closed in \((H, (\tau|H)^o)\) and so \(B\) is \(g^*\)-closed in \((H, (\tau|H)^o)\). Therefore, using Theorem 2.7 \(B\) is \(\xi\)-closed in \((H, \tau|H)\).

3 On \(\xi^*\)-closed sets and \(\xi^{**}\)-closed sets

We introduce two classes of “\(\xi\)-closed sets” and investigate some properties.

**Definition 3.1** (i) A subset \(A\) is called \(\xi^*\)-closed in \((X, \tau)\) if \(\tau^o-Cl(A) \subseteq Int(U)\) whenever \(A \subseteq U\) and \(U\) is go-open in \((X, \tau)\).

(ii) A subset \(A\) is called \(\xi^{**}\)-closed in \((X, \tau)\) if \(\tau^{**-Cl(A)} \subseteq Int(Cl(U))\) whenever \(A \subseteq U\) and \(U\) is go-open in \((X, \tau)\).

(iii) The complement of a \(\xi^*\)-closed set (resp. \(\xi^{**}\)-closed set) of \((X, \tau)\) is called a \(\xi^*\)-open (resp. \(\xi^{**}\)-open) set in \((X, \tau)\).

**Theorem 3.2** (i) Every \(\xi^*\)-closed set is \(\xi\)-closed.

(ii) Every \(\xi\)-closed set is \(\xi^{**}\)-closed.
Proof. (i) The proof is obvious. (ii) Let $A$ be a $\xi$-closed set of a space $(X, \tau)$. Let $U$ be a go-open set of $(X, \tau)$ such that $A \subseteq U$. Then, we have that $\tau^\alpha \text{-Cl}(A) \subseteq U$. We recall that every go-closed set is $\omega$go-closed ($=\text{preclosed}$) and so every go-open set is preopen ([6, Theorems 2.2, 2.3(ii), Remark 2.4], cf.Definition 2.1(iii)). Therefore, we have that $\tau^\alpha \text{-Cl}(A) \subseteq U \subseteq \text{Int}(\text{Cl}(U))$ and so $A$ is $\xi^{**}$-closed. □

Remark 3.3 (i) The converses of Theorem 3.2 are not true in general by the following examples. Let $(X, \tau)$ be a space such that $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. Then, a subset $\{b\}$ is $\xi$-closed; it is not $\xi^*$-closed. A subset $\{a\}$ is $\xi^{**}$-closed; it is not $\xi$-closed.

(ii) The following examples show that $\alpha$-closedness and $\xi^{**}$-closedness are independent. Let $(X, \tau)$ be a space of (i) above. A subset $\{b\}$ is $\alpha$-closed; it is not $\xi^*$-closed. Let $(X, \tau)$ be a space in Remark 2.4(i). A subset $\{a, b, c\}$ is $\xi^*$-closed; it is not $\alpha$-closed.

Remark 3.4 Theorem 3.2, Theorem 2.3 and Remark 3.3 show the following diagram of implications. Remark 3.3 and Remark 2.4(i) show that all implications are not reversible.

$$
\begin{array}{c}
\alpha \text{-closed} \\
\text{Not} \downarrow \\
\xi \text{-closed} \rightarrow \xi^{**} \text{-closed} \\
\xi^* \text{-closed}
\end{array}
$$

Theorem 3.5 Let $A$ be a subset of $(X, \tau)$.

(i) The union of two $\alpha$-closed sets is $\rho$-closed, where $\rho \in \{\xi^*, \xi^{**}\}$.

(ii) If $A$ is $\rho$-closed in $(X, \tau)$ and $A \subseteq B \subseteq \tau^\alpha \text{-Cl}(A)$, then $B$ is $\rho$-closed, where $\rho \in \{\xi^*, \xi^{**}\}$.

(iii) If $A$ is $\xi^*$-closed (resp. $\xi^{**}$-closed), then $\tau^\alpha \text{-Cl}(A) \setminus A$ does not contain non-empty go-closed set (resp. go-closed and semi-open set).

(iv) For each $x \in X, \{x\}$ is go-closed or its complement $X \setminus \{x\}$ is $\xi^*$-closed in $(X, \tau)$.

(v) For each $x \in X, \{x\}$ is go-closed and open, or its complement $X \setminus \{x\}$ is $\xi^{**}$-closed in $(X, \tau)$.

Proof. (i)-(iii) The proofs are obvious. (iv) Suppose that $\{x\}$ is not go-closed in $(X, \tau)$ (i.e.$X \setminus \{x\}$ is not go-open). Then, $X$ is a unique go-open set containing $X \setminus \{x\}$. Thus $X \setminus \{x\}$ is $\xi^*$-closed in $(X, \tau)$. (v) Suppose that $\{x\}$ is not go-closed in $(X, \tau)$. By similar argument of the proof of (iv), it is shown that $X \setminus \{x\}$ is $\xi^{**}$-closed in $(X, \tau)$. Suppose that $\{x\}$ is not open. Let $U$ be a go-open set containing $X \setminus \{x\}$. If $U = X$, then $\tau^\alpha \text{-Cl}(X \setminus \{x\}) \subseteq \text{Int}(\text{Cl}(U)) = X$. If $U = X \setminus \{x\}$, then $\tau^\alpha \text{-Cl}(X \setminus \{x\}) \subseteq X = \text{Int}(X) = \text{Int}(\text{Cl}(U))$. Thus, $X \setminus \{x\}$ is $\xi^{**}$-closed in $(X, \tau)$. □

We have the following property on $\xi^*$-closedness and $\xi^{**}$-closedness in a subspace, respectively.

Theorem 3.6 Let $B$ and $H$ be subsets of $(X, \tau)$ such that $B \subseteq H$.

(i) If $B$ is $\rho$-closed in $(H, \tau|H)$ and $H$ is open and closed in $(X, \tau)$, then $B$ is $\rho$-closed in $(X, \tau)$, where $\rho \in \{\xi^*, \xi^{**}\}$.

(ii) Suppose that, for $(X, \tau^\alpha)$ and $H$,

($*$) $\text{GO}(H, \tau^\alpha|H) \subseteq \{H \cap O \mid O \in \text{GO}(X, \tau^\alpha)\}$ holds.

If $B$ is $\rho$-closed in $(X, \tau)$ and $H$ is open in $(X, \tau)$, then $B$ is $\rho$-closed in $(H, \tau|H)$, where $\rho \in \{\xi^*, \xi^{**}\}$.
Proof (i) Case 1. \( \rho = \xi^* \): Let \( U \) be a \( \gamma \)-open set in \((X, \tau)\) (i.e. \( U \in GO(X, \tau^o)\)), cf. [16, Theorem 2.3] Definition 2.1(iii)) such that \( B \subseteq U \). Then, we have that \( U \cap H \subseteq GO(H, (\tau|H)^o) \), because \( U \in GO(X, \tau^o) \) and \( \tau^o|H = (\tau|H)^o \) for \( \tau \in \tau \) (e.g.,[4, Lemma 2.4(ii)]). Since \( B \) is \( \xi^* \)-closed in \((H, \tau|H)\) and \( B \subseteq U \cap H \), \( \tau^o|H)-Cl(B) \subseteq (\tau|H)-Int(U \cap H) \) and so \( \tau^o|H)-Cl(B) \cap H \subseteq Int(U \cap H) \cap H \). Put \( V := Int(U \cap H) \cup (X \setminus \tau^o|H)-Cl(B) \). Then, it is shown that \( H \subseteq V \) and \( V \in \tau^o \) and so \( V \) is \( \gamma \)-open in \((X, \tau)\). It follows from assumption that \( \tau^o|H)-Cl(B) \subseteq \tau^o|H)-Cl(H) \subseteq V \subseteq Int(U) \cup (X \setminus \tau^o|H)-Cl(B) \) and hence \( \tau^o|H)-Cl(B) \subseteq \tau^o|H)-Cl(H) \) holds (i.e., \( B \) is \( \xi^* \)-closed in \((X, \tau)\)). Case 2. \( \rho = \xi^{**} \): Let \( U \) be a \( \gamma \)-open set (i.e. \( U \in GO(X, \tau^o)\)) such that \( B \subseteq U \). Since \( U \cap H \subseteq GO(H, (\tau|H)^o) \) and \( B \subseteq U \cap H \), we have that \( H \cap \tau^o|H)-Cl(B) = \tau^o|H)-Cl(B) \subseteq (\tau|H)-Int((\tau|H)-Cl(H \cap U)) = (\tau|H)-Int(H \cap Cl(H \cap U)) \subseteq H \cap Int(Cl(H \cap U)) \subseteq H \cap Int(Cl(H \cap U)) \) hold. Put \( W := \int(Cl(H \cap U)) \cup (X \setminus \tau^o|H)-Cl(B) \). Then, \( H \subseteq W \) and \( W \in \tau^o \) and so \( W \) is \( \gamma \)-open in \((X, \tau)\). Since \( H \) is \( \xi^{**} \)-closed, \( \tau^o|H)-Cl(B) \subseteq \tau^o|H)-Cl(H) \subseteq W \subseteq Int(Cl(U)) \cup (X \setminus \tau^o|H)-Cl(B) \) and hence \( \tau^o|H)-Cl(B) \subseteq \tau^o|H)-Cl(H) \) holds (i.e. \( B \) is \( \xi^{**} \)-closed in \((X, \tau)\)).

(ii) Case 1. \( \rho = \xi^* \): Let \( V \) be a \( \gamma \)-open set of \((H, \tau|H)\) (i.e., \( \gamma \subseteq GO(H, (\tau|H)^o)\)) such that \( B \subseteq U \). Then, \( V \subseteq GO(H, \tau^o|H) \). Using (**), there exists a set \( O \subseteq GO(X, \tau^o) \) such that \( V = O \cap H \). Since \( B \subseteq O \) and \( O \in \xi^* \)-closed in \((X, \tau)\), we have that \( \tau^o|H)-Cl(B) = \tau^o|H)-Cl(O) = H \cap \tau^o|H)-Cl(O) \subseteq H \cap Int(O) = H \cap Int(H \cap Int(O)) = (\tau|H)-Int(H \cap Int(O)) \subseteq (\tau|H)-Int(V) \subseteq \tau^o|H)-Cl(B) \). Thus \( B \subseteq \xi^* \)-closed in \((X, \tau)\). Case 2. \( \rho = \xi^{**} \): Let \( V \) be a \( \gamma \)-open set of \((H, \tau|H)\) (i.e. \( \gamma \subseteq GO(H, (\tau|H)^o)\)) such that \( B \subseteq U \). Then, using (**), there exists a set \( O \subseteq GO(X, \tau^o) \) such that \( V = O \cap H \). Since \( B \subseteq O \) and \( O \subseteq \xi^{**} \)-closed in \((X, \tau)\), we have that \( \tau^o|H)-Cl(B) \subseteq \tau^o|H)-Cl(O) \subseteq (\tau|H)-Int(H \cap Int(O)) \subseteq (\tau|H)-Int(H \cap Cl(H \cap O)) \subseteq (\tau|H)-Int(H \cap Cl(O)) \subseteq (\tau|H)-Int((\tau|H)-Cl(V)) \) and so \( \tau^o|H)-Cl(B) \subseteq (\tau|H)-Int((\tau|H)-Cl(V)) \) and so \( B \subseteq \xi^{**} \)-closed in \((H, \tau|H)\).

4 Topologies induced from families of \( \rho \)-closed sets, where \( \rho \in \{\xi, \xi^*, \xi^{**}\} \) We can introduce topologies from \( \rho \)-closed sets, where \( \rho \in \{\xi, \xi^*, \xi^{**}\} \).

Definition 4.1 For a subset \( E \) of \((X, \tau)\), we define the following closures: \( \rho Cl_{\#}(E) := \cap\{A \mid A \subseteq \rho Cl_{\#}(E) \} \) and \( E \subseteq \xi^* Cl_{\#}(E) \subseteq \xi Cl_{\#}(E) \subseteq Cl_{\#}(E) \) hold.

Theorem 4.2 Let \( E \) and \( F \) be subsets of \((X, \tau)\).

(i) \( E \subseteq \xi^* Cl_{\#}(E) \subseteq \xi Cl_{\#}(E) \subseteq Cl_{\#}(E) \) and \( E \subseteq \xi^* Cl_{\#}(E) \subseteq \tau^o|H)-Cl(O) \subseteq Cl_{\#}(E) \) hold.

(ii) For each \( \rho \in \{\xi, \xi^*, \xi^{**}\} \), \( \rho Cl_{\#}(\emptyset) = \emptyset \) and \( \rho Cl_{\#}(X) = X \) hold.

(iii) If \( E \subseteq F \), then \( \rho Cl_{\#}(E) \subseteq \rho Cl_{\#}(F) \) holds for each \( \rho \in \{\xi, \xi^*, \xi^{**}\} \).

(iv) For each \( \rho \in \{\xi, \xi^*, \xi^{**}\} \), \( Cl_{\#}(E \cup F) = Cl_{\#}(E) \cup \rho Cl_{\#}(F) \) holds.

(v) If \( E \) is \( \rho \)-closed, then \( Cl_{\#}(E) = E \) holds, whereas \( \rho \in \{\xi, \xi^*, \xi^{**}\} \).

(vi) For each \( \rho \in \{\xi, \xi^*, \xi^{**}\} \), \( Cl_{\#}(\rho Cl_{\#}(E)) = Cl_{\#}(E) \) holds.

(vii) For each \( \rho \in \{\xi, \xi^*, \xi^{**}\} \), \( Cl_{\#}(\bullet) \) is a Kuratowski closure operator on \( X \).

Proof. (i) The implications are obtained by Theorem 2.3 and Theorem 3.2 respectively. (ii)-(iii) They are obvious from definitions. (iv) By (iii), it is enough to prove that \( \rho Cl_{\#}(E \cup F) \subseteq \rho Cl_{\#}(E) \cup \rho Cl_{\#}(F) \) holds. Let \( x \notin \rho Cl_{\#}(E) \cup \rho Cl_{\#}(F) \). Then, there exist \( \rho \)-closed subsets \( A \) and \( B \) such that \( x \notin A, x \notin B, E \subseteq A \) and \( F \subseteq B \). By Remark 2.8 and Theorem 3.5(i), it is obtained that \( A \cup B \) is \( \rho \)-closed. Since \( E \cup F \subseteq A \cup B \) and \( x \notin A \cup B \), we have that \( x \notin \rho Cl_{\#}(E \cup F) \). (v) It is obvious from definition. (vi) Using (i) it suffices to prove an inclusion: \( \rho Cl_{\#}(\rho Cl_{\#}(E)) \subseteq \rho Cl_{\#}(E) \). Let \( x \notin \rho Cl_{\#}(E) \). Then, there exists a \( \rho \)-closed set \( A \) such that \( E \subseteq A \) and \( x \notin A \). Then, by (v), \( \rho Cl_{\#}(E) \subseteq A \) and hence \( x \notin \rho Cl_{\#}(E) \). (vii) It is obvious from (i),(ii),(iv) and (vi).
Definition 4.3 For a space \((X, \tau)\) and a \(\rho \in \{\xi, \xi^*, \xi^{**}\}\), we define the following families:
\[ \rho \tau_\# := \{U \mid \rho \text{Cl}_\#(X \setminus U) = X \setminus U\} \]

Corollary 4.4 For any topology \(\tau\), the following properties hold.
(i) Three families of subsets \(\xi \tau_\#, \xi^* \tau_\#\) and \(\xi^{**} \tau_\#\) are topologies of \(X\).
(ii) \(\xi^* \tau_\# \subseteq \xi \tau_\# \subseteq \xi^{**} \tau_\# = P(X)\) and \(\tau \subseteq \tau^\alpha \subseteq \xi \tau_\#\).

Proof. (i) By Theorem 4.2(vii), they are topologies of \(X\). (ii) The inclusions are obtained by Theorem 4.2 and Definition 4.3. We claim that \(P(X) \subseteq \xi \tau_\#\) holds. Let \(A \in P(X)\). Using Theorem 3.5(v), any singleton is \(\rho \text{O}\)-closed and open, or \(\xi^{**}\)-open in \((X, \tau)\). Thus any singleton is \(\xi^{**}\)-open in \((X, \tau)\), because an open set is \(\xi^{**}\)-open in \((X, \tau)\) (cf. Theorem 2.3(i), Theorem 3.2(ii)). Then, we have that \(\{x\} \in \xi \tau_\#\) for each \(x \in A\). By (i), it is shown that \(A = \bigcup\{\{x\} \mid x \in A\} \in \xi \tau_\#\) and so \(P(X) \subseteq \xi \tau_\#\). □

Remark 4.5 Let \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}\). Then, a subset \(A = \{a, c\}\) of a space \((X, \tau)\) is not a \(\xi^{**}\)-open set; \(A \in \xi \tau_\#\) (cf. Corollary 4.4(ii)).

In the proof of the following proposition, we use the following notations: \(\xi \text{O}(X, \tau) = \{U \mid U \in \xi \text{O}(X) \subseteq \xi \tau_\#\}\); \(\xi^{**}\text{O}(X, \tau) = \{V \mid V \in \xi^{**}\text{O}(X) \subseteq \xi^{**}\tau_\#\}\). The example of Remark 4.5 shows that \(\xi^{**}\text{O}(X, \tau) \neq \xi^{**} \tau_\#\) in general.

Proposition 4.6 For any space \((X, \tau)\), the following properties hold.
(i) Every \(\xi\)-closed set is \(\alpha\)-closed in \((X, \tau)\) if and only if \(\xi \tau_\# = \tau^\alpha\).
(ii) Every \(\xi\)-closed set is closed in \((X, \tau)\) if and only if \(\xi \tau_\# = \tau\).
(iii) Every \(\xi^{**}\)-closed set is \(\alpha\)-closed in \((X, \tau)\) if and only if \(\tau^\alpha = P(X)\).
(iv) Every \(\xi^{**}\)-closed set is closed in \((X, \tau)\) if and only if \(\tau = P(X)\).

Proof. (i) (Necessity) Since every \(\xi\)-closed set is \(\alpha\)-closed in \((X, \tau)\), we have that \(\xi \text{O}(X, \tau) = \tau^\alpha\) and so, for any subset \(E\) of \(X\), \(\xi \text{Cl}_\#(E) = \cap\{F \mid E \subseteq F, X \setminus F \in \xi \text{O}(X, \tau)\} = \cap\{F \mid E \subseteq F, X \setminus F \in \tau^\alpha\text{-Cl}(E)\}\). Hence, \(\xi \tau_\# = \tau^\alpha\). (Sufficiency) Let \(A\) be a \(\xi\)-closed set of \((X, \tau)\). Then, \(A = \xi \text{Cl}_\#(A)\) and so \(X \setminus A \in \xi \tau_\#\). By assumption, \(X \setminus A \in \tau^\alpha\) and hence \(A\) is \(\alpha\)-closed in \((X, \tau)\). (ii) (Necessity) Since every \(\xi\)-closed set is closed in \((X, \tau)\), we have that \(\xi \text{O}(X, \tau) = \tau\) and so, for any subset \(E\) of \(X\), \(\xi \text{Cl}_\#(E) = \cap\{F \mid E \subseteq F, X \setminus F \in \xi \text{O}(X, \tau)\} = \cap\{F \mid E \subseteq F, X \setminus F \in \tau\text{-Cl}(E)\}\). Hence, \(\xi \tau_\# = \tau\). (Sufficiency) Let \(A\) be a \(\xi\)-closed set of \((X, \tau)\). Then, \(A = \xi \text{Cl}_\#(A)\) and so \(X \setminus A \in \xi \tau_\#\). By assumption, \(X \setminus A \in \tau\) and hence \(A\) is \(\alpha\)-closed in \((X, \tau)\). (iii) (Necessity) Since every \(\xi^{**}\)-closed set is \(\alpha\)-closed in \((X, \tau)\), we have that \(\xi^{**}\text{O}(X, \tau) = \tau^\alpha\) and so, for any subset \(E\) of \(X\), \(\xi^{**}\text{Cl}_\#(E) = \cap\{F \mid E \subseteq F, X \setminus F \in \xi^{**}\text{O}(X, \tau)\} = \cap\{F \mid E \subseteq F, X \setminus F \in \tau^\alpha\text{-Cl}(E)\}\). Hence, \(\xi^{**} \tau_\# = \tau^\alpha\). (Sufficiency) Let \(A\) be a \(\xi^{**}\)-closed set of \((X, \tau)\). Then, \(A = \xi^{**}\text{Cl}_\#(A)\) and so \(X \setminus A \in \xi^{**} \tau_\#\). By assumption, \(X \setminus A \in \tau\) and hence \(A\) is closed in \((X, \tau)\). □

The above Proposition 4.6 suggests new separation axioms \(T_{\xi}\) and \(T_{\xi^{**}}\) which are defined and investigated in the following section.
New separation axioms $T_\xi$ and $T_{\xi'}$

In this section we also use the following notations: $\xi O(X,\tau) := \{U|U \text{ is } \xi \text{-open in } (X,\tau)\}; \xi^{**}O(X,\tau) := \{V|V \text{ is } \xi^{**} \text{-open in } (X,\tau)\}.

Definition 5.1 (i) A space $(X,\tau)$ is a $T_\xi$-space if every $\xi$-closed set is $\alpha$-closed (i.e. $\tau^\alpha = \xi O(X,\tau)$).

(ii) A space $(X,\tau)$ is a $T_{\xi'}$-space if every $\xi^{**}$-closed set is $\alpha$-closed (i.e. $\tau^\alpha = \xi^{**}O(X,\tau)$).

(iii) [27, Definition 4.1] A space $(X,\tau)$ is a $T_{1\frac{1}{2}}^{**}$-space if every $g^{**}$-closed set is closed.

Theorem 5.2 For a space $(X,\tau)$, the following properties are equivalent:

1. $(X,\tau)$ is a $T_\xi$-space;
2. $\tau^\alpha = \tau$ holds;
3. Every singleton of $X$ is $g$-closed or $\alpha$-open in $(X,\tau)$;
4. Every singleton of $X$ is $g$-closed or open in $(X,\tau)$;
5. A space $(X,\tau^\alpha)$ is $T_{1\frac{1}{2}}^{**}$.

Proof. (1) $\iff$ (2) It is Proposition 4.6(i). (1) $\Rightarrow$ (3) Let $x \in X$. Then, by Lemma 2.9, $X = X_{\xi O} \cup X_{\xi O}$. Suppose that $\{x\}$ is not $g$-closed. Then $x \in X_{\xi O}$ (i.e. $\{x\} \in \xi O(X,\tau)$).

Using assumption that $\tau^\alpha = \xi O(X,\tau)$, we have that $\{x\}$ is $\alpha$-open. (3) $\iff$ (4) The proof is obvious, because a singleton $\{x\}$ is open if and only if it is $\alpha$-open. (3) $\Rightarrow$ (5) It is shown that a subset $A$ is $g$-closed in $(X,\tau)$ if and only if $A$ is $g$-closed in $(X,\tau^\alpha)$ [16, Theorem 2.3]. Moreover, $A$ is $\alpha$-open in $(X,\tau)$ (i.e. $A \in \tau^\alpha$) if and only if $A$ is open in $(X,\tau^\alpha)$. Then, by (3), every singleton $\{x\}$ is $g$-closed or open in $(X,\tau^\alpha)$. By [27, Theorem 4.15], $(X,\tau^\alpha)$ is $T_{1\frac{1}{2}}^{**}$. (5) $\Rightarrow$ (1) We claim that every $\xi$-closed set is $\alpha$-closed in $(X,\tau)$. Let $A$ be a $\xi$-closed set in $(X,\tau)$. Then, by Theorem 2.7 and (5), $A$ is $g^{**}$-closed in $(X,\tau^\alpha)$ and so $A$ is closed in $(X,\tau^\alpha)$ (i.e. $A \in \alpha$-closed in $(X,\tau)$).

Every $T_{3\frac{1}{2}}$-space is a $T_{1\frac{1}{2}}$-space ([8, Corollary 4.7]). A space $(X,\tau)$ is called a $T_{3\frac{1}{2}}$-space [8] if every $\delta$-generalized closed subset is $\delta$-closed in $(X,\tau)$. A space $(X,\tau)$ is called a $T_{1\frac{1}{2}}$-space [15] if every $g$-closed subset is closed in $(X,\tau)$. It is well known that a space $(X,\tau)$ is $T_{3\frac{1}{2}}$ if and only if every singleton $\{x\}$ is regular open or closed in $(X,\tau)$ ([8, Theorem 4.3]). Moreover, a space $(X,\tau)$ is $T_{1\frac{1}{2}}$ if and only if every singleton $\{x\}$ is open or closed in $(X,\tau)$ ([9, Theorem 2.6]). The digital line $(\mathbb{Z},\kappa)$ is $T_{3\frac{1}{2}}$ ([8, Example 4.6; Theorem 4.3]) and so it is $T_{1\frac{1}{2}}$ ([8, Corollary 4.7]). A space $(X,\tau)$ is $T_{1\frac{1}{2}}$ if an induced space $(X,\tau^\alpha)$ is $T_{1\frac{1}{2}}$ ([16]). Digital objects are related to some low separation axioms.

The following result (iii) of Corollary 5.3 is probably unexpected:

Corollary 5.3 (i) Every $T_{1\frac{1}{2}}$-space is $T_\xi$.

(ii) Every $T_{\xi'}$-space is $T_\xi$.

(iii) A space $(X,\tau)$ is $T_\xi$ if and only if $\tau = P(X)$.

Proof. (i) Suppose that $(X,\tau)$ is $T_{1\frac{1}{2}}$. Then, for a point $x \in X$, $\{x\}$ is closed or open, by [9, Theorem 2.6]. Using Theorem 5.2, $(X,\tau)$ is $T_\xi$. (ii) It is obvious from Theorem 3.2 and Definition 5.1. (iii) A space $(X,\tau)$ is $T_{1\frac{1}{2}}$ if and only if $\tau^\alpha = P(X)$ holds (cf. Proposition 4.6(iii)). And, it is shown that $\tau^\alpha = P(X)$ if and only if $\tau = P(X)$.

Remark 5.4 Moreover, we have the following diagrams of implications: (i) $\alpha T_1 \to \alpha T_{1\frac{1}{2}} \to T_\xi$ (cf. [16, Theorem 5.4(iii)], Theorem 5.2); a space $(X,\tau)$ is called an $\alpha T_1$ (resp. $\alpha T_{1\frac{1}{2}}$) [16] if an induced space $(X,\tau^\alpha)$ is $T_1$ (resp. $T_{1\frac{1}{2}}$). (ii) $\alpha T_m \to \alpha T_{1\frac{1}{2}} \to \alpha T_{1\frac{1}{2}} \to T_\xi$ (cf. (i) above, [16, Theorem 5.4]); a space $(X,\tau)$ is $\alpha T_{1\frac{1}{2}}$ (resp. $\alpha T_m$) [16] if every $g^{**}$-closed set is $\alpha$-closed (resp. closed).
Theorem 5.5 If \((X, \tau)\) is \(T_\xi\) and a subset \(H\) is open in \((X, \tau)\), then \((H, \tau|H)\) is \(T_\xi\).

Proof. Let \(x \in H\). By Theorem 5.2, the singleton \(\{x\}\) is \(g\)-closed or \(\alpha\)-open in \((X, \tau)\) and so \(\{x\}\) is \(g\)-closed or open in \((X, \tau^\alpha)\). Then, using [15, Theorem 2.9], \(\{x\}\) is \(g\)-closed or open in \((H, \tau^\alpha|H)\). Therefore, by Theorem 5.2, \((H, \tau^\alpha|H)\) is \(T_\xi\), because \(\tau^\alpha|H = (\tau|H)^\alpha\) for \(H \in \tau\).

Remark 5.6 (i) The digital line \((\mathbf{Z}, \kappa)\) is \(T_\xi\), because it is \(T_{1/2}\) (cf.[8, Example 4.6], Corollary 5.3(i)). (ii) (cf.Theorem 7.1(i)) The digital plane \((\mathbf{Z}^2, \kappa^2)\) is a \(T_\xi\)-space (cf.Section 7 below); it is not \(T_{1/2}\). Thus, the converse of Corollary 5.3 does not true in general. (iii) The converse of Corollary 5.3(ii) does not true in general (cf.Remark 7.2).

6 Some functions and groups

Definition 6.1 Let \(f : (X, \tau) \rightarrow (Y, \sigma)\) be a function between spaces and \(\rho \in \{\xi, \xi^*, \xi^{**}\}\).

A function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is said to be:

(i) \(\rho\)-continuous if for every closed set \(F\) of \((Y, \sigma)\), \(f^{-1}(F)\) is \(\rho\)-closed in \((X, \tau)\);

(ii) \(\rho\)-irresolute if for every \(\rho\)-closed set \(B\) of \((Y, \sigma)\), \(f^{-1}(B)\) is \(\rho\)-closed in \((X, \tau)\);

(iii) \(\rho\)-open if for every open set \(U\) of \((X, \tau)\), \(f(U)\) is \(\rho\)-open in \((Y, \sigma)\);

(iv) \(\rho\)-closed if for every closed set \(C\) of \((X, \tau)\), \(f(C)\) is \(\rho\)-closed in \((Y, \sigma)\);

(v) \(\rho\)-homeomorphism if \(f\) is a bijective \(\rho\)-continuous and \(f^{-1}\) is \(\rho\)-continuous;

(vi) \(\rho\)-homeomorphism if \(f\) is a bijective \(\rho\)-irresolute and \(f^{-1}\) is \(\rho\)-irresolute.

We recall the following definitions and properties: a function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is said to be \(\alpha\)-continuous ([22], [19]) (resp. \(\alpha\)-irresolute [18], \(g\alpha\)-irresolute [4, Definition 2.1(ii)]) if for every closed (resp. \(\alpha\)-closed, \(g\alpha\)-closed) set \(F\) of \((Y, \sigma)\), \(f^{-1}(F)\) is \(\alpha\)-closed (resp. \(\alpha\)-open, \(g\alpha\)-open) in \((X, \tau)\).

Lemma 6.2 (i) ([23, Theorem 4.13]) If \(f : (X, \tau) \rightarrow (Y, \sigma)\) is almost-open and \(\alpha\)-continuous, then \(f\) is \(\alpha\)-irresolute.

(ii) Especially, if \(f : (X, \tau) \rightarrow (Y, \sigma)\) is open and continuous, then \(f\) is \(\alpha\)-irresolute.

(iii) If \(f : (X, \tau) \rightarrow (Y, \sigma)\) is a homeomorphism, then \(f : (X, \tau) \rightarrow (Y, \sigma)\) and \(f^{-1} : (Y, \sigma) \rightarrow (X, \tau)\) are \(\alpha\)-irresolute and \(g\alpha\)-irresolute.

Proof. (i) This is Theorem 4.13 in [23]. We recall definition of \(\alpha\)-continuous functions [26]: a function \(f : (X, \tau) \rightarrow (Y, \sigma)\) is \(\alpha\)-almost-open if \(f(U)\) is open in \((Y, \sigma)\) for every regular open set \(U\) of \((X, \tau)\) (eg.[23, p.124]). (ii) Every open function is almost-open and every \(\alpha\)-continuous function is \(\alpha\)-continuous. Thus (ii) is obtained by (i). (iii) Since \(f\) and \(f^{-1}\) are open and continuous, by (ii) \(f : (X, \tau) \rightarrow (Y, \sigma)\) and \(f^{-1} : (Y, \sigma) \rightarrow (X, \tau)\) are \(\alpha\)-irresolute. Thus, the induced functions \(f : (X, \tau^\alpha) \rightarrow (Y, \sigma^\alpha)\) and \(f^{-1} : (Y, \sigma^\alpha) \rightarrow (X, \tau^\alpha)\) are homeomorphisms. By [15, Theorem 6.1], \(f(A)\) is \(g\alpha\)-closed in \((Y, \sigma^\alpha)\) for every \(g\alpha\)-closed set \(A\) of \((X, \tau^\alpha)\). And, by [15, Theorem 6.3], \(f^{-1}(B)\) is \(g\alpha\)-closed in \((X, \tau^\alpha)\) for every \(g\alpha\)-closed set \(B\) of \((Y, \sigma^\alpha)\). Thus we have that the set \(f(A)\) is \(g\alpha\)-closed in \((Y, \sigma)\) for every \(g\alpha\)-closed set \(A\) of \((X, \tau)\) and \(f^{-1}(B)\) is \(g\alpha\)-closed in \((X, \tau)\) for every \(g\alpha\)-closed set \(B\) of \((Y, \sigma)\). Therefore, \(f : (X, \tau) \rightarrow (Y, \sigma)\) and \(f^{-1} : (Y, \sigma) \rightarrow (X, \tau)\) are \(g\alpha\)-irresolute.

Theorem 6.3 (i) Every \(\alpha\)-continuous function is \(\xi\)-continuous.

(ii) Every \(\xi^*\)-continuous function is \(\xi\)-continuous.

(iii) Every \(\xi\)-continuous function is \(\xi^{**}\)-continuous.

(iv) Every \(\rho\)-irresolute function is \(\rho\)-continuous for each \(\rho \in \{\xi, \xi^{**}\}\).
Theorem 2.3(i), $f^{-1}(F)$ is $\xi$-closed and so $f$ is $\xi$-continuous. (ii)-(vi) They are obvious from definitions.

(vii) Let $f : (X, \tau) \to (Y, \sigma)$ be a homeomorphism. First, we claim that $f^{-1} : (Y, \sigma) \to (X, \tau)$ is $\rho$-irresolute for each $\rho \in \{\xi, \xi^*, \xi^{**}\}$.

**Case 1.** $\rho = \xi^{**}$: Let $A$ be a $\xi^{**}$-closed set of $(X, \tau)$. To prove that $f(A)$ is $\xi^{**}$-closed in $(Y, \sigma)$, let $U$ be a $\mathfrak{g}$-open set in $(Y, \sigma)$ such that $f(A) \subseteq U$. Then, by Lemma 6.2(iii), $f^{-1}(U)$ is $\mathfrak{g}$-open in $(X, \tau)$. Thus, we have that $f(\mathfrak{g}^{-}\text{Cl}(A)) \subseteq f(\text{Int}(\text{Cl}(f^{-1}(U))))$. Since $f : (X, \tau) \to (Y, \sigma)$ is a homeomorphism, we have that $\mathfrak{g}^{-}\text{Cl}(f(A)) \subseteq \mathfrak{g}^{-}\text{Cl}(f(\text{Int}(\text{Cl}(B))))$ for any subset $B$ of $(X, \tau)$, we have that $\sigma^{-}\text{Cl}(f(A)) \subseteq \text{Int}(f(\text{Cl}(U)))$. Thus, $f(A)$ is $\xi^{**}$-closed in $(Y, \sigma)$.

**Case 2.** $\rho = \xi^*$: Let $A$ be a $\xi^*$-closed set of $(X, \tau)$. Let $U$ be a $\mathfrak{g}$-open set in $(Y, \sigma)$ such that $f(A) \subseteq U$. Then, by Lemma 6.2(iii), $f^{-1}(U)$ is $\mathfrak{g}$-open in $(X, \tau)$. Thus, we have that $f(\mathfrak{g}^{-}\text{Cl}(A)) \subseteq f(\mathfrak{g}^{-}\text{Int}(U)))$. Since $f : (X, \tau) \to (Y, \sigma)$ is a homeomorphism, we have that $\mathfrak{g}^{-}\text{Cl}(f(A)) \subseteq \mathfrak{g}^{-}\text{Cl}(f(\text{Int}(\text{Cl}(B))))$ for any subset $B$ of $(X, \tau)$, we have that $\sigma^{-}\text{Cl}(f(A)) \subseteq \text{Int}(f(\text{Cl}(U)))$. Thus, $f(A)$ is $\xi^*$-closed in $(Y, \sigma)$.

**Case 3.** $\rho = \xi$: Let $A$ be a $\xi$-closed set of $(X, \tau)$. Let $U$ be a $\mathfrak{g}$-open set in $(Y, \sigma)$ such that $f(A) \subseteq U$. Then, by Lemma 6.2(iii), $f^{-1}(U)$ is $\mathfrak{g}$-open in $(X, \tau)$. Thus, we have that $f(\mathfrak{g}^{-}\text{Cl}(A)) \subseteq f(\mathfrak{g}^{-}\text{Int}(U)))$. Since $f : (X, \tau) \to (Y, \sigma)$ is a homeomorphism, we have that $\mathfrak{g}^{-}\text{Cl}(f(A)) \subseteq \mathfrak{g}^{-}\text{Cl}(f(\text{Int}(\text{Cl}(B))))$ for any subset $B$ of $(X, \tau)$, we have that $\sigma^{-}\text{Cl}(f(A)) \subseteq \text{Int}(f(\text{Cl}(U)))$. Thus, $f(A)$ is $\xi$-closed in $(Y, \sigma)$.

Therefore, we claim that $f^{-1} : (Y, \sigma) \to (X, \tau)$ is $\rho$-irresolute for each $\rho \in \{\xi, \xi^*, \xi^{**}\}$. Since $f^{-1} : (Y, \sigma) \to (X, \tau)$ is a homeomorphism, we show similarly that $f : (X, \tau) \to (Y, \sigma)$ is $\rho$-irresolute and hence $f$ is a $\rho$-homeomorphism for each $\rho \in \{\xi, \xi^*, \xi^{**}\}$.

(viii) Assume that $\rho = \xi$ (resp. $\xi^{**}$). Let $F$ be a closed set of $(Y, \sigma)$. Then, $F$ is $\xi$-closed (resp. $\xi^{**}$-closed) of $(Y, \sigma)$ (cf. Theorem 2.3(iii) (resp. Theorem 3.2(ii)). Then, $f^{-1}(F)$ is $\xi$-closed (resp. $\xi^{**}$-closed) in $(Y, \sigma)$, because $f : (X, \tau) \to (Y, \sigma)$ is $\xi$-irresolute (resp. $\xi^{**}$-irresolute). Thus, we have that $f : (X, \tau) \to (Y, \sigma)$ is $\xi$-continuous (resp. $\xi^{**}$-continuous). Similarly, it is shown that $f^{-1} : (Y, \sigma) \to (X, \tau)$ is $\xi$-continuous (resp. $\xi^{**}$-continuous). $\Box$
the $\alpha$-continuity and the $\xi^*$-continuity are independent to each others. (v) The above (i)-(iv) and Theorem 6.3(i)-(iii) show that the following diagram of implications holds and all implications are not reversible.

\[
\begin{array}{ccc}
\alpha \text{-continuous} & \land & \forall f \text{ is } \xi \text{-continuous} & \rightarrow & \xi^* \text{-continuous} \\
\land & \downarrow & & & \\
\xi^* \text{-continuous} & & & & \\
\end{array}
\]

The following theorem is a pasting lemma for $\rho$-continuous (resp. $\rho$-irresolute) functions for each $\rho \in \{\xi, \xi^*, \xi^{**}\}$. Let $X = A \cup B$ and $f : A \rightarrow Y$ and $h : B \rightarrow Y$ be two functions. It is called that $f$ and $h$ are compatible if $f(x) = h(x)$ for every $x \in A \cap B$. The combination $f \land h : X \rightarrow Y$ is defined by $(f \land h)(x) = f(x)$ for every $x \in A$ and $(f \land h)(x) = h(x)$ for every $x \in B$.

**Theorem 6.5** Let $\rho \in \{\xi, \xi^*, \xi^{**}\}$. Suppose that $A$ and $B$ are open and closed subset of $(X, \tau)$ such that $X = A \cup B$. Let $f : (A, \tau|A) \rightarrow (Y, \sigma)$ and $h : (B, \tau|B) \rightarrow (Y, \sigma)$ be compatible functions.

(i) If $f$ and $h$ are $\rho$-continuous, then its combination $f \land h : (X, \tau) \rightarrow (Y, \sigma)$ is also $\rho$-continuous.

(ii) If $f$ and $h$ are $\rho$-irresolute, then its combination $f \land h : (X, \tau) \rightarrow (Y, \sigma)$ is also $\rho$-irresolute.

**Proof.** (i) Let $F$ be a closed set of $(Y, \sigma)$. Then, $(f \land h)^{-1}(F) = f^{-1}(F) \cup h^{-1}(F)$ and $f^{-1}(F)$ (resp. $h^{-1}(F)$) is $\rho$-closed in $(A, \tau|A)$ (resp. $(B, \tau|B)$). By Corollary 2.13(i) and Remark 2.8(ii) for $\rho = \xi$; Theorem 3.6(i) and Theorem 3.5(i) for each $\rho \in \{\xi^*, \xi^{**}\}$, $(f \land h)^{-1}(F)$ is $\rho$-closed in $(X, \tau)$. Therefore, $f \land h$ is $\rho$-continuous. (ii) Let $F$ be a $\rho$-closed set of $(Y, \sigma)$. Since $f^{-1}(F)$ (resp. $h^{-1}(F)$) is $\rho$-closed in $(A, \tau|A)$ (resp. $(B, \tau|B)$), (iii) is proved by an argument similar to that in (i) above. \(\square\)

We construct some groups corresponding to a space $(X, \tau)$.

**Definition 6.6** For a space $(X, \tau)$ and $\rho \in \{\xi, \xi^*, \xi^{**}\}$, we define the following collections of functions:

(i) $ph(X, \tau) = \{f : (X, \tau) \rightarrow (X, \tau) \text{ is a } \rho\text{-homeomorphism}\}$;

(ii) $pch(X, \tau) = \{f : (X, \tau) \rightarrow (X, \tau) \text{ is a } \rho\text{-homeomorphism}\}$;

(iii) $h(X, \tau) = \{f : (X, \tau) \rightarrow (X, \tau) \text{ is a homeomorphism}\}$.

**Theorem 6.7** (i) For each $\rho \in \{\xi, \xi^*, \xi^{**}\}$, $h(X, \tau) \subseteq pch(X, \tau)$ holds.

(ii) For each $\rho \in \{\xi, \xi^*, \xi^{**}\}$, $pch(X, \tau) \subseteq ph(X, \tau)$ holds.

(iii) The set $pch(X, \tau)$ forms a group containing $h(X, \tau)$ as its subgroup for each $\rho \in \{\xi, \xi^*, \xi^{**}\}$.

**Proof.** (i) (resp. (ii)) It is obtained by Theorem 6.3(vii) (resp. Theorem 6.3(viii)). (iii) A binary operation $\beta : pch(X, \tau) \times pch(X, \tau) \rightarrow pch(X, \tau)$ is well defined by $\beta(u, v) = u \circ v$ (the composition of functions) for any $u, v \in pch(X, \tau)$ (cf. Theorem 6.3(v)). Then, it is shown that $pch(X, \tau)$ forms a group under $\beta$. Using (i), $h(X, \tau)$ is a subgroup of $pch(X, \tau)$. \(\square\)
Corollary 6.8 Assume \( \rho \in \{ \xi, \xi^*, \xi^{**} \} \).

(i) If there exists a \( \rho \)-homeomorphism \( f : (X, \tau) \to (Y, \sigma) \), then there exists a group isomorphism: \( \rho \text{ch}(X, \tau) \cong \rho \text{ch}(Y, \sigma) \).

(ii) Especially, if there exists a homeomorphism \( f : (X, \tau) \to (Y, \sigma) \), then there exists a group isomorphism: \( \rho \text{ch}(X, \tau) \cong \rho \text{ch}(Y, \sigma) \).

Proof. (i) Let \( f : (X, \tau) \to (Y, \sigma) \) be a \( \rho \)-homeomorphism. Then, a group isomorphism \( f_* : \rho \text{ch}(X, \tau) \cong \rho \text{ch}(Y, \sigma) \) is well defined by \( f_*(u) := f \circ u \circ f^{-1} \), using Theorem 6.3(v).

(ii) It is obvious by (i) and Theorem 6.3(vii). \( \square \)

In the below remark, we use the following notations: for a space \( (X, \tau) \), \( \rho \text{C}(X, \tau) = \{ F \mid F \) is \( \rho \)-closed in \( (X, \tau) \)\} where \( \rho \in \{ \xi, \xi^*, \xi^{**} \} \).

Remark 6.9 For the following spaces \( (X, \tau) \) and \( (Y, \sigma) \), we get the group structures of \( \rho \text{ch}(X, \tau) \) and \( \rho \text{ch}(Y, \sigma) \), where \( \rho \in \{ \xi, \xi^*, \xi^{**} \} \). Let \( X = \{ a, b, c, d \}, \tau = \{ \emptyset, \{ a \}, \{ c \}, \{ a, c \}, X \} \) and \( \sigma = \{ \emptyset, \{ b \}, \{ c \}, \{ b, c \}, \{ b, c, d \}, Y \} \).

(i) \( \xi \text{ch}(X, \tau) = \emptyset, \xi \text{ch}(Y, \sigma) = \emptyset \).

(ii) \( \xi \text{ch}(Y, \sigma) \) is not continuous; it is \( \xi \)-irresolute. Let \( \xi \text{ch}(X, \tau) = \emptyset \).

Remark 6.10 The converse of Corollary 6.8 is not true. Let \( (X, \tau) \) and \( (Y, \sigma) \) be the spaces of Remark 6.9 above. Let \( f : (X, \tau) \to (Y, \sigma) \) be a function defined by \( f(x) = x \) for any \( x \in \{ a, b \}, f(c) = d, f(d) = c \). Then, we observe that, for each \( \rho \in \{ \xi, \xi^*, \xi^{**} \}, f \) induces an isomorphism \( f_* : \rho \text{ch}(X, \tau) \cong \rho \text{ch}(Y, \sigma) \) such that \( f_*(a) = h, f_*(b) = a, h(a) = d, h(b) = c, h(c) = b, h(d) = c \).

Remark 6.11 (i) The following examples show that the continuity and \( \xi^* \)-continuity (cf. Definition 6.1(i)) are independent. Let \( X = \{ a, b, c \} \) and \( \tau = \{ \emptyset, \{ a \}, \{ a, b \}, X \} \). Let \( f : (X, \tau) \to (X, \tau) \) be a continuous function defined by \( f(a) = f(b) = b, f(c) = c \). Then, \( f \) is not \( \xi^* \)-continuous. Let \( g : (X, \tau) \to (X, \tau) \) be a function defined by \( g(a) = a, g(b) = c, g(c) = b \). Then, \( g \) is not continuous; it is \( \xi^* \)-continuous. Indeed, \( \xi^* \text{C}(X, \tau) = \emptyset \).

(ii) The following functions \( f : (X, \tau) \to (Y, \sigma) \) and \( g : (X, \tau) \to (Y, \sigma) \) show that the converse of Theorem 6.3(iv) is not true.

Case 1. \( \rho = \xi \). Let \( X = \{ a, b, c \}, \tau = \{ \emptyset, \{ a \}, \{ a, b \}, X \} \) and \( \sigma = \{ \emptyset, \{ a \}, \{ a, b \}, Y \} \). Let \( f : (X, \tau) \to (Y, \sigma) \) be a function defined by \( f(x) = x \) for any \( x \in X \). Then, \( f \) is not \( \xi \)-irresolute; it is \( \xi \)-continuous. Indeed, \( \xi \text{C}(X, \tau) = \emptyset \).

Remark 6.12 (i) The following examples show that the continuity and \( \xi^* \)-continuity (cf. Definition 6.1(i)) are independent. Let \( X = \{ a, b, c \} \) and \( \tau = \{ \emptyset, \{ a \}, \{ a, b \}, X \} \). Let \( f : (X, \tau) \to (X, \tau) \) be a continuous function defined by \( f(a) = f(b) = b, f(c) = c \). Then, \( f \) is not \( \xi \)-irresolute; it is \( \xi \)-continuous. Indeed, \( \xi \text{C}(X, \tau) = \emptyset \).
\[ \xi \text{-closed sets in topological spaces and digital planes} \]

**Case 2.** \( \rho = \xi^* \): Let \((X, \tau)\) and \((Y, \sigma)\) be topological spaces defined in Remark 6.9 and \(g : (X, \tau) \to (Y, \sigma)\) a function defined as \(g(a) = b, g(b) = a, g(c) = d, g(d) = c\). Then, \(g\) is not \(\xi^*\)- irresolute. This space is a mathematical model of the computer screen. Let \((\xi, \xi^*)\) need not to be true. Let \((Y, \sigma)\) be a space of Remark 6.9 and \(f : (Y, \sigma) \to (Y, \sigma)\) a function defined by \(f(a) = d, f(d) = a, f(x) = x\) for any \(x \in \{b, c\}\). Then, \(f\) is \(\rho\)-homeomorphism, where \(\rho \in \{\xi, \xi^*\}: f\) is not a homeomorphism. We note that \(f\) is not \(\xi^*\)- irresolute. (iv) The converse of Theorem 6.3(viii) for \(\rho = \xi^*\) need not to be true. Let \((X, \tau)\) and \((Y, \sigma)\) be the spaces of Remark 6.9 and \(g : (X, \tau) \to (Y, \sigma)\) a function defined in (ii) Case 2 above. Then, \(g\) is not a \(\xi^*\)-homeomorphism; it is a \(\xi^*\)-homeomorphism.

7 \(\rho\)-closed sets of the digital plane where \(\rho \in \{\xi, \xi^*\}\) In this section, we show that the digital plane \((\mathbb{Z}^2, \kappa^2)\) is a \(T_{\xi}\)-space (cf. Remark 5.6(ii)) and investigate characterizations of \(\xi\)-closed sets and \(\xi^*\)-closed sets of the digital plane. First, we recall related definitions and some properties of the digital plane. The digital line is the set of the integers, \(\mathbb{Z}\), equipped with the topology \(\kappa\) having \(\{\{2m - 1, 2m, 2m + 1\}|m \in \mathbb{Z}\}\) as a subbase. It is denoted by \((\mathbb{Z}, \kappa)\). A singleton \((2n + 1)\) is open and a subset \(\{2n - 1, 2n, 2n + 1\}\) is the smallest open set containing \(2n\), where \(s, n \in \mathbb{Z}\). The digital line \((\mathbb{Z}, \kappa)\) is a typical example of the \(T_{1/2}\)-space which is not \(T_1\) (cf. [15] [9]), because every singleton of \((\mathbb{Z}, \kappa)\) is open or closed. Furthermore, it is shown, in [8, Example 4.6] that \((\mathbb{Z}, \kappa)\) is \(T_{3/4}\). Let \((\mathbb{Z}^2, \kappa^2)\) be the topological product of two copies of the digital line \((\mathbb{Z}, \kappa)\), where \(\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}\) and \(\kappa^2 = \kappa \times \kappa\). In this paper, the space \((\mathbb{Z}^2, \kappa^2)\) is called the digital plane (cf. [12, p.10], [13, p.907], [6, Section 6], [24, Section 5]). This space is a mathematical model of the computer screen. Let \((\mathbb{Z}^2, \kappa^2) = (x \{x\} \text{ is open in } (\mathbb{Z}^2, \kappa^2)\). Then, \((\mathbb{Z}^2, \kappa^2) = \{(2n + 1, 2m + 1)\} n, m \in \mathbb{Z}\) and it is open and dense in \((\mathbb{Z}^2, \kappa^2)\).

**Theorem 7.1** (cf. Remark 5.6(ii)) (i) The digital plane \((\mathbb{Z}^2, \kappa^2)\) is a \(T_{\xi}\)-space.

(ii) A subspace \((H, \kappa^2|H)\) of \((\mathbb{Z}^2, \kappa^2)\) is a \(T_{\xi}\)-space, where \(H = (\mathbb{Z}^2, \kappa^2)\).

**Proof.** (i) Let \(x\) be a point of \((\mathbb{Z}^2, \kappa^2)\).

**Case 1.** \(x = (2m + 1, 2n + 1)\), where \(n, m \in \mathbb{Z}\): The singleton \(\{x\}\) is open in \((\mathbb{Z}^2, \kappa^2)\).

**Case 2.** \(x = (2m, 2n)\), where \(n, m \in \mathbb{Z}\): The singleton \(\{x\}\) is closed in \((\mathbb{Z}^2, \kappa^2)\) and so it is closed (cf. Theorem 2.3(ii)).

**Case 3.** \(x = (2m, 2n + 1)\), where \(n, m \in \mathbb{Z}\): The singleton \(\{x\}\) is go-closed in \((\mathbb{Z}^2, \kappa^2)\). Indeed, it is shown, that for any go-open set \(U\) including \(\{x\}, (\kappa^2)_{\alpha}\)-\(Cl(\{x\}) = \{x\} \cup Cl(\{x\}) = \{x\} \cup Cl(\{x\}) = \{x\} \cup Cl(\{x\}) = \{x\} \cup Cl(\{x\}) = \{x\} \cup Cl(\{x\}) = \{x\} \cup Cl(\{x\}) = \{x\} \subseteq U\) and so \(\{x\}\) is \(\xi\)-closed (cf. [12, Theorem 1.5(c)]). Then, by Theorem 2.3(ii), \(\{x\}\) is go-closed in \((\mathbb{Z}^2, \kappa^2)\).

**Case 4.** \(x = (2m + 1, 2n)\), where \(n, m \in \mathbb{Z}\): The proof for this case is similar to Case 3 above.

Thus we have that every singleton \(\{x\}\) of \((\mathbb{Z}^2, \kappa^2)\) is go-closed or open. Therefore, using Theorem 5.2, \((\mathbb{Z}^2, \kappa^2)\) is a \(T_{\xi}\)-space. (ii) A singleton \((\{x_1, x_2\})\) is open in \((\mathbb{Z}^2, \kappa^2)\) if and only if \(x_1\) and \(x_2\) are odd integers. Thus, \(H\) is open in \((\mathbb{Z}^2, \kappa^2)\). By (i) and Theorem 5.5, \((H, \kappa^2|H)\) is \(T_{\xi}\). □

**Remark 7.2** (cf. Remark 5.6(iii)) The converse of Corollary 5.3(ii) does not true in general. By definition, Corollary 5.3(iii) and Theorem 7.1, it is shown that \(\kappa^2 \neq P(\mathbb{Z}^2)\) and so \((\mathbb{Z}^2, \kappa^2)\) is not \(T_{\xi^*}\); it is \(T_{\xi}\).

We recall the following notations of families of subsets, definitions and a fact: For a space \((X, \tau)\), \(PO(X, \tau)\) (resp. \(\xi O(X, \tau), \tau^n, SO(X, \tau), GoO(X, \tau)\)) denotes the family of
all preopen (resp. $\xi$-open, $\alpha$-open, semi-open, go-open) subsets of $(X, \tau)$ and $PC(X, \tau)$ (resp. $\xi C(X, \tau), SC(X, \tau), GoC(X, \tau)$) denotes the family of all preclosed (resp. $\xi$-closed, semi-closed, go-closed) subsets of $(X, \tau)$. A subset $A$ of $(X, \tau)$ is called preopen (resp. semi-open) in $(X, \tau)$, if $A \subseteq Int(Cl(A))$ (resp. $A \subseteq Cl(Int(A))$) holds in $(X, \tau)$. It is well known that $\tau^\alpha = PO(X, \tau) \cap SO(X, \tau)$ holds for any space $(X, \tau)$([23, Lemma 3.1], [25]).

**Theorem 7.3** For the digital plane $(Z^2, \kappa^2)$, the following properties hold:
(i) $\xi O(Z^2, \kappa^2) = PO(Z^2, \kappa^2)$; $\xi C(Z^2, \kappa^2) = PC(Z^2, \kappa^2)$.
(ii) $\xi O(Z^2, \kappa^2) \subseteq SO(Z^2, \kappa^2)$; $\xi C(Z^2, \kappa^2) \subseteq SC(Z^2, \kappa^2)$.
(iii) $\xi O(Z^2, \kappa^2) = PO(Z^2, \kappa^2) = (\kappa^2)^{\alpha} = GoO(Z^2, \kappa^2)$.

**Proof.** (i) We recall that
(*) $GoO(Z^2, \kappa^2) = PO(Z^2, \kappa^2) = (\kappa^2)^{\alpha}$ hold ([6, Theorem 6.1 (ii)]).

By Theorem 2.3(ii) and (*), $\xi C(Z^2, \kappa^2) \subseteq GoC(Z^2, \kappa^2) = PC(Z^2, \kappa^2)$. Conversely, by Theorem 2.3(i) and (*), $PO(Z^2, \kappa^2) = (\kappa^2)^{\alpha} \subseteq \xi O(Z^2, \kappa^2)$. Thus we have that $\xi C(Z^2, \kappa^2) = PC(Z^2, \kappa^2)$ and $\xi O(Z^2, \kappa^2) = PO(Z^2, \kappa^2)$ hold. (ii) By [6, Theorem 6.1(ii)], $PO(Z^2, \kappa^2) \subseteq SO(Z^2, \kappa^2)$ holds. Thus, (ii) is obtained by (i). (iii) It is proved by using (i) (ii) above and [6, Theorem 6.1(ii)].

To investigate further properties of $\xi$-closed sets and $\xi^{**}$-closed sets on the digital plane $(Z^2, \kappa^2)$, we prepare the following two propositions (Proposition 7.4, Proposition 7.5 below). Let $(X, \tau)$ be a space and $E$ a subset of $(X, \tau)$. First, we recall the following properties on the preclosure $pCl(A) := \{F | A \subseteq F, F$ is preclosed in $(X, \tau)\}$, the $\alpha$-closure $\tau^\alpha$-Cl$(A)$ and the $\tau$-closure Cl$(A)$ in $(X, \tau)$: the following properties are known.

(*) $pCl(E) \subseteq \tau^{\alpha}$-Cl$(E) \subseteq Cl(E)$ hold for any subset $E$.

(**) If $E$ is $\alpha$-open, then $pCl(E) = \tau^\alpha$-Cl$(E) = Cl(E)$ hold; for an open set $E$, the equality of (ii) is also true, because any open set is $\alpha$-open.

(***) If $E$ is preopen, then $\tau^{\alpha}$-Cl$(E) = Cl(E)$ hold; we note that $pCl(E) \neq Cl(E)$ in general. Indeed, let $(X, \tau)$ be a space, where $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a, b\}, X\}$. For a preopen set $E = \{a\}, pCl(E) = \{a\} \subseteq X = Cl(E) = \tau^{\alpha}$-Cl$(E)$ hold. However, for a preopen set $E$ of the digital plane $(Z^2, \kappa^2)$, we have the following relationships on closures of $E$.

**Proposition 7.4** (i) For a subset $E$ of $(Z^2, \kappa^2), pCl(E) = (\kappa^2)^{\alpha}$-Cl$(E)$ holds.

(ii) For a preopen set $E$ of $(Z^2, \kappa^2), pCl(E) = (\kappa^2)^{\alpha}$-Cl$(E) = Cl(E)$ hold.

**Proof.** (i) It is obtained from a fact that $PO(Z^2, \kappa^2) = (\kappa^2)^{\alpha}$ (cf. Theorem 7.3(iii)). (ii) By using (i) and the $\alpha$-closure’s formula [2, Theorem 1.5(c)], it is shown that $pCl(E) = (\kappa^2)^{\alpha}$-Cl$(E) = E \cup Cl(Int(Cl(E))) \supseteq E \cup Cl(Cl(E)) = Cl(E)$. Therefore, using (*), we have that $pCl(E) = (\kappa^2)^{\alpha}$-Cl$(E) = Cl(E)$ hold. □

Recall the set $(Z^2)_{m, n} = \{x | \{x\}$ is open in $(Z^2, \kappa^2)\} = \{(2m + 1, 2s + 1)|m, s \in Z\}$ and it is open and dense in $(Z^2, \kappa^2)$.

**Proposition 7.5** For a non-empty subset $E$ of $(Z^2, \kappa^2)$, the following properties are equivalent:

(1) $E$ is dense in $(Z^2, \kappa^2)$;
(2) $Cl(E) = Int(Cl(E))$ holds;
(3) $E \supseteq (Z^2)_{m, n}$ holds.

**Proof.** (1)⇒(2) Since $Cl(E) = Z^2, Cl(E) = Z^2 = Int(Cl(E))$ holds. (2)⇒(3) By (2), $Cl(E)$ is open and closed in $(Z^2, \kappa^2)$ and $Cl(E) \neq \emptyset$. Since $(Z^2, \kappa^2)$ is connected [1, p.67], $Cl(E) = Z^2$. Let $x$ be any point of $(Z^2)_{m, n}$. Then, $\{x\}$ is open and $x \in Cl(E)$. Thus we have that $x \in E$ and hence $(Z^2)_{m, n} \subseteq E$. (3)⇒(1) It follows from (3) that $Cl(E) \supseteq Cl((Z^2)_{m, n}) = Z^2$ and so $Cl(E) = Z^2$. □
Remark 7.6 For the following space \((X, \tau)\) and a set \(E\) of \(X\), the implication (2)\(\Rightarrow\)(1) in Proposition 7.5 does not hold in general. Let \(X = \{a, b, c\}\) and \(\tau = \emptyset, \{a\}, \{b, c\}, X\). Then, a subset \(E = \{a\}\) is not dense in \((X, \tau)\); \(Cl(E) = Int(Cl(E)) = \{a\}\) holds.

Theorem 7.7 Let \(E\) be a subset of \((\mathbb{Z}^2, \kappa^2)\).

(i) If \(E\) is dense in \((\mathbb{Z}^2, \kappa^2)\), then \(E\) is \(\xi^*\)-closed.

(ii) If \(E \supseteq (\mathbb{Z}^2)_{\kappa^2}\), then \(E\) is \(\xi^*\)-closed in \((\mathbb{Z}^2, \kappa^2)\).

(iii) If \(E\) is a preopen and dense subset of \((\mathbb{Z}^2, \kappa^2)\) and \(E \neq \mathbb{Z}^2\), then \(E\) is not \(\xi\)-closed.

(iv) If \(E \supseteq (\mathbb{Z}^2)_{\kappa^2}\) and \(E \neq \mathbb{Z}^2\), then \(E\) is not \(\xi\)-closed in \((\mathbb{Z}^2, \kappa^2)\).

Proof. (i) Let \(U\) be a go-open set containing \(E\). Then, \(\mathbb{Z}^2 = Int(Cl(U)) \subseteq Int(Cl(E))\) and so \(Int(Cl(U)) = \mathbb{Z}^2\). Thus we have that \((\kappa^2)^\alpha-Cl(E) = E \cup Cl(Int(Cl(U)) = E \cup \mathbb{Z}^2 = \mathbb{Z}^2 = Int(Cl(U))\) and so \(E\) is \(\xi^*\)-closed. (ii) By Proposition 7.5, \(E\) is dense in \((\mathbb{Z}^2, \kappa^2)\), because \(E\) is \(\xi^*\)-closed. (iii) Suppose that \(E\) is \(\xi\)-closed. Since \(E \subseteq PO(\mathbb{Z}^2, \kappa^2) = GaO(\mathbb{Z}^2, \kappa^2)\) (cf. Theorem 7.3(iii)) and \(E \subseteq \mathbb{Z}\), we have that \((\kappa^2)^\alpha-Cl(E) = Cl(E) \subseteq E\) using Proposition 7.4(ii). Therefore, \(E = \mathbb{Z}^2\), because \(Cl(E) = \mathbb{Z}^2\). This is a contradiction. (iv) It follows from assumption that \(Cl(E) \supseteq Cl((\mathbb{Z}^2)_{\kappa^2}) = \mathbb{Z}^2\) and so \(E\) is dense and \(E\) is preopen. Using (iii), \(E\) is not \(\xi\)-closed. □

Theorem 7.8 (i) If a non-empty subset \(E\) is preopen and it is not dense in \((\mathbb{Z}^2, \kappa^2)\), then \(E\) is not \(\xi^*\)-closed.

(ii) Every non-empty proper subset of \((\mathbb{Z}^2)_{\kappa^2}\) is not \(\xi^*\)-closed in \((\mathbb{Z}^2, \kappa^2)\).

Proof. (i) Suppose that \(E\) is \(\xi^*\)-closed. Since \(E \subseteq GaO(\mathbb{Z}^2, \kappa^2)\) and \(E \subseteq \mathbb{E}\), by using Proposition 7.4(ii), it is shown that \((\kappa^2)^\alpha-Cl(E) = Cl(E) \subseteq Int(Cl(E))\) and so \(Cl(E) = Int(Cl(E))\). By Proposition 7.5, \(E\) is dense. This is a contradiction. (ii) Let \(E\) be a non-empty proper subset of \((\mathbb{Z}^2)_{\kappa^2}\). Then, \(E\) is open and so preopen and there exists a point \(x_0\) such that \(x_0 \in (\mathbb{Z}^2)_{\kappa^2}\) and \(x_0 \notin E\). We claim that \(E\) is not dense. Indeed, suppose that \(E\) is dense. Since \(x_0\) is open and \(x_0 \in Cl(E)\), we have that \(x_0 \in E\). This is a contradiction. Therefore, the set \(E\) is non-empty, preopen and \(E\) is not dense. By using (i) above, \(E\) is not \(\xi^*\)-closed. □

Remark 7.9 The following example shows that Theorem 7.8 for a space does not hold in general. Let \((X, \tau)\) a space of Remark 7.6 and a subset \(E = \{a\}\). Then, \(E\) is preopen and non-dense subset of \((X, \tau)\); the set \(E\) is \(\xi^*\)-closed.

Theorem 7.10 For a non-empty subset \(E\) of \((\mathbb{Z}^2, \kappa^2)\), the following properties are equivalent:

1. \(E\) is \(\xi^*\)-closed and preopen in \((\mathbb{Z}^2, \kappa^2)\);
2. \(E\) is dense in \((\mathbb{Z}^2, \kappa^2)\);
3. \(E \supseteq (\mathbb{Z}^2)_{\kappa^2}\) holds.

Proof. (1)\(\Rightarrow\)(2) Since \(E \subseteq E \subseteq PO(\mathbb{Z}^2, \kappa^2) = GaO(\mathbb{Z}^2, \kappa^2)\) and \(E\) is \(\xi^*\)-closed, \(Cl(E) \subseteq Int(Cl(E))\) holds (cf. Proposition 7.4(ii)). By Proposition 7.5, it is shown that \(E\) is dense in \((\mathbb{Z}^2, \kappa^2)\). (2)\(\Rightarrow\)(1) Using Theorem 7.7(i), \(E\) is \(\xi^*\)-closed. Since \(E\) is dense, \(E \subseteq Cl(E) = \mathbb{Z}^2 = Int(Cl(E))\) holds. (2)\(\Leftrightarrow\)(3) This is shown in Proposition 7.5. □

Remark 7.11 (i) For a topological space, the implication (1)\(\Rightarrow\)(2) in Theorem 7.10 does not hold in general. Let \((X, \tau)\) be a space, where \(X = \{a, b, c\}\) and \(\tau = \emptyset, \{a\}, \{b\}, \{a, b\},\{a, c\}, X\) and \(E = \{a, c\}\). Then, \(E \subseteq PO(X, \tau) \cap \xi^*C(X, \tau)\); \(E\) is not dense in \((X, \tau)\).

(ii) By Theorem 7.10, the following subset \(E\) is \(\xi^*\)-closed and \(\xi\)-open in \((\mathbb{Z}^2, \kappa^2)\); \(E = (\mathbb{Z}^2)_{\kappa^2} \cup A\), where \(A\) is any non-empty set. The set \((\mathbb{Z}^2)_{\kappa^2}\) is called as the open screen of the digital plane.
References


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