ξ-CLOSED SETS IN TOPOLOGICAL SPACES AND DIGITAL PLANES

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Abstract. In this paper, we introduce and investigate three classes of subsets called ξ-closed sets, ξ^*- closed sets and ξ^{**}-closed sets in topological spaces. As applications we introduce two separation axioms T_ξ and $T_{\xi^{**}}$ of topological spaces and we construct a group of ρc-homeomorphisms which contains the group of all homeomorphisms as a subgroup, where $\rho \in \{\xi, \xi^*, \xi^{**}\}$. A discussion of ρ-closed sets in the digital plane concludes the paper, where $\rho \in \{\xi, \xi^{**}\}$. The digital plane is a T_ξ-space; it is not $T_{\xi^{**}}$.

1 Introduction In 1970, Levine [15] introduced and investigated the notion of *generalized closed sets* in a topological space and one of $T_{1/2}$*-spaces*. By [15, Theorem 5.3, Corollary 5.6], it was shown that the class of the $T_{1/2}$-spaces is placed between the class of the T_0-spaces and one of the T_1-spaces. In 1977, Dunham [9, Theorem 2.5] proved that a topological space is $T_{1/2}$ if and only if every singleton is open or closed. We know that the *digital line* is a typical example of the $T_{1/2}$-spaces (eg.[8, Example 4.6], [12]). Using the concept of α-sets (= α-open sets) [21], in 1993 Balachandran, Devi and Maki defined the concept of *generalized ρ-closed sets* (cf. Definition 2.1(ii)-(iii)) analogous to generalized closed sets [15], where $\rho \in \{\alpha, \alpha^*, \alpha^{**}\}$. Recently, Devi, Bhuvaneswari and Maki define a weak form of generalized ρ-closed sets and investigate their behaviours in the digital plane, where $\rho \in \{\alpha, \alpha^*, \alpha^{**}\}$. Moreover, Veera Kumar [27] define and investigate the notion of *g^*-closed sets* (cf.Definition 2.1(vi)) which is placed between the class of the closed sets and one of the generalized closed sets [15].

In Section 2 of this paper, we introduce a new class of generalized closed sets which is called *ξ-closed sets* (cf.Definition 2.2) and investigate some basic properties of them. In Section 3, ξ^*-closed sets and ξ^{**}-closed sets are introduced. Some implications of their generalized closed sets (cf. Remark 3.4) and some properties of their behabiours to a subspace are investigated. In Section 4, new topologies induced from families of ρ-closed sets, where $\rho \in \{\xi, \xi^*, \xi^{**}\}$. In Section 5, we introduce new separation axioms T_ξ and $T_{\xi^{**}}$ analogous to the axiom $T_{1/2}$ [15]. The digital plane is an example of T_ξ-spaces; it is not a $T_{\xi^{**}}$-space (cf.Remark 5.6; Theorem 7.1 and Remark 7.2 in Section 7). In Section 6, using ρ-closed sets, where $\rho \in \{\xi, \xi^*, \xi^{**}\}$, new classes of functions and some groups are introduced (cf.Definition 6.1, Definition 6.6). Their groups are new topological invariants (cf.Corollary 6.8(ii)). In Section 7, it is proved that the digital plane is a T_ξ-space (Theorem 7.1); a discussion of ξ-closed sets and ξ^{**}-closed sets in the digital plane concludes the paper.

Throughout this paper, (X, τ) and (Y, σ) represent nonempty topological spaces on which no separation axioms are assumed, unless otherwise mentioned. Throughout this paper, "a space" means a topological space. For some undefined or related concepts, the reader is reffered to [11], [20] in TOPOLOGY ATLAS, URL:http://at.yorku.ca./topology/.

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2 On ξ-closed sets The purpose of this section is to introduce and investigate the notion of the ξ-closed sets and some relationships between well known *generalized closed sets*. A subset A is called α*-open* [21] in (X, τ) if $A \subseteq Int(Cl(Int(A)))$ holds; the complement of an α-open set is called α*-closed*. The family of all α-open sets in (X, τ) is denoted by τ^α. The α-closure of a subset A is denoted by $\tau^\alpha\text{-}Cl(A) = \cap\{F | X \setminus F \in \tau^\alpha \text{ and } A \subseteq F\}$.

Definition 2.1 We recall the following definitions which are used in this paper. Let A be a subset of (X, τ).

(i) The set A is called *generalized closed* [15] (briefly, g*-closed*) in (X, τ), if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ); the complement of a g-closed set of (X, τ) is called g*-open* in (X, τ). **(ii)** A is called $g\alpha$*-closed* [16] (resp. $g\alpha^*$*-closed,* $g\alpha^{**}$*-closed*) in (X, τ), if $\tau^\alpha\text{-}Cl(A) \subseteq U$ (resp. $\tau^\alpha\text{-}Cl(A) \subseteq Int(U)$, $\tau^\alpha\text{-Cl}(A) \subseteq Int(Cl(U))$) whenever $A \subseteq U$ and U is α-open in (X, τ). **(iii)** The complement of a $g\alpha$-closed set (resp. $g\alpha^*$-closed set, $g\alpha^{**}$-closed set) of (X, τ) is called $g\alpha$*-open* (resp. $g\alpha^*$*-open,* $g\alpha^{**}$*-open*) in (X, τ). **(iv)** A is called αg*-closed* [17] in (X, τ), if $\tau^\alpha\text{-}Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). **(v)** A is called gs*-closed* [3] in (X, τ), if $sCl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ), where $sCl(A) = \cap\{F | F$ is a semi-closed set of (X, τ) such that $A \subseteq F\}$ is the semi-closure of A. A subset F is called *semi-closed* [14] in (X, τ), if $Int(Cl(F)) \subseteq F$ holds in (X, τ). **(vi)** A is called g^**-closed* [27] in (X, τ), if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X, τ). Every closed set is g^*-closed; every g^*-closed set is g-closed [27, Theorems 3.2, 3.4].

Definition 2.2 A subset A is called ξ*-closed* in (X, τ) if $\tau^\alpha\text{-}Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is a $g\alpha$-open set of (X, τ) (cf.Definition 2.1(iii)). The complement of a ξ-closed set of (X, τ) is called ξ*-open* in (X, τ).

Theorem 2.3 (i) *Every closed set and every* α*-closed set is* ξ*-closed.*
(ii) *Every* ξ*-closed set is* $g\alpha$*-closed,* $g\alpha^{**}$*-closed,* αg*-closed and* gs*-closed.* □

Remark 2.4 The converse of Theorem 2.3(i) (resp. (ii)) is not true in general by the following example (i) (resp. (ii)). **(i)** Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{c, d\}, X\}$. For a space (X, τ), a subset $\{a, b, c\}$ is ξ-closed; it is neither closed nor α-closed. Indeed, $\tau^\alpha = \{\emptyset, \{c, d\}, \{b, c, d\}, \{a, c, d\}, X\}$. **(ii)** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{b\}, \{a, c\}, X\}$. For a space (X, τ), a subset $\{a\}$ is $g\alpha$-closed, $g\alpha^{**}$-closed, αg-closed and gs-closed; it is not ξ-closed.

Remark 2.5 By Theorem 2.3, we obtain the following diagram of implications. Remark 2.4 shows that implications are not reversible.

$$\textit{closed} \to \alpha\textit{-closed} \to \xi\textit{-closed} \to g\alpha\textit{-closed}$$

Remark 2.6 The following examples show that the ξ-closedness is independent from the $g\alpha^*$-closedness, g-closedness and g^*-closedness. **(i)** In the same space (X, τ) of Remark 2.4(ii), a subset $\{a\}$ is $g\alpha^*$-closed; it is not ξ-closed. **(ii)** In the same space (X, τ) of Remark 2.4(i), a subset $\{a\}$ is ξ-closed; it is not $g\alpha^*$-closed. **(iii)** Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{b\}, \{a, b\}, X\}$. For a space (X, τ), a subset $\{a\}$ is ξ-closed;it is neither g-closed nor g^*-closed. A subset $\{b, c\}$ is g-closed and g^*-closed; it is not ξ-closed.

Theorem 2.7 *A subset* A *of* X *is* ξ*-closed in* (X, τ) *if and only if* A *is* g^**-closed in a space* (X, τ^α).

Proof. We recall that a subset B is $g\alpha$-open in (X, τ) if and only if B is g-open in (X, τ^α) ([16, Theorem 2.3], cf.Definition 2.1(iii)). Thus, Theorem 2.7 is proved by definitions. □

Remark 2.8 *t26* By using Theorem 2.7, we can translate theorems in [27] into ξ-closedness version. For examples, we have the following:

(i) A set A is ξ-closed in (X,τ) if and only if τ^{α}-$Cl(A)\setminus A$ does not contains any non-empty $g\alpha$-closed set of (X,τ) ([27, Theorem 3.14]).

(ii) The union of two ξ-closed sets is ξ-closed ([27, Remark 3.12]).

We have an alternative characterization of ξ-closed sets (cf.Theorem 2.10 below). We prepare the following notations: $G_{\alpha}O(X,\tau) := \{U|\ U$ is $g\alpha$-open in $(X,\tau)\}$; $G_{\alpha}C(X,\tau) := \{F|\ F$ is $g\alpha$-closed in $(X,\tau)\}$; $X_{\mathcal{G}_{\alpha}\mathcal{C}} := \{x|\ \{x\}\in G_{\alpha}C(X,\tau)\}$; $X_{\xi\mathcal{O}} := \{x|\ \{x\}$ is ξ-open in $(X,\tau)\}$; $G_{\alpha}O(X,\tau)$-$Ker(A) = \cap\{U|\ U$ is $g\alpha$-open in (X,τ) and $A\subseteq U\}$ for a subset A of X (briefly, $G_{\alpha}O$-$Ker(A)$).

Lemma 2.9 *For any space* $(X,\tau), X = X_{\mathcal{G}_{\alpha}\mathcal{C}}\cup X_{\xi\mathcal{O}}$ *holds.*

Proof. Let $x\in X$. Suppose that $\{x\}$ is not $g\alpha$-closed in (X,τ) (i.e.$X\setminus\{x\}$ is not $g\alpha$-open). Then, X is a unique $g\alpha$-open set containing $X\setminus\{x\}$. Thus $X\setminus\{x\}$ is ξ-closed in (X,τ) and so $\{x\}$ is ξ-open. Therefore, $x\in X_{\mathcal{G}_{\alpha}\mathcal{C}}\cup X_{\xi\mathcal{O}}$ holds. □

Theorem 2.10 *For a subset A of (X,τ), the following properties are equivalent:*

(1) *A is ξ-closed;*

(2) *τ^{α}-$Cl(A)\subseteq G_{\alpha}O$-$Ker(A)$ holds;*

(3) (i) *τ^{α}-$Cl(A)\cap X_{\mathcal{G}_{\alpha}\mathcal{C}}\subseteq A$ and* (ii) *τ^{α}-$Cl(A)\cap X_{\xi\mathcal{O}}\subseteq G_{\alpha}O$-$Ker(A)$ hold.*

Proof. **(1)⇒(2)** Let $x\notin G_{\alpha}O$-$Ker(A)$. Then, there exists a set $U\in G_{\alpha}O(X,\tau)$ such that $x\notin U$ and $A\subseteq U$. Since A is ξ-closed, τ^{α}-$Cl(A)\subseteq U$ and so $x\notin\tau^{\alpha}$-$Cl(A)$. **(2)⇒(3) (i)** First we claim that $G_{\alpha}O$-$Ker(A)\cap X_{\mathcal{G}_{\alpha}\mathcal{C}}\subseteq A$. Indeed, let $x\in G_{\alpha}O$-$Ker(A)\cap X_{\mathcal{G}_{\alpha}\mathcal{C}}$ and assume that $x\notin A$. Since the set $X\setminus\{x\}\in G_{\alpha}O(X,\tau)$ and $A\subseteq X\setminus\{x\}, G_{\alpha}O$-$Ker(A)\subseteq X\setminus\{x\}$. Then, we have that $x\in X\setminus\{x\}$ and so this is a contradiction. Thus, we show that τ^{α}-$Ker(A)\cap X_{\mathcal{G}_{\alpha}\mathcal{C}}\subseteq A$. By using (2), τ^{α}-$Cl(A)\cap X_{\mathcal{G}_{\alpha}\mathcal{C}}\subseteq G_{\alpha}O$-$Ker(A)\cap X_{\mathcal{G}_{\alpha}\mathcal{C}}\subseteq A$. **(ii)** It is obtained by (2). **(3)⇒(2)** By Lemma 2.9 and (3), τ^{α}-$Cl(A) = \tau^{\alpha}$- $Cl(A)\cap X = \tau^{\alpha}$-$Cl(A)\cap(X_{\mathcal{G}_{\alpha}\mathcal{C}}\cup X_{\xi\mathcal{O}}) = (\tau^{\alpha}$-$Cl(A)\cap X_{\mathcal{G}_{\alpha}\mathcal{C}})\cup(\tau^{\alpha}$-$Cl(A)\cap X_{\xi\mathcal{O}})\subseteq A\cup G_{\alpha}O$-$Ker(A) = G_{\alpha}O$-$Ker(A)$. That is, τ^{α}-$Cl(A)\subseteq G_{\alpha}O$-$Ker(A)$ holds. **(2)⇒(1)** Let $U\in G_{\alpha}O(X,\tau)$ such that $A\subseteq U$. Then, we have that $G_{\alpha}O$-$Ker(A)\subseteq U$ and so, by (2), τ^{α}-$Cl(A)\subseteq U$. Therefore, A is ξ-closed. □

Corollary 2.11 *Let $\mathcal{P} := \{A|\tau^{\alpha}$-$Cl(A)\cap X_{\xi\mathcal{O}}\subseteq G_{\alpha}O$-$Ker(A)\}$.*

(i) *If $\cap_{i\in\Sigma}A_i\in\mathcal{P}$ and A_i is ξ-closed set in (X,τ) for each $i\in\Sigma$, then $\cap_{i\in\Sigma}A_i$ is ξ-closed in (X,τ).*

(ii) *If $\mathcal{P} = P(X)$ and A_i is ξ-closed set in (X,τ) for each $i\in\Sigma$, then $\cap_{i\in\Sigma}A_i$ is ξ-closed in (X,τ).*

(iii) *If $X_{\xi\mathcal{O}} = \emptyset$ and A_i is ξ-closed set in (X,τ) for each $i\in\Sigma$, then $\cap_{i\in\Sigma}A_i$ is ξ-closed in (X,τ).*

(iv) *If τ^{α}-$Cl(A_i)\cap X_{\xi\mathcal{O}}\subseteq A_i$ and A_i is a ξ-closed set in (X,τ) for each $i\in\Sigma$, then $\cap_{i\in\Sigma}A_i$ is ξ-closed in (X,τ).*

Proof. **(i)** By Theorem 2.10, τ^{α}-$Cl(A_i)\cap X_{\mathcal{G}_{\alpha}\mathcal{C}}\subseteq A_i$ for each $i\in\Sigma$. Then, we have that τ^{α}-$Cl(\cap_{i\in\Sigma}A_i)\cap X_{\mathcal{G}_{\alpha}\mathcal{C}}\subseteq\cap_{i\in\Sigma}A_i$. Using assumption and Theorem 2.10(3), $\cap_{i\in\Sigma}A_i$ is ξ-closed. **(ii)-(iv)** By (i), they are proved. □

The following theorem is concerned on a property of g^{*}-closedness in a subspace. As a corollary, we have a property of ξ-closedness in a subspace (cf. Corollary 2.13 below).

We recall the following notations and some properties. For a space (X,τ) and a subset H of (X,τ), $GO(X,\tau) := \{U|\ U$ is g-open in $(X,\tau)\}; GO(H,\tau|H) := \{V|\ V$ is g-open in $(H,\tau|H)\}$. If $U \in GO(X,\tau)$ and $V \in GO(X,\tau)$, then $U \cap V \in GO(X,\tau)$ ([15, Theorem 2.4]). If $U \in GO(X,\tau), H \in \tau$ and $X \setminus H \in \tau$, then $U \cap H \in GO(H,\tau|H)$ ([24, Lemma 2.10(ii)]). If $U \in GO(X,\tau)$ and $V \in \tau$, then $U \cup V \in GO(X,\tau)$ ([15, Corollary 2.7]).

Theorem 2.12 *Let B and H be subsets in (X,τ) such that $B \subseteq H$.*

(i) *If B is g^*-closed in $(H,\tau|H)$ and H is open and closed in (X,τ), then B is g^*-closed in (X,τ).*

(ii) *Suppose that, for (X,τ) and H,*

$(*)$ *$GO(H,\tau|H) \subseteq \{H \cap O|\ O \in GO(X,\tau)\}$ holds.*

If B is g^*-closed in (X,τ), then B is g^*-closed in $(H,\tau|H)$.

Proof. **(i)** Let $O \in GO(X,\tau)$ such that $B \subseteq O$. We have that $H \cap O \in GO(H,\tau|H)$ and $B \subseteq H \cap O$. Then, $H \cap Cl(B) \subseteq H \cap O$ holds. It is shown that $H \subseteq O \cup (X \setminus Cl(B))$ and the subset $O \cup (X \setminus Cl(B)) \in GO(X,\tau)$. Since H is g^*-closed in $(X,\tau), Cl(H) \subseteq O \cup (X \setminus Cl(B))$ and so $Cl(B) \subseteq O \cup (X \setminus Cl(B))$. Therefore, we have that $Cl(B) \subseteq O$ and so B is g^*-closed in (X,τ). **(ii)** Let $V \in GO(H,\tau|H)$ such that $B \subseteq V$. Using assumption $(*)$, there exists a subset $O \in GO(X,\tau)$ such that $V = H \cap O$. Then, we have that $Cl(B) \subseteq O$ and so $(\tau|H)$-$Cl(B) = Cl(B) \cap H \subseteq O \cap H = V$. Therefore, B is g^*-closed in $(H,\tau|H)$. □

Using Theorem 2.12 for (X,τ^α) and Theorem 2.7 , we prove the following property on ξ-closedness in a subspace.

Corollary 2.13 *Let B and H be subsets of (X,τ) such that $B \subset H$.*

(i) *If B is ξ-closed in $(H,\tau|H)$ and H is open and closed in (X,τ), then B is ξ-closed in (X,τ).*

(ii) *Suppose that, for (X,τ^α) and H,*

$(**)$ *$GO(H,\tau^\alpha|H) \subseteq \{H \cap O|\ O \in GO(X,\tau^\alpha)\}$ holds.*

If B is ξ-closed in (X,τ) and H is open in (X,τ), then B is ξ-closed in $(H,\tau|H)$.

Proof **(i)** Using Theorem 2.7, we have that B is ξ-closed in $(H,\tau|H)$ if and only if B is g^*-closed in $(H,(\tau|H)^\alpha)$. Then, the set B is g^*-closed in $(H,\tau^\alpha|H)$, because $(\tau|H)^\alpha = \tau^\alpha|H$ holds if $H \in \tau$ (eg.[16, Lemma 2.4 (ii)]). Using Theorem 2.12(i) for (X,τ^α), B is g^*-closed in (X,τ^α) because H is open and closed in (X,τ^α). Therefore, using Theorem 2.7, B is ξ-closed in (X,τ). **(ii)** By Theorem 2.12(ii) for (X,τ^α), it is shown that B is g^*-closed in $(H,\tau^\alpha|H)$ and so B is g^*-closed in $(H,(\tau|H)^\alpha)$. Therefore, using Theorem 2.7 B is ξ-closed in $(H,\tau|H)$. □

3 On ξ^*-closed sets and ξ^{**}-closed sets

We introduce two classes of "ξ-closed sets" and investigate some properties.

Definition 3.1 (i) A subset A is called *ξ^*-closed* in (X,τ) if τ^α-$Cl(A) \subseteq Int(U)$ whenever $A \subseteq U$ and U is $g\alpha$-open in (X,τ).

(ii) A subset A is called *ξ^{**}-closed* in (X,τ) if τ^α-$Cl(A) \subseteq Int(Cl(U))$ whenever $A \subseteq U$ and U is $g\alpha$-open in (X,τ).

(iii) The complement of a ξ^*-closed set (resp. ξ^{**}-closed set) of (X,τ) is called a *ξ^*-open* (resp. *ξ^{**}-open*) set in (X,τ).

Theorem 3.2 (i) *Every ξ^*-closed set is ξ-closed.*

(ii) *Every ξ-closed set is ξ^{**}-closed.*

Proof. **(i)** The proof is obvious. **(ii)** Let A be a ξ-closed set of a space (X, τ). Let U be a $g\alpha$-open set of (X, τ) such that $A \subseteq U$. Then, we have that τ^{α}-$Cl(A) \subseteq U$. We recall that every $g\alpha$-closed set is $wg\alpha$-closed (=preclosed) and so every $g\alpha$-open set is preopen ([6, Theorems 2.2, 2.3(ii), Remark 2.4], cf.Definition 2.1(iii)). Therefore, we have that τ^{α}-$Cl(A) \subseteq U \subseteq Int(Cl(U))$ and so A is ξ^{**}-closed. □

Remark 3.3 **(i)** The converses of Theorem 3.2 are not true in general by the following examples. Let (X, τ) be a space such that $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. Then, a subset $\{b\}$ is ξ-closed; it is not ξ^*-closed. A subset $\{a\}$ is ξ^{**}-closed;it is not ξ-closed.
(ii) The following examples show that α-closedness and ξ^{**}-closedness are independent. Let (X, τ) be a space of (i) above. A subset $\{b\}$ is α-closed; it is not ξ^*-closed. Let (X, τ) be a space in Remark 2.4(i). A subset $\{a, b, c\}$ is ξ^*-closed; it is not α-closed.

Remark 3.4 Theorem 3.2, Theorem 2.3 and Remark 3.3 show the following diagram of implications. Remark 3.3 and Remark 2.4(i) show that all implications are not reversible.

$$\begin{array}{ccc} \alpha\text{-}closed & & \\ & \searrow & \\ \not\downarrow\not\uparrow & & \xi\text{-}closed \;\rightarrow\; \xi^{**}\text{-}closed \\ & \nearrow & \\ \xi^{*}\text{-}closed & & \end{array}$$

Theorem 3.5 *Let A be a subset of (X, τ).*

(i) *The union of two ρ-closed sets is ρ-closed, where $\rho \in \{\xi^*, \xi^{**}\}$.*

(ii) *If A is ρ-closed in (X, τ) and $A \subseteq B \subseteq \tau^{\alpha}$-$Cl(A)$, then B is ρ-closed, where $\rho \in \{\xi^*, \xi^{**}\}$.*

(iii) *If A is ξ^*-closed (resp. ξ^{**}-closed), then τ^{α}-$Cl(A) \setminus A$ does not contain non-empty $g\alpha$-closed set (resp. $g\alpha$-closed and semi-open set).*

(iv) *For each $x \in X, \{x\}$ is $g\alpha$-closed or its complement $X \setminus \{x\}$ is ξ^*-closed in (X, τ).*

(v) *For each $x \in X, \{x\}$ is $g\alpha$-closed and open, or its complement $X \setminus \{x\}$ is ξ^{**}-closed in (X, τ).*

Proof. **(i)-(iii)** The proofs are obvious. **(iv)** Suppose that $\{x\}$ is not $g\alpha$-closed in (X, τ) (i.e.$X \setminus \{x\}$ is not $g\alpha$-open). Then, X is a unique $g\alpha$-open set containing $X \setminus \{x\}$. Thus $X \setminus \{x\}$ is ξ^*-closed in (X, τ). **(v)** Suppose that $\{x\}$ is not $g\alpha$-closed in (X, τ). By similar argument of the proof of (iv), it is shown that $X \setminus \{x\}$ is ξ^{**}-closed in (X, τ). Suppose that $\{x\}$ is not open. Let U be a $g\alpha$-open set containing $X \setminus \{x\}$. If $U = X$, then τ^{α}-$Cl(X \setminus \{x\}) \subseteq Int(Cl(U)) = X$. If $U = X \setminus \{x\}$, then τ^{α}-$Cl(X \setminus \{x\}) \subseteq X = Int(X) = Int(Cl(U))$. Thus, $X \setminus \{x\}$ is ξ^{**}-closed in (X, τ). □

We have the following property on ξ^*-closedness and ξ^{**}-closedness in a subspace, respectively.

Theorem 3.6 *Let B and H be subsets of (X, τ) such that $B \subset H$.*

(i) *If B is ρ-closed in $(H, \tau|H)$ and H is open and closed in (X, τ), then B is ρ-closed in (X, τ), where $\rho \in \{\xi^*, \xi^{**}\}$.*

(ii) *Suppose that, for (X, τ^{α}) and H,*

$(**)$ *$GO(H, \tau^{\alpha}|H) \subseteq \{H \cap O |\ O \in GO(X, \tau^{\alpha})\}$ holds.*

If B is ρ-closed in (X, τ) and H is open in (X, τ), then B is ρ-closed in $(H, \tau|H)$, where $\rho \in \{\xi^*, \xi^{**}\}$.

Proof **(i) Case 1.** $\rho = \xi^*$: Let U be a $g\alpha$-open set in (X, τ) (i.e.$U \in GO(X, \tau^\alpha)$, cf. [16, Theorem 2.3] Definition 2.1(iii)) such that $B \subseteq U$. Then, we have that $U \cap H \in GO(H, (\tau|H)^\alpha)$, because $U \in GO(X, \tau^\alpha)$ and $\tau^\alpha|H = (\tau|H)^\alpha$ for $H \in \tau$ (eg.[4, Lemma 2.4(ii)]). Since B is ξ^*-closed in $(H, \tau|H)$ and $B \subseteq U \cap H$, $(\tau|H)^\alpha$-$Cl(B) \subseteq (\tau|H)$-$Int(U \cap H)$ and so $(\tau^\alpha$-$Cl(B) \cap H \subseteq Int(U \cap H) \cap H$. Put $V := Int(U \cap H) \cup (X \setminus \tau^\alpha$-$Cl(B))$. Then, it is shown that $H \subseteq V$ and $V \in \tau^\alpha$ and so V is $g\alpha$-open in (X, τ). It follows from assumption that τ^α-$Cl(B) \subseteq \tau^\alpha$-$Cl(H) \subseteq V \subseteq Int(U) \cup (X \setminus \tau^\alpha$-$Cl(B))$ and hence τ^α-$Cl(B) \subseteq Int(U)$ holds (i.e., B is ξ^*-closed in (X, τ)). **Case 2.** $\rho = \xi^{**}$: Let U be a $g\alpha$-open set (i.e. $U \in GO(X, \tau^\alpha)$) such that $B \subseteq U$. Since $U \cap H \in GO(H, (\tau|H)^\alpha)$ and $B \subseteq U \cap H$, we have that $H \cap \tau^\alpha$-$Cl(B) = (\tau^\alpha|H)$-$Cl(B) \subseteq (\tau|H)$-$Int((\tau|H)$-$Cl(H \cap U)) = (\tau|H)$-$Int(H \cap Cl(H \cap U)) = H \cap Int(H \cap Cl(H \cap U)) \subseteq H \cap (Int(Cl(H \cap U)))$ hold. Put $W := Int(Cl(H \cap U)) \cup (X \setminus \tau^\alpha$-$Cl(B))$. Then, $H \subseteq W$ and $W \in \tau^\alpha$ and so W is $g\alpha$-open in (X, τ). Since H is ξ-closed, τ^α-$Cl(B) \subseteq \tau^\alpha$-$Cl(H) \subseteq W \subseteq Int(Cl(U)) \cup (X \setminus \tau^\alpha$-$Cl(B))$ and hence τ^α-$Cl(B) \subseteq Int(Cl(U))$ holds (i.e.B is ξ^{**}-closed in (X, τ)).

(ii) Case 1. ρ=ξ^*: Let V be a $g\alpha$-open set of $(H, \tau|H)$ (i.e., $V \in GO(H, (\tau|H)^\alpha)$) such that $B \subseteq V$. Then, $V \in GO(H, \tau^\alpha|H)$. Using $(**)$, there exists a set $O \in GO(X, \tau^\alpha)$ such that $V = O \cap H$. Since $B \subseteq O$ and B is ξ^*-closed in (X, τ), we have that $(\tau|H)^\alpha$-$Cl(B) = (\tau^\alpha|H)$-$Cl(B) = H \cap \tau^\alpha$-$Cl(B) \subseteq H \cap Int(O) = H \cap Int(H \cap Int(O)) = (\tau|H)$-$Int(H \cap Int(O)) \subseteq (\tau|H)$-$Int(V)$ and so B is ξ^*-closed in $(H, \tau|H)$. **Case 2.** ρ=ξ^{**}: Let V be a $g\alpha$-open set of $(H, \tau|H)$ (i.e. $V \in GO(H, (\tau|H)^\alpha)$) such that $B \subseteq V$. Then, using $(**)$, there exists a set $O \in GO(X, \tau^\alpha)$ such that $V = O \cap H$. Since $B \subseteq O$ and B is ξ^{**}-closed in (X, τ), we have that τ^α-$Cl(B) \subseteq Int(Cl(O))$. Then, we have that $(\tau|H)^\alpha$-$Cl(B) = H \cap \tau^\alpha$-$Cl(B) \subseteq H \cap Int(Cl(O)) = H \cap Int(H \cap Cl(O)) = (\tau|H)$-$Int(H \cap Cl(O)) = (\tau|H)$-$Int(H \cap H \cap Cl(O)) \subseteq (\tau|H)$-$Int(H \cap Cl(H \cap O)) \subseteq (\tau|H)$-$Int((\tau|H)$-$Cl(V))$ and so $(\tau|H)^\alpha$-$Cl(B) \subseteq (\tau|H)$-$Int((\tau|H)$-$Cl(V))$ and so B is ξ^{**}-closed in $(H, \tau|H)$. □

4 Topologies induced from families of ρ-closed sets, where $\rho \in \{\xi, \xi^*, \xi^{**}\}$

We can introduce topologies from ρ-closed sets, where $\rho \in \{\xi, \xi^*, \xi^{**}\}$.

Definition 4.1 For a subset E of (X, τ), we define the following closures: ${}_\rho Cl_\#(E) := \cap\{A|\ A$ is a ρ-closed set in (X, τ) and $E \subseteq A\}$ for each $\rho \in \{\xi, \xi^*, \xi^{**}\}$.

Theorem 4.2 *Let E and F be subsets of (X, τ).*
(i) $E \subseteq {}_{\xi^{**}}Cl_\#(E) \subseteq {}_\xi Cl_\#(E) \subseteq {}_{\xi^*}Cl_\#(E)$
and $E \subseteq {}_{\xi^{**}}Cl_\#(E) \subseteq {}_\xi Cl_\#(E) \subseteq \tau^\alpha$-$Cl(E) \subseteq Cl(E)$ *hold.*
(ii) *For each* $\rho \in \{\xi, \xi^*, \xi^{**}\}$, ${}_\rho Cl_\#(\emptyset) = \emptyset$ *and* ${}_\rho Cl_\#(X) = X$ *hold.*
(iii) *If* $E \subseteq F$, *then* ${}_\rho Cl_\#(E) \subseteq {}_\rho Cl_\#(F)$ *holds for each* $\rho \in \{\xi, \xi^*, \xi^{**}\}$.
(iv) *For each* $\rho \in \{\xi, \xi^*, \xi^{**}\}$, ${}_\rho Cl_\#(E \cup F) = {}_\rho Cl_\#(E) \cup {}_\rho Cl_\#(F)$ *holds.*
(v) *If E is ρ-closed, then* ${}_\rho Cl_\#(E) = E$ *holds, where* $\rho \in \{\xi, \xi^*, \xi^{**}\}$.
(vi) *For each* $\rho \in \{\xi, \xi^*, \xi^{**}\}$, ${}_\rho Cl_\#({}_\rho Cl_\#(E)) = {}_\rho Cl_\#(E)$ *holds.*
(vii) *For each* $\rho \in \{\xi, \xi^*, \xi^{**}\}$, ${}_\rho Cl_\#(\bullet)$ *is a Kuratowski closure operator on* X.

Proof. **(i)** The implications are obtained by Theorem 2.3 and Theorem 3.2 respectively. **(ii)-(iii)** They are obvious from definitions. **(iv)** By (iii), it is enough to prove that ${}_\rho Cl_\#(E \cup F) \subseteq {}_\rho Cl_\#(E) \cup {}_\rho Cl_\#(F)$ holds. Let $x \notin {}_\rho Cl_\#(E) \cup {}_\rho Cl_\#(F)$. Then, there exist ρ-closed subsets A and B such that $x \notin A, x \notin B, E \subseteq A$ and $F \subseteq B$. By Remark 2.8 and Theorem 3.5(i), it is obtained that $A \cup B$ is ρ-closed. Since $E \cup F \subseteq A \cup B$ and $x \notin A \cup B$, we have that $x \notin {}_\rho Cl_\#(E \cup F)$. **(v)** It is obvious from definition. **(vi)** Using (i) it suffices to prove an inclusion: ${}_\rho Cl_\#({}_\rho Cl_\#(E)) \subseteq {}_\rho Cl_\#(E)$. Let $x \notin {}_\rho Cl_\#(E)$. Then, there exists a ρ-closed set A such that $E \subseteq A$ and $x \notin A$. Then, by (v), ${}_\rho Cl_\#(E) \subseteq A$ and hence $x \notin {}_\rho Cl_\#(E)$. **(vii)** It is obvious from (i),(ii),(iv) and (vi). □

Definition 4.3 For a space (X, τ) and a $\rho \in \{\xi, \xi^*, \xi^{**}\}$, we define the following families: ${}_\rho\tau_\# := \{U|\ {}_\rho Cl_\#(X \setminus U) = X \setminus U\}$.

Corollary 4.4 *For any topology τ, the following properties hold.*
(i) *Three families of subsets ${}_\xi\tau_\#$, ${}_{\xi^*}\tau_\#$ and ${}_{\xi^{**}}\tau_\#$ are topologies of X.*
(ii) ${}_{\xi^*}\tau_\# \subseteq {}_\xi\tau_\# \subseteq {}_{\xi^{**}}\tau_\# = P(X)$ *and* $\tau \subseteq \tau^\alpha \subseteq {}_\xi\tau_\#$.

Proof. **(i)** By Theorem 4.2(vii), they are topologies of X. **(ii)** The inclusions are obtained by Theorem 4.2 and Definition 4.3. We claim that $P(X) \subseteq {}_{\xi^{**}}\tau_\#$ holds. Let $A \in P(X)$. Using Theorem 3.5(v), any singleton is $g\alpha$ -closed and open, or ξ^{**}-open in (X, τ). Thus any singleton is ξ^{**}-open in (X, τ), because an open set is ξ^{**}-open in (X, τ) (cf.Theorem 2.3(i),Theorem 3.2(ii)). Then, we have that $\{x\} \in {}_{\xi^{**}}\tau_\#$ for each $x \in A$. By (i), it is shown that $A = \cup\{\{x\}|x \in A\} \in {}_{\xi^{**}}\tau_\#$ and so $P(X) \subseteq {}_{\xi^{**}}\tau_\#$. □

Remark 4.5 Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Then, a subset $A = \{a, c\}$ of a space (X, τ) is not a ξ^{**}-open set; $A \in {}_{\xi^{**}}\tau_\#$ (cf. Corollary 4.4(ii)).

In the proof of the following proposition, we use the following notations: $\xi O(X, \tau) = \{U|\ U$ is ξ-open in $(X, \tau)\}; \xi^{**}O(X, \tau) = \{V|\ V$ is ξ^{**}-open in $(X, \tau)\}$. The example of Remark 4.5 shows that $\xi^{**}O(X, \tau) \neq_{\xi^{**}} \tau_\#(= P(X))$ in general.

Proposition 4.6 *For any space (X, τ), the following properties hold.*
(i) *Every ξ-closed set is α-closed in (X, τ) if and only if ${}_\xi\tau_\# = \tau^\alpha$.*
(ii) *Every ξ-closed set is closed in (X, τ) if and only if ${}_\xi\tau_\# = \tau$.*
(iii) *Every ξ^{**}-closed set is α-closed in (X, τ) if and only if $\tau^\alpha = P(X)$.*
(iv) *Every ξ^{**}-closed set is closed in (X, τ) if and only if $\tau = P(X)$.*

*Proof.***(i) (Necessity)** Since every ξ-closed set is α-closed in (X, τ), we have that $\xi O(X, \tau) = \tau^\alpha$ and so, for any subset E of X, ${}_\xi Cl_\#(E) = \cap\{F|\ E \subseteq F, X\setminus F \in \xi O(X, \tau)\} = \cap\{F|\ E \subseteq F, X \setminus F \in \tau^\alpha\} = \tau^\alpha\text{-}Cl(E)$ holds. Therefore, we have that ${}_\xi\tau_\# = \tau^\alpha$. **(Sufficiency)** Let A be a ξ-closed set of (X, τ). Then, $A = {}_\xi Cl_\#(A)$ and so $X \setminus A \in {}_\xi\tau_\#$. By assumption, $X \setminus A \in \tau^\alpha$ and hence A is α-closed in (X, τ). **(ii) (Necessity)** Since every ξ-closed set is closed in (X, τ), we have that $\xi O(X, \tau) = \tau$ and so, for any subset E of X, ${}_\xi Cl_\#(E) = \cap\{F|\ E \subseteq F, X \setminus F \in \xi O(X, \tau)\} = \cap\{F|\ E \subseteq F, X \setminus F \in \tau\} = \tau\text{-}Cl(E)$ holds. Therefore, we have that ${}_\xi\tau_\# = \tau$. **(Sufficiency)** Let A be a ξ-closed set of (X, τ). Then, $A = {}_\xi Cl_\#(A)$ and so $X \setminus A \in {}_\xi\tau_\#$. By assumption, $X \setminus A \in \tau$ and hence A is closed in (X, τ). **(iii) (Necessity)** Since every ξ^{**}-closed set is α-closed in (X, τ), we have that $\xi^{**}O(X, \tau) = \tau^\alpha$ and so, for any subset E of X, ${}_{\xi^{**}}Cl_\#(E) = \tau^\alpha\text{-}Cl(E)$ holds. Therefore, using Corollary 4.4, we have that $\tau^\alpha = P(X)$. **(Sufficiency)** Let A be a ξ^{**}-closed set of (X, τ). Then, $A = {}_{\xi^{**}}Cl_\#(A)$ and so $X \setminus A \in {}_{\xi^{**}}\tau_\#$. By assumption, $X \setminus A \in \tau^\alpha$ and hence A is α-closed in (X, τ). **(iv) (Necessity)** Since every ξ^{**}-closed set is closed in (X, τ), we have that $\xi^{**}O(X, \tau) = \tau$ and so, for any subset E of X, ${}_{\xi^{**}}Cl_\#(E) = \cap\{F|\ E \subseteq F, X\setminus F \in \xi^{**}O(X, \tau)\} = \cap\{F|\ E \subseteq F, X\setminus F \in \tau\} = \tau\text{-}Cl(E)$ holds. Therefore, using Corollary 4.4, we have that ${}_{\xi^{**}}\tau_\# = \tau$ and hence $\tau = P(X)$. **(Sufficiency)** Let A be a ξ^{**}-closed set of (X, τ). Then, $A = {}_{\xi^{**}}Cl_\#(A)$ and so $X \setminus A \in {}_{\xi^{**}}\tau_\#$. By assumption, $X \setminus A \in \tau$ and hence A is closed in (X, τ). □

The above Proposition 4.6 suggests new separation axioms T_ξ and $T_{\xi^{**}}$ which are defined and investigated in the following section.

5 New separation axioms T_ξ and $T_{\xi^{**}}$

In this section we also use the following notations: $\xi O(X,\tau) := \{U | U$ is ξ-open in $(X,\tau)\}$; $\xi^{**}O(X,\tau) := \{V|\ V$ is ξ^{**}-open in $(X,\tau)\}$.

Definition 5.1 (i) A space (X,τ) is a *T_ξ-space* if every ξ-closed set is α-closed (i.e.$\tau^\alpha = \xi O(X,\tau)$).
(ii) A space (X,τ) is a *$T_{\xi^{**}}$-space* if every ξ^{**}-closed set is α-closed (i.e.$\tau^\alpha = \xi^{**}O(X,\tau)$).
(iii) [27, Definition 4.1] A space (X,τ) is a *$T^*_{1/2}$-space* if every g^*-closed set is closed.

Theorem 5.2 *For a space (X,τ), the following properties are equivalent:*
(1) *(X,τ) is a T_ξ-space;*
(2) *${}_\xi\tau_\# = \tau^\alpha$ holds;*
(3) *Every singleton of X is $g\alpha$-closed or α-open in (X,τ);*
(4) *Every singleton of X is $g\alpha$-closed or open in (X,τ);*
(5) *A space (X,τ^α) is $T^*_{1/2}$.*

Proof. (**1**) $\Leftrightarrow$ (**2**) It is Proposition 4.6(i). (**1**) $\Rightarrow$ (**3**) Let $x \in X$. Then, by Lemma 2.9, $X = X_{\mathcal{G}_\alpha\mathcal{C}} \cup X_{\xi\mathcal{O}}$. Suppose that $\{x\}$ is not $g\alpha$-closed. Then $x \in X_{\xi\mathcal{O}}$(i.e.$\{x\} \in \xi O(X,\tau)$). Using assumption that $\tau^\alpha = \xi O(X,\tau)$, we have that $\{x\}$ is α-open. (**3**) $\Leftrightarrow$ (**4**) The proof is obvious, because a singleton $\{x\}$ is open if and only if it is α-open. (**3**) $\Rightarrow$ (**5**) It is shown that a subset A is $g\alpha$-closed in (X,τ) if and only if A is g-closed in (X,τ^α) [16, Theorem 2.3]. Moreover, A is α-open in (X,τ) (i.e.$A \in \tau^\alpha$) if and only if A is open in (X,τ^α). Then, by (3), every singleton $\{x\}$ is g-closed or open in (X,τ^α). By [27, Theorem 4.15], (X,τ^α) is $T^*_{1/2}$. (**5**) $\Rightarrow$ (**1**) We claim that every ξ-closed set is α-closed in (X,τ). Let A be a ξ-closed set in (X,τ). Then, by Theorem 2.7 and (5), A is g^*-closed in (X,τ^α) and so A is closed in (X,τ^α) (i.e.A is α-closed in (X,τ)). □

Every $T_{3/4}$-space is a $T_{1/2}$-space ([8, Corollary 4.7]). A space (X,τ) is called a $T_{3/4}$-space [8] if every δ-generalzed closed subset is δ-closed in (X,τ). A space (X,τ) is called a $T_{1/2}$-space [15] if every g-closed subset is closed in (X,τ). It is well known that a space (X,τ) is $T_{3/4}$ if and only if every singleton $\{x\}$ is regular open or closed in (X,τ) ([8, Theorem 4.3]). Moreover, a space (X,τ) is $T_{1/2}$ if and only if every singleton $\{x\}$ is open or closed in (X,τ) ([9, Theorem 2.6]). The digital line $(\mathbf{Z},\kappa)$ is $T_{3/4}$ ([8, Example 4.6;Theorem 4.3]) and so it is $T_{1/2}$ ([8, Corollary 4.7]). A space (X,τ) is ${}_\alpha T_{1/2}$ if an induced space (X,τ^α) is $T_{1/2}$ ([16]). Digital objects are related to some low separation axioms.

The following result (iii) of Corollary 5.3 is probably unexpected:

Corollary 5.3 (i) *Every $T_{1/2}$-space is T_ξ.*
(ii) *Every $T_{\xi^{**}}$-space is T_ξ.*
(iii) *A space (X,τ) is $T_{\xi^{**}}$ if and only if $\tau = P(X)$.*

Proof. **(i)** Suppose that (X,τ) is $T_{1/2}$. Then, for a point $x \in X, \{x\}$ is closed or open, by [9, Theorem 2.6]. Using Theorem 5.2,(X,τ) is T_ξ. **(ii)** It is obvious from Theorem 3.2 and Definition 5.1. **(iii)** A space (X,τ) is $T_{\xi^{**}}$ if and only if $\tau^\alpha = P(X)$ holds (cf.Proposition 4.6(iii)). And, it is shown that $\tau^\alpha = P(X)$ if and only if $\tau = P(X)$. □

Remark 5.4 Moreover, we have the following diagrams of implications: **(i)** ${}_\alpha T_1 \to {}_\alpha T_{1/2} \to T_\xi$ (cf.[16, Theorem 5.4(iii)], Theorem 5.2); a space (X,τ) is called an ${}_\alpha T_1$ (resp. ${}_\alpha T_{1/2}$) [16] if an induced space (X,τ^α) is T_1 (resp. $T_{1/2}$). **(ii)** ${}_\alpha T_m \to {}_\alpha T^*_{1/2} \to {}_\alpha T_{1/2} \to T_\xi$ (cf.(i) above, [16, Theorem 5.4]); a space (X,τ) is ${}_\alpha T^*_{1/2}$ (resp. ${}_\alpha T_m$) [16] if every $g\alpha^{**}$-closed set is α-closed (resp. closed).

Theorem 5.5 *If (X, τ) is T_ξ and a subset H is open in (X, τ), then $(H, \tau|H)$ is T_ξ.*

Proof. Let $x \in H$. By Theorem 5.2, the singleton $\{x\}$ is gα-closed or α-open in (X, τ) and so $\{x\}$ is g-closed or open in (X, τ^α). Then, using [15, Theorem 2.9], $\{x\}$ is g-closed or open in $(H, \tau^\alpha|H)$. Therefore, by Theorem 5.2, $(H, (\tau|H)^\alpha)$ is T_ξ, because $\tau^\alpha|H = (\tau|H)^\alpha$ for $H \in \tau$. □

Remark 5.6 **(i)** The digital line $(\mathbf{Z}, \kappa)$ is T_ξ, because it is $T_{1/2}$ (cf.[8, Example 4.6], Corollary 5.3(i)). **(ii)** (cf.Theorem 7.1(i)) The digital plane $(\mathbf{Z}^2, \kappa^2)$ is a T_ξ-space (cf.Section 7 below); it is not $T_{1/2}$. Thus, the converse of Corollary 5.3(i) does not true in general. **(iii)** The converse of Corollary 5.3(ii) does not true in general (cf.Remark 7.2).

6 Some functions and groups

Definition 6.1 Let $f : (X, \tau) \to (Y, \sigma)$ be a function between spaces and $\rho \in \{\xi, \xi^*, \xi^{**}\}$. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be:

(i) *ρ-continuous* if for every closed set F of (Y, σ), $f^{-1}(F)$ is ρ-closed in (X, τ);
(ii)*ρ-irresolute* if for every ρ-closed set B of (Y, σ), $f^{-1}(B)$ is ρ-closed in (X, τ);
(iii) *ρ-open* if for every open set U of (X, τ), $f(U)$ is ρ-open in (Y, σ);
(iv) *ρ-closed* if for every closed set C of (X, τ), $f(C)$ is ρ-closed in (Y, σ);
(v) *ρ-homeomorphism* if f is a bijective ρ-continuous and f^{-1} is ρ-continuous;
(vi) *ρc-homeomorphism* if f is a bijective ρ-irresolute and f^{-1} is ρ-irresolute.

We recall the following definitions and properties: a function $f : (X, \tau) \to (Y, \sigma)$ is said to be *α-continuous* ([22], [19]) (resp. *α-irresolute* [18], *$g\alpha$-irresolute* [4, Definition 2.1(ii)]) if for every closed (resp. α-closed, $g\alpha$-closed) set F of (Y, σ), $f^{-1}(F)$ is α-closed (resp. α-closed, $g\alpha$-closed) in (X, τ). In [22], the α-continuous function was firstly called *strongly semi-continuous* (cf. [23]). It is easily shown that $f : (X, \tau) \to (Y, \sigma)$ is $g\alpha$-irresolute if and only if for every $g\alpha$-open set U of (Y, σ), $f^{-1}(U)$ is $g\alpha$-open in (X, τ).

Lemma 6.2 (i)([23, Theorem 4.13]) *If $f : (X, \tau) \to (Y, \sigma)$ is almost-open and α-continuous, then f is α-irresolute.*

(ii) *Especially, if $f : (X, \tau) \to (Y, \sigma)$ is open and continuous, then f is α-irresolute.*

(iii) *If $f : (X, \tau) \to (Y, \sigma)$ is a homeomorphism, then $f : (X, \tau) \to (Y, \sigma)$ and $f^{-1} : (Y, \sigma) \to (X, \tau)$ are α-irresolute and $g\alpha$-irresolute.*

Proof. **(i)** This is Theorem 4.13 in [23]. We recall definition of *almost-open* functions [26]: a function $f : (X, \tau) \to (Y, \sigma)$ is almost-open if $f(U)$ is open in (Y, σ) for every regular open set U of (X, τ) (eg.[23, p.124]). **(ii)** Every open function is almost-open and every continuous function is α-continuous. Thus (ii) is obtained by (i). **(iii)** Since f and f^{-1} are open and continuous, by (ii) $f : (X, \tau) \to (Y, \sigma)$ and $f^{-1} : (Y, \sigma) \to (X, \tau)$ are α-irresolute. Thus, the induced functions $f : (X, \tau^\alpha) \to (Y, \sigma^\alpha)$ and $f^{-1} : (Y, \sigma^\alpha) \to (X, \tau^\alpha)$ are homeomorphisms. By [15, Theorem 6.1], $f(A)$ is g-closed in (Y, σ^α) for every g-closed set A of (X, τ^α). And, by [15, Theorem 6.3], $f^{-1}(B)$ is g-closed in (X, τ^α) for every g-closed set B of (Y, σ^α). Thus we have that the set $f(A)$ is $g\alpha$-closed in (Y, σ) for every $g\alpha$-closed set A of (X, τ) and $f^{-1}(B)$ is $g\alpha$-closed in (X, τ) for every $g\alpha$-closed set B of (Y, σ). Therefore, $f : (X, \tau) \to (Y, \sigma)$ and $f^{-1} : (Y, \sigma) \to (X, \tau)$ are $g\alpha$-irresolute. □

Theorem 6.3 (i) *Every α-continuous function is ξ-continuous.*

(ii) *Every ξ^*-continuous function is ξ-continuous.*

(iii) *Every ξ-continuous function is ξ^{**}-continuous.*

(iv) *Every ρ-irresolute function is ρ-continuous for each $\rho \in \{\xi, \xi^{**}\}$.*

(v) *If functions* $f : (X, \tau) \to (Y, \sigma)$ *and* $g : (Y, \sigma) \to (Z, \eta)$ *are* ρ*-irresolute, then the composition* $g \circ f : (X, \tau) \to (Z, \eta)$ *is* ρ*-irresolute, for each* $\rho \in \{\xi, \xi^*, \xi^{**}\}$.

(vi) *The following properties are equivalent for a bijection* $f : (X, \tau) \to (Y, \sigma)$ *and* $\rho \in \{\xi, \xi^*, \xi^{**}\}$.

(1) f *is* ρ*-open and* ρ*-continuous;*

(2) f *is a* ρ*-homeomorphism;*

(3) f *is* ρ*-closed and* ρ*-continuous.*

(vii) *If* $f : (X, \tau) \to (Y, \sigma)$ *is a homeomorphism, then* $f : (X, \tau) \to (Y, \sigma)$ *and* $f^{-1} : (Y, \sigma) \to (X, \tau)$ *are* ρ*-irresolute* (*i.e.* $f : (X, \tau) \to (Y, \sigma)$ *is a* ρc*-homeomorphism*) *for each* $\rho \in \{\xi, \xi^*, \xi^{**}\}$.

(viii) *If* $f : (X, \tau) \to (Y, \sigma)$ *is a* ρc*-homeomorphism, then* $f : (X, \tau) \to (Y, \sigma)$ *is a* ρ*-homeomorphism for each* $\rho \in \{\xi, \xi^{**}\}$.

Proof. **(i)** Suppose that f is α-continuous. Let F be a closed set in (Y, σ). Then, by Theorem 2.3(i), $f^{-1}(F)$ is ξ-closed and so f is ξ-continuous. **(ii)-(vi)** They are obvious from definitions.

(vii) Let $f : (X, \tau) \to (Y, \sigma)$ be a homeomorphism. First, we claim that $f^{-1} : (Y, \sigma) \to (X, \tau)$ is ρ-irresolute for each $\rho \in \{\xi, \xi^*, \xi^{**}\}$.

Case 1. $\rho = \xi^{**}$: Let A be a ξ^{**}-closed set of (X, τ). To prove that $f(A)$ is ξ^{**}-closed in (Y, σ), let U be a $g\alpha$-open set in (Y, σ) such that $f(A) \subseteq U$. Then, by Lemma 6.2(iii), $f^{-1}(U)$ is $g\alpha$-open in (X, τ). Thus, we have that $f(\tau^\alpha\text{-}Cl(A)) \subseteq f(Int(Cl(f^{-1}(U))))$. Since $f : (X, \tau) \to (Y, \sigma)$ is a homeomorphism and $\tau^\alpha\text{-}Cl(B) = A \cup Cl(Int(Cl(B)))$ for any subset B of (X, τ)([2, Theorem 1.5(c)]), we have that $\sigma^\alpha\text{-}Cl(f(A)) \subseteq Int(Cl(U))$. Thus, $f(A)$ is ξ^{**}- closed in (Y, σ).

Case 2. $\rho = \xi^*$: Let A be a ξ^*-closed set of (X, τ). Let U be a $g\alpha$-open set in (Y, σ) such that $f(A) \subseteq U$. Then, by Lemma 6.2(iii), $f^{-1}(U)$ is $g\alpha$-open in (X, τ). Thus, we have that $f(\tau^\alpha\text{-}Cl(A)) \subseteq f(f^{-1}(Int(U)))$. Since $f : (X, \tau) \to (Y, \sigma)$ is a homeomorphism, we have that $\sigma^\alpha\text{-}Cl(f(A)) \subseteq Int(U)$ and so $f(A)$ is ξ^*-closed in (Y, σ).

Case 3. $\rho = \xi$: Let A be a ξ-closed set of (X, τ). Let U be a $g\alpha$-open set in (Y, σ) such that $f(A) \subseteq U$. Then, by Lemma 6.2(iii), $f^{-1}(U)$ is $g\alpha$-open in (X, τ). Thus, we have that $f(\tau^\alpha\text{-}Cl(A)) \subseteq f(f^{-1}(U))$ and so $\sigma^\alpha\text{-}Cl(f(A)) \subseteq U$. Then, $f(A)$ is ξ- closed in (Y, σ).

Therefore, we claimed that $f^{-1} : (Y, \sigma) \to (X, \tau)$ is ρ-irresolute for each $\rho \in \{\xi, \xi^*, \xi^{**}\}$. Since $f^{-1} : (Y, \sigma) \to (X, \tau)$ is a homeomorphism, we show similarly that $f : (X, \tau) \to (X, \tau)$ is ρ-irresolute and hence f is a ρc-homeomorphism for each $\rho \in \{\xi, \xi^*, \xi^{**}\}$.

(viii) Assume that $\rho = \xi$ (resp. ξ^{**}). Let F be a closed set of (Y, σ). Then, F is ξ-closed (resp. ξ^{**}-closed) of (Y, σ) (cf. Theorem 2.3(i) (resp. Theorem 3.2(ii)). Then, $f^{-1}(F)$ is ξ-closed (resp. ξ^{**}-closed) in (Y, σ), because $f : (X, \tau) \to (Y, \sigma)$ is ξ-irresolute (resp. ξ^{**}-irresolute). Thus, we have that $f : (X, \tau) \to (Y, \sigma)$ is ξ-continuous (resp. ξ^{**}-continuous). Similarly, it is shown that $f^{-1} : (Y, \sigma) \to (X, \tau)$ is ξ-continuous (resp. ξ^{**}-continuous). □

Remark 6.4 **(i)** The following example shows that the converse of Theorem 6.3(i) need not to be true. Let $X = \{a, b, c, d\}, \tau = \{\emptyset, \{c, d\}, X\}, Y = \{p, q\}$ and $\sigma = \{\emptyset, \{p\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(a) = f(b) = f(c) = q$ and $f(d) = p$. Then, f is ξ^*-continuous (and so ξ-continuous); it is not α-continuous. **(ii)** The converse of Theorem 6.3(ii) need not to be true. Indeed, let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{a, c\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(x) = x$ for any $x \in X$. Then, f is α-continuous (and so ξ-continuous); it is not ξ^*-continuous. **(iii)** The converse of Theorem 6.3(iii) need not to be true. Indeed, let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b, c\}, X\}$ and $\sigma = \{\emptyset, \{a, c\}, Y\}$. Then, a function $f : (X, \tau) \to (Y, \sigma)$ defined by $f(x) = x$ for any $x \in X$, is ξ^{**}-continuous; it is not ξ-continuous. **(iv)** The above (i) and (ii) show that the α-continuity and the ξ^*-continuity are independent to each others. **(v)** The above (i)-(iv) and Theorem 6.3(i)-(iii) show that the following diagram of implications holds and all implications are not reversible.

$$\begin{array}{ccccc}
\alpha\textit{-continuous} & & & & \\
 & \searrow & & & \\
\not\downarrow\not\uparrow & & \xi\textit{-continuous} & \rightarrow & \xi^{**}\textit{-continuous} \\
 & \nearrow & & & \\
\xi^*\textit{-continuous} & & & &
\end{array}$$

The following theorem is a pasting lemma for ρ-continuous (resp. ρ-irresolute) functions for each $\rho \in \{\xi, \xi^*, \xi^{**}\}$. Let $X = A \cup B$ and $f : A \to Y$ and $h : B \to Y$ be two functions. It is called that f and h are compatible if $f(x) = h(x)$ for every $x \in A \cap B$. The combination $f \nabla h : X \to Y$ is defined by $(f \nabla h)(x) = f(x)$ for every $x \in A$ and $(f \nabla h)(x) = h(x)$ for every $x \in B$.

Theorem 6.5 *Let $\rho \in \{\xi, \xi^*, \xi^{**}\}$. Suppose that A and B are open and closed subset of (X, τ) such that $X = A \cup B$. Let $f : (A, \tau|A) \to (Y, \sigma)$ and $h : (B, \tau|B) \to (Y, \sigma)$ be compatible functions.*

(i) *If f and h are ρ-continuous, then its combination $f \nabla h : (X, \tau) \to (Y, \sigma)$ is also ρ-continuous.*

(ii) *If f and h are ρ-irresolute, then its combination $f \nabla h : (X, \tau) \to (Y, \sigma)$ is also ρ-irresolute.*

Proof. **(i)** Let F be a closed set of (Y, σ). Then, $(f \nabla h)^{-1}(F) = f^{-1}(F) \cup h^{-1}(F)$ and $f^{-1}(F)$ (resp. $h^{-1}(F)$) is ρ-closed in $(A, \tau|A)$ (resp. $(B, \tau|B)$). By Corollary 2.13(i) and Remark 2.8(ii) for $\rho = \xi$; Theorem 3.6(i) and Theorem 3.5(i) for each $\rho \in \{\xi^*, \xi^{**}\}$, $(f \nabla h)^{-1}(F)$ is ρ-closed in (X, τ). Therefore, $f \nabla h$ is ρ-continuous. **(ii)** Let F be a ρ-closed set of (Y, σ). Since $f^{-1}(F)$ (resp. $h^{-1}(F)$) is ρ-closed in $(A, \tau|A)$ (resp. $(B, \tau|B)$), (ii) is proved by an argument similar to that in (i) above. □

We construct some groups corresponding to a space (X, τ).

Definition 6.6 For a space (X, τ) and $\rho \in \{\xi, \xi^*, \xi^{**}\}$, we define the following collections of functions:

(i) $\rho h(X, \tau) = \{f |\ f : (X, \tau) \to (X, \tau)$ is a ρ-homeomorphism$\}$;

(ii) $\rho ch(X, \tau) = \{f |\ f : (X, \tau) \to (X, \tau)$ is a ρc-homeomorphism$\}$;

(iii) $h(X, \tau) = \{f |\ f : (X, \tau) \to (X, \tau)$ is a homeomorphism$\}$.

Theorem 6.7 (i) *For each $\rho \in \{\xi, \xi^*, \xi^{**}\}$, $h(X, \tau) \subseteq \rho ch(X, \tau)$ holds.*

(ii) *For each $\rho \in \{\xi, \xi^{**}\}$, $\rho ch(X, \tau) \subseteq \rho h(X, \tau)$ holds.*

(iii) *The set $\rho ch(X, \tau)$ forms a group containing $h(X, \tau)$ as its subgroup for each $\rho \in \{\xi, \xi^*, \xi^{**}\}$.*

Proof. **(i)** (resp. **(ii)**) It is obtained by Theorem 6.3(vii) (resp. Theorem 6.3(viii)). **(iii)** A binary operation $\beta : \rho ch(X, \tau) \times \rho ch(X, \tau) \to \rho ch(X, \tau)$ is well defined by $\beta(u, v) = u \circ v$ (the composition of functions) for any $u, v \in \rho ch(X, \tau)$ (cf.Theorem 6.3(v)). Then, it is shown that $\rho ch(X, \tau)$ forms a group under β. Using (i), $h(X, \tau)$ is a subgroup of $\rho ch(X, \tau)$. □

Corollary 6.8 *Assume* $\rho \in \{\xi, \xi^*, \xi^{**}\}$.

(i) *If there exists a* ρ*c-homeomorphism* $f : (X, \tau) \to (Y, \sigma)$*, then there exists a group isomorphism:* $\rho ch(X, \tau) \cong \rho ch(Y, \sigma)$ *holds.*

(ii) *Especially, if there exists a homeomorphism* $f : (X, \tau) \to (Y, \sigma)$*, then there exists a group isomorphism:* $\rho ch(X, \tau) \cong \rho ch(Y, \sigma)$ *holds.*

Proof. **(i)** Let $f : (X, \tau) \to (Y, \sigma)$ be a ρc-homeomorphism. Then, a group isomorphism $f_* : \rho ch(X, \tau) \cong \rho ch(Y, \sigma)$ is well defined by $f_*(u) := f \circ u \circ f^{-1}$, using Theorem 6.3(v). **(ii)** It is obvious by (i) and Theorem 6.3(vii). □

In the below remark, we use the following notations: for a space (X, τ), $\rho C(X, \tau) = \{F |\ F \text{ is } \rho\text{-closed in } (X, \tau)\}$ where $\rho \in \{\xi, \xi^*, \xi^{**}\}$.

Remark 6.9 For the following spaces (X, τ) and (Y, σ), we get the group structures of $\rho ch(X, \tau)$ and $\rho ch(Y, \sigma)$, where $\rho \in \{\xi^*, \xi, \xi^{**}\}$. Let $X = Y = \{a, b, c, d\}, \tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{b, c, d\}, Y\}$.

(i) $\xi^* ch(X, \tau) = \xi ch(X, \tau) = \xi^{**} ch(X, \tau) = h(X, \tau) = \{1_X, h_1, h_2, h_3\}$ and $h_i^2 = 1 (i = 1, 2, 3)$ hold, where $h_1, h_2, h_3 : (X, \tau) \to (X, \tau)$ are functions defined by $h_1(a) = c, h_1(c) = a, h_1(x) = x$ for $x \in \{b, d\}; h_2(x) = x$ for any $x \in \{a, c\}, h_2(b) = d, h_2(d) = b; h_3(a) = c, h_3(b) = d, h_3(c) = a, h_3(d) = b$ and $1_X : (X, \tau) \to (X, \tau)$ is the identity.

(ii) $\xi^* ch(Y, \sigma) = h(Y, \sigma) = \{1_Y, h_4\} \cong Z_2; \xi ch(Y, \sigma) = \xi^{**} ch(Y, \sigma) = \{1_Y, h_4, h_5, h_6\}$ and $h_i^2 = 1 (i = 4, 5, 6)$ hold, where $h_4(x) = x$ for any $x \in \{a, d\}, h_4(b) = c, h_4(c) = b; h_5(x) = x$ for any $x \in \{b, c\}, h_5(a) = d, h_5(d) = a; h_6(a) = d, h_6(b) = c, h_6(c) = b, h_6(d) = a$ and $1_Y : (Y, \sigma) \to (Y, \sigma)$ is the identity.

Indeed, we have the following properties:
$\xi^* C(X, \tau) = \{\emptyset, \{b, d\}, \{b, c, d\}, \{a, b, d\}, X\}$;
$\xi C(X, \tau) = \{\emptyset, \{b\}, \{d\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\}$;
$\xi^{**} C(X, \tau) = P(X) \setminus \{\{a\}, \{c\}\}$ and
$\xi^* C(Y, \sigma) = \{\emptyset, \{a\}, \{a, d\}, \{a, c, d\}, \{a, b, d\}, Y\}$;
$\xi C(Y, \sigma) = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{a, c, d\}, \{a, b, d\}, Y\}$;
$\xi^{**} C(Y, \sigma) = P(Y) \setminus \{\{b\}, \{c\}\}$.

Remark 6.10 The converse of Corollary 6.8 is not true. Let (X, τ) and (Y, σ) be the spaces of Remark 6.9 above. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(x) = x$ for any $x \in \{a, b\}, f(c) = d, f(d) = c$. Then, we observe that, for each $\rho \in \{\xi, \xi^{**}\}, f$ induces an isomorphism $f_* : \rho ch(X, \tau) \cong \rho ch(Y, \sigma)$ such that $f_*(h_1) = h_5, f_*(h_2) = h_4, f_*(h_3) = h_6$. Moreover, it is observed that f is not a ρc-homeomorphism, where $\rho \in \{\xi^*, \xi, \xi^{**}\}$, and f is not a homeomorphism.

Remark 6.11 **(i)** The following examples show that the continuity and ξ^*-continuity (cf. Definition 6.1(i)) are independent. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$. Let $f : (X, \tau) \to (X, \tau)$ be a continuous function defined by $f(a) = f(b) = b, f(c) = c$. Then, f is not ξ^*-contiuous. Let $g : (X, \tau) \to (X, \tau)$ be a function defined by $g(a) = a, g(b) = c, g(c) = b$. Then, g is not continuous; it is ξ^*-continuous. Indeed, it is observed that $\xi^* C(X, \tau) = \{\emptyset, \{b\}, \{b, c\}, X\}$.

(ii) The following functions $f : (X, \tau) \to (Y, \sigma)$ and $g : (X, \tau) \to (Y, \sigma)$ show that the converse of Theorem 6.3(iv) is not true.

Case 1. $\rho = \xi$: Let $X = Y = \{a, b, c\}, \tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{a, b\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(x) = x$ for any $x \in X$. Then, f is not ξ-irresolute; it is ξ-continuous. Indeed, $\xi C(X, \tau) = \{\emptyset, \{c\}, \{b, c\}, \{a, c\}, X\}$ and $\xi C(Y, \sigma) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, Y\}$ hold.

Case 2. $\rho = \xi^{**}$: Let (X, τ) and (Y, σ) be topological spaces defined in Remark 6.9 and $g : (X, \tau) \to (Y, \sigma)$ a function defined as $g(a) = b, g(b) = a, g(c) = d, g(d) = c$. Then, g is not ξ^{**}-irresolute; it is ξ^{**}-continuous.

(iii) The converse of Theorem 6.3(vii) for $\rho \in \{\xi, \xi^{**}\}$ need not to be true. Let (Y, σ) be a space of Remark 6.9 and $f : (Y, \sigma) \to (Y, \sigma)$ a function defined by $f(a) = d, f(d) = a, f(x) = x$ for any $x \in \{b, c\}$. Then, f is ρc-homeomorphism, where $\rho \in \{\xi, \xi^{**}\}$; f is not a homeomorphism. We note that f is not ξ^*-irresolute. **(iv)** The converse of Theorem 6.3(viii) for $\rho = \xi^{**}$ need not to be true. Let (X, τ) and (Y, σ) be the spaces of Remark 6.9 and $g : (X, \tau) \to (Y, \sigma)$ a function defined in (ii) Case 2 above. Then, g is not a ξ^{**}c-homeomorphism; it is a ξ^{**}-homeomorphism.

7 ρ-closed sets of the digital plane where $\rho \in \{\xi, \xi^{**}\}$ In this section, we show that the digital plane $(\mathbf{Z}^2, \kappa^2)$ is a T_ξ-space (cf. Remark 5.6(ii)) and investigate characterizations of ξ-closed sets and ξ^{**}-closed sets of the digital plane. First, we recall related definitions and some propeties of the digital plane. The digital line is the set of the integers, $\mathbf{Z}$, equipped with the topology κ having $\{\{2m-1, 2m, 2m+1\} | m \in \mathbf{Z}\}$ as a subbase. It is denoted by $(\mathbf{Z}, \kappa)$. A singleton $\{2n+1\}$ is open and a subset $\{2n-1, 2n, 2n+1\}$ is the smallest open set containing $2n$, where $s, n \in \mathbf{Z}$. The digital line $(\mathbf{Z}, \kappa)$ is a typical example of the $T_{1/2}$-space which is not T_1 (cf.[15] [9]), because every singleton of $(\mathbf{Z}, \kappa)$ is open or closed. Furthermore, it is shown, in [8, Example 4.6] that $(\mathbf{Z}, \kappa)$ is $T_{3/4}$. Let $(\mathbf{Z}^2, \kappa^2)$ be the topological product of two copies of the digital line $(\mathbf{Z}, \kappa)$, where $\mathbf{Z}^2 = \mathbf{Z} \times \mathbf{Z}$ and $\kappa^2 = \kappa \times \kappa$. In this paper, the space $(\mathbf{Z}^2, \kappa^2)$ is called the *digital plane* (cf.[12, p.10], [13, p.907], [6, Section 6], [24, Section 5], [10]). This space is a mathematical model of the computer screen. Let $(\mathbf{Z}^2)_{\kappa^2} := \{x | \ \{x\}$ is open in $(\mathbf{Z}^2, \kappa^2)\}$. Then, $(\mathbf{Z}^2)_{\kappa^2} = \{(2n+1, 2m+1) | \ n, m \in \mathbf{Z}\}$ and it is open and dense in $(\mathbf{Z}^2, \kappa^2)$.

Theorem 7.1 *t170* (cf. Remark 5.6(ii)) (i) *The digital plane $(\mathbf{Z}^2, \kappa^2)$ is a T_ξ-space.*
(ii) *A subspace $(H, \kappa^2|H)$ of $(\mathbf{Z}^2, \kappa^2)$ is a T_ξ-space, where $H = (\mathbf{Z}^2)_{\kappa^2}$.*

Proof. **(i)** Let x be a point of $(\mathbf{Z}^2, \kappa^2)$.

Case 1. $x = (2m+1, 2n+1)$, where $n, m \in \mathbf{Z}$: The singleton $\{x\}$ is open in (Z^2, κ^2).

Case 2. $x = (2m, 2n)$, where $n, m \in \mathbf{Z}$: The singleton $\{x\}$ is closed in (Z^2, κ^2) and so it is gα-closed (cf. Theorem 2.3(ii)).

Case 3. $x = (2m, 2n+1)$, where $n, m \in \mathbf{Z}$: The singleton $\{x\}$ is gα-closed in (Z^2, κ^2). Indeed, it is shown that, for any gα-open set U including $\{x\}, (\kappa^2)^\alpha\text{-}Cl(\{x\}) = \{x\} \cup Cl(Int(Cl(\{x\}))) = \{x\} \cup Cl(Int(\{2m\} \times \{2n, 2n+1, 2n+2\})) = \{x\} \cup Cl(\emptyset) = \{x\} \subseteq U$ and so $\{x\}$ is ξ-closed (cf. [2, Theorem 1.5(c)]). Then, by Theorem 2.3(ii), $\{x\}$ is gα-closed in $(\mathbf{Z}^2, \kappa^2)$.

Case 4. $x = (2m+1, 2n)$, where $n, m \in \mathbf{Z}$: The proof for this case is similar to Case 3 above.

Thus we have that every singleton $\{x\}$ of $(\mathbf{Z}^2, \kappa^2)$ is gα-closed or open. Therefore, using Theorem 5.2, $(\mathbf{Z}^2, \kappa^2)$ is a T_ξ-space. **(ii)** A singleton $\{(x_1, x_2)\}$ is open in $(\mathbf{Z}^2, \kappa^2)$ if and only if x_1 and x_2 are odd integers. Thus, H is open in $(\mathbf{Z}^2, \kappa^2)$. By (i) and Theorem 5.5, $(H, \kappa^2|H)$ is T_ξ. $\square$

Remark 7.2 (cf.Remark 5.6(iii)) The converse of Corollary 5.3(ii) does not true in general. By definition, Corollary 5.3(iii) and Theorem 7.1, it is shown that $\kappa^2 \neq P(\mathbf{Z}^2)$ and so $(\mathbf{Z}^2, \kappa^2)$ is not $T_{\xi^{**}}$; it is T_ξ.

We recall the following notations of families of subsets, definitions and a fact: For a space $(X, \tau), PO(X, \tau)$ (resp. $\xi O(X, \tau), \tau^\alpha, SO(X, \tau), G\alpha O(X, \tau)$) denotes the family of all preopen (resp. ξ-open, α-open, semi-open, gα-open) subsets of (X,τ)) and $PC(X,\tau)$ (resp. $\xi C(X,\tau), SC(X,\tau), G\alpha C(X,\tau)$) denotes the family of all preclosed (resp. ξ-closed, semi-closed, gα-closed) subsets of (X,τ). A subset A of (X,τ) is called *preopen* (resp. *semi-open*) in (X,τ), if $A \subseteq Int(Cl(A))$ (resp. $A \subseteq Cl(Int(A))$) holds in (X,τ). It is well known that $\tau^\alpha = PO(X,\tau) \cap SO(X,\tau)$ holds for any space (X,τ)([23, Lemma 3.1], [25]).

Theorem 7.3 *For the digital plane* $(\mathbf{Z}^2,\kappa^2)$*, the following properties hold:*
(i) $\xi O(\mathbf{Z}^2,\kappa^2) = PO(\mathbf{Z}^2,\kappa^2)$; $\xi C(\mathbf{Z}^2,\kappa^2) = PC(\mathbf{Z}^2,\kappa^2)$.
(ii) $\xi O(\mathbf{Z}^2,\kappa^2) \subseteq SO(\mathbf{Z}^2,\kappa^2)$; $\xi C(\mathbf{Z}^2,\kappa^2) \subseteq SC(\mathbf{Z}^2,\kappa^2)$.
(iii) $\xi O(\mathbf{Z}^2,\kappa^2) = PO(\mathbf{Z}^2,\kappa^2) = (\kappa^2)^\alpha = G\alpha O(\mathbf{Z}^2,\kappa^2)$.

Proof. **(i)** We recall that
$(*)$ $G\alpha O(\mathbf{Z}^2,\kappa^2) = PO(\mathbf{Z}^2,\kappa^2) = (\kappa^2)^\alpha$ hold ([6, Theorem 6.1 (ii)]).

By Theorem 2.3(ii) and $(*)$, $\xi C(\mathbf{Z}^2,\kappa^2) \subseteq G\alpha C(\mathbf{Z}^2,\kappa^2) = PC(\mathbf{Z}^2,\kappa^2)$. Convercely, by Theorem 2.3(i) and $(*)$, $PO(\mathbf{Z}^2,\kappa^2) = (\kappa^2)^\alpha \subseteq \xi O(\mathbf{Z}^2,\kappa^2)$. Thus we have that $\xi C(\mathbf{Z}^2,\kappa^2) = PC(\mathbf{Z}^2,\kappa^2)$ and $\xi O(\mathbf{Z}^2,\kappa^2) = PO(\mathbf{Z}^2,\kappa^2)$ hold. **(ii)** By [6, Theorem 6.1(i)], $PO(\mathbf{Z}^2,\kappa^2) \subseteq SO(\mathbf{Z}^2,\kappa^2)$ holds. Thus, (ii) is obtained by (i). **(iii)** It is proved by using (i) (ii) above and [6, Theorem 6.1(ii)]. □

To investigate further properties of ξ-closed sets and ξ^{**}-closed sets on the digital plane $(\mathbf{Z}^2,\kappa^2)$, we prepare the following two propositions (Proposition 7.4, Proposition 7.5 below). Let (X,τ) be a space and E a subset of (X,τ). First, we recall the following properties on the preclosure $pCl(A) := \cap\{F | A \subseteq F, F \text{ is preclosed in } (X,\tau)\}$, the α-closure τ^α-$Cl(A)$ and the τ-closure $Cl(A)$ in (X,τ): the following properties are known.

$(*)$ $pCl(E) \subseteq \tau^\alpha\text{-}Cl(E) \subseteq Cl(E)$ hold for any subset E.

$(**)$ If E is α-open, then $pCl(E) = \tau^\alpha\text{-}Cl(E) = Cl(E)$ hold; for an open set E, the equality of (ii) is also true, because any open set is α-open.

$(***)$ If E is preopen, then $\tau^\alpha\text{-}Cl(E) = Cl(E)$ hold; we note that $pCl(E) \neq Cl(E)$ in general. Indeed, let (X,τ) be a space, where $X = \{a,b,c\}$ and $\tau = \{\emptyset, \{a,b\}, X\}$. For a preopen set $E = \{a\}, pCl(E) = \{a\} \subsetneq X = Cl(E) = \tau^\alpha\text{-}Cl(E)$ hold. However, for a preopen set E of the digital plane $(\mathbf{Z}^2,\kappa^2)$, we have the following relationships on closures of E.

Proposition 7.4 (i) *For a subset* E *of* $(\mathbf{Z}^2,\kappa^2), pCl(E) = (\kappa^2)^\alpha\text{-}Cl(E)$ *holds.*
(ii) *For a preopen set* E *of* $(\mathbf{Z}^2,\kappa^2), pCl(E) = (\kappa^2)^\alpha\text{-}Cl(E) = Cl(E)$ *hold.*

Proof. **(i)** It is obtained from a fact that $PO(\mathbf{Z}^2,\kappa^2) = (\kappa^2)^\alpha$ (cf.Theorem 7.3(iii)). **(ii)** By using (i) and the α-closure's formula [2, Theorem 1.5(c)], it is shown that $pCl(E) = (\kappa^2)^\alpha\text{-}Cl(E) = E \cup Cl(Int(Cl(E))) \supseteq E \cup Cl(E) = Cl(E)$. Therefore, using $(*)$ above, we have that $pCl(E) = (\kappa^2)^\alpha\text{-}Cl(E) = Cl(E)$ hold. □

Recall the set $(\mathbf{Z}^2)_{\kappa^2} = \{x | \ \{x\} \text{ is open in } (\mathbf{Z}^2,\kappa^2)\} = \{(2m+1, 2s+1) | m,s \in \mathbf{Z}\}$ and it is open and dense in $(\mathbf{Z}^2,\kappa^2)$.

Proposition 7.5 *For a non-empty subset* E *of* $(\mathbf{Z}^2,\kappa^2)$*, the following properties are equivalent:*
(1) E *is dense in* $(\mathbf{Z}^2,\kappa^2)$*;*
(2) $Cl(E) = Int(Cl(E))$ *holds;*
(3) $E \supseteq (\mathbf{Z}^2)_{\kappa^2}$ *holds.*

Proof. **(1)⇒(2)** Since $Cl(E) = \mathbf{Z}^2, Cl(E) = \mathbf{Z}^2 = Int(Cl(E))$ holds. **(2)⇒(3)** By (2), $Cl(E)$ is open and closed in $(\mathbf{Z}^2,\kappa^2)$ and $Cl(E) \neq \emptyset$. Since $(\mathbf{Z}^2,\kappa^2)$ is connected [1, p.67], $Cl(E) = \mathbf{Z}^2$. Let x be any point of $(\mathbf{Z}^2)_{\kappa^2}$. Then, $\{x\}$ is open and $x \in Cl(E)$. Thus we have that $x \in E$ and hence $(\mathbf{Z}^2)_{\kappa^2} \subseteq E$. **(3)⇒(1)** It follows from (3) that $Cl(E) \supseteq Cl((\mathbf{Z}^2)_{\kappa^2}) = \mathbf{Z}^2$ and so $Cl(E) = \mathbf{Z}^2$. □

Remark 7.6 For the following space (X, τ) and a set E of X, the implication (2)$\Rightarrow$(1) in Proposition 7.5 does not hold in general. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then, a subset $E = \{a\}$ is not dense in $(X, \tau); Cl(E) = Int(Cl(E)) = \{a\}$ holds.

Theorem 7.7 *Let E be a subset of $(\mathbf{Z}^2, \kappa^2)$.*
(i) *If E is dense in $(\mathbf{Z}^2, \kappa^2)$, then E is ξ^{**}-closed.*
(ii) *If $E \supseteq (\mathbf{Z}^2)_{\kappa^2}$, then E is ξ^{**}-closed in $(\mathbf{Z}^2, \kappa^2)$.*
(iii) *If E is a preopen and dense subset of $(\mathbf{Z}^2, \kappa^2)$ and $E \neq \mathbf{Z}^2$, then E is not ξ-closed.*
(iv) *If $E \supseteq (\mathbf{Z}^2)_{\kappa^2}$ and $E \neq \mathbf{Z}^2$, then E is not ξ-closed in $(\mathbf{Z}^2, \kappa^2)$.*

Proof. **(i)** Let U be a gα-open set containing E. Then, $\mathbf{Z}^2 = Int(Cl(E)) \subseteq Int(Cl(U))$ and so $Int(Cl(U)) = \mathbf{Z}^2$. Thus we have that $(\kappa^2)^\alpha$-$Cl(E) = E \cup Cl(Int(Cl(E))) = E \cup \mathbf{Z}^2 = \mathbf{Z}^2 = Int(Cl(U))$ and so E is ξ^{**}-closed. **(ii)** By Proposition 7.5, E is dense in $(\mathbf{Z}^2, \kappa^2)$, because $E \neq \emptyset$. Using (i), E is ξ^{**}-closed. **(iii)** Suppose that E is ξ-closed. Since $E \in PO(\mathbf{Z}^2, \kappa^2) = G\alpha O(\mathbf{Z}^2, \kappa^2)$ (cf. Theorem 7.3(iii)) and $E \subseteq E$, we have that $(\kappa^2)^\alpha$-$Cl(E) = Cl(E) \subseteq E$ using Proposition 7.4(ii). Therefore, $E = \mathbf{Z}^2$, because $Cl(E) = \mathbf{Z}^2$. This is a contradiction. **(iv)** It follows from assumption that $Cl(E) \supseteq Cl((\mathbf{Z}^2)_{\kappa^2}) = \mathbf{Z}^2$ and so E is dense and E is preopen. Using (iii), E is not ξ-closed. □

Theorem 7.8 (i) *If a non-empty subset E is preopen and it is not dense in $(\mathbf{Z}^2, \kappa^2)$, then E is not ξ^{**}-closed.*
(ii) *Every non-empty proper subset of $(\mathbf{Z}^2)_{\kappa^2}$ is not ξ^{**}-closed in $(\mathbf{Z}^2, \kappa^2)$.*

Proof. **(i)** Suppose that E is ξ^{**}-closed. Since $E \in G\alpha O(\mathbf{Z}^2, \kappa^2)$ and $E \subseteq E$, by using Proposition 7.4(ii), it is shown that $(\kappa^2)^\alpha$-$Cl(E) = Cl(E) \subset Int(Cl(E))$ and so $Cl(E) = Int(Cl(E))$. By Proposition 7.5, E is dense. This is a contradiction. **(ii)** Let E be a non-empty proper subset of $(\mathbf{Z}^2)_{\kappa^2}$. Then, E is open and so preopen and there exists a point x_0 such that $x_0 \in (\mathbf{Z}^2)_{\kappa^2}$ and $x_0 \notin E$. We claim that E is not dense. Indeed, suppose that E is dense. Since $\{x_0\}$ is open and $x_0 \in Cl(E)$, we have that $x_0 \in E$. This is a contradiction. Therefore, the set E is non-empty, preopen and E is not dense. By using (i) above, E is not ξ^{**}-closed. □

Remark 7.9 The following example shows that Theorem 7.8 for a space does not hold in general. Let (X, τ) be a space of Remark 7.6 and a subset $E = \{a\}$. Then, E is preopen and non-dense subset of (X, τ); the set E is ξ^{**}-closed.

Theorem 7.10 *For a non-empty subset E of $(\mathbf{Z}^2, \kappa^2)$, the following properties are equivalent:*
(1) *E is ξ^{**}-closed and preopen in $(\mathbf{Z}^2, \kappa^2)$;*
(2) *E is dense in $(\mathbf{Z}^2, \kappa^2)$;*
(3) *$E \supseteq (\mathbf{Z}^2)_{\kappa^2}$ holds.*

Proof. **(1)$\Rightarrow$(2)** Since $E \subset E \in PO(\mathbf{Z}^2, \kappa^2) = G\alpha O(\mathbf{Z}^2, \kappa^2)$ and E is ξ^{**}-closed, $Cl(E) \subseteq Int(Cl(E))$ holds (cf. Proposition 7.4(ii)). By Proposition 7.5, it is shown that E is dense in $(\mathbf{Z}^2, \kappa^2)$. **(2)$\Rightarrow$(1)** Using Theorem 7.7(i), E is ξ^{**}-closed. Since E is dense, $E \subseteq Cl(E) = \mathbf{Z}^2 = Int(Cl(E))$ holds. **(2)$\Leftrightarrow$ (3)** This is shown in Proposition 7.5. □

Remark 7.11 **(i)** For a topological space, the implication (1)$\Rightarrow$(2) in Theorem 7.10 does not hold in general. Let (X, τ) be a space, where $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $E = \{a, c\}$. Then, $E \in PO(X, \tau) \cap \xi^{**}C(X, \tau)$; E is not dense in (X, τ). **(ii)** By Theorem 7.10, the following subset E is ξ^{**}-closed and ξ-open in $(\mathbf{Z}^2, \kappa^2) : E = (\mathbf{Z}^2)_{\kappa^2} \cup A$, where A is any non-empty set. The set $(\mathbf{Z}^2)_{\kappa^2}$ is called as the *open screen* of the digital plane.

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