ON NEW CHARACTERIZATIONS OF SEMI-T_i-SPACES, WHERE $i = 0, 1/2$

N. Araki, M. Ganster, H. Maki and A. Nishi

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Abstract. In this paper, we characterize semi-T_0-spaces [11] and semi-$T_{1/2}$-spaces [3] by using the notion of semi-λ-closed sets [6].

1 Introduction and results In 1997, Arenas, Dontchev and Ganster [2] introduced the notion of λ-closed sets in a topological space and using the λ-closedness they characterized T_i-spaces, where $i \in \{0, 1/4, 1/2, 1\}$. The purpose of this paper is to characterize two separation axioms semi-T_0 [11] and semi-$T_{1/2}$-spaces [3] by using the notion of semi-λ-closed sets [6]. Throughout this paper, "a space" means a topological space which lacks any separation axioms unless explicitely stated. Let (X, τ) be a space. Recall that a subset A of X is said to be *semi-open* in (X, τ) if $A \subset ClInt(A)$ holds. Let $SO(X, \tau)$ be the family of all semi-open sets in (X, τ). A set A is said to be *semi-closed* if $X \setminus A \in SO(X, \tau)$. Let $SC(X, \tau)$ be the family of all semi-closed sets of (X, τ). The *semi-closure* $sCl(E)$ (resp. *semi-kernel* $sKer(E)$) of a subset E is defined by $sCl(E) := \bigcap\{F | E \subset F, F \in SC(X, \tau)\}$ (resp. $sKer(E) := \bigcap\{U | E \subset U, U \in SO(X, \tau)\}$). It is well known that $sCl(E) = E \cup IntCl(E)$ holds for any subset E of (X, τ) ([9, Lemma 1]). In [6], the notion of semi-λ-closed sets is introduced and investigated. A subset A of a space (X, τ) is *semi-λ-closed* if and only if $A = sKer(A) \cap sCl(A)$ ([6, Proposition 2.6]). The complement of a semi-λ-closed set is called *semi-λ-open*. Let $SLO(X, \tau)$ be the family of all semi-λ-open sets of (X, τ). In [6, p.263], it was stated that $SLO(X, \tau)$ is always a topology on X.

A space (X, τ) is *semi-T_0* [11] if, for each $x, y \in X$ such that $x \neq y$ there exists a semi-open set containing x but not y or a semi-open set containing y but not x. A space (X, τ) is called *semi-$T_{1/2}$* [3] if every sg-closed set is semi-closed. A space (X, τ) is semi-$T_{1/2}$ if and only if every singleton of X is semi-open(=open) or semi-closed ([13, Theorem 4.8] [4, Definition 3.2]). Using the concept of semi-λ-closed sets [6], we characterize semi-T_0-spaces and semi-$T_{1/2}$-spaces. Main theorems are as follows:

Theorem 1.1 *The following properties are equivalent:*
(1) *A space (X, τ) is semi-T_0;*
(2) *Every singleton of X is semi-λ-closed in (X, τ);*
(3) *Every finite subset of X is semi-λ-closed in (X, τ);*
(4) *For every finite set F of X and for every point $y \notin F$ there exists a set V_y containing F and disjoint from $\{y\}$ such that $V_y \in SO(X, \tau)$ or $V_y \in SC(X, \tau)$.*
(5) *An induced space $(X, SLO(X, \tau))$ is T_1.*

Theorem 1.2 *The following properties are equivalent:*
(1) *A space (X, τ) is semi-$T_{1/2}$;*
(2) *Every subset of X is semi-λ-closed in (X, τ);*
(3) *$SLO(X, \tau) = P(X)$ holds.*

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2 A lemma and a corollary We first prepare a lemma from a generalized point of view in the light of [7]. We recall the following concepts from [7, Section 4]: For a space (X, τ), let $\mathcal{E}_X$ and $\mathcal{E}'_X$ be subfamilies of the power set $P(X)$ of X satisfying the following properties:

($\mathcal{A}$) $\{\emptyset, X\} \subset \mathcal{E}_X$ and $\{\emptyset, X\} \subset \mathcal{E}'_X$.

Sometimes, $\mathcal{E}_X$ and $\mathcal{E}'_X$ are denoted by $\mathcal{E}$ and $\mathcal{E}'$, respectively. Then, for a subset A of X, the following two sets are well defined:

$\mathcal{E}_X$-$Ker(A) := \bigcap\{U | A \subset U, U \in \mathcal{E}_X\}$,

$\mathcal{E}_X$-$Cl(A) := \bigcap\{F | A \subset F, X \setminus F \in \mathcal{E}_X\}$.

A subset A is said to be an *$\mathcal{E}_X$-closed* in (X, τ) if $X \setminus A \in \mathcal{E}_X$. A subset B is said to be an *$\mathcal{E}_X$-Λ-set* in (X, τ) if $B = \mathcal{E}_X$-$Ker(B)$. We note that, in [7, p.63], the above teminology, "$\mathcal{E}_X$-Λ-sets" was used as "$\mathcal{E}_X$-λ-sets". For an ordered pair $(\mathcal{E}_X, \mathcal{E}'_X)$, a subset A is said to be *$(\mathcal{E}_X, \mathcal{E}'_X)$-$\lambda$-closed* in (X, τ) if A is an intersection of an $\mathcal{E}_X$-Λ-set and an $\mathcal{E}'_X$-closed set. The following (i) and (ii) are useful for the present paper ([7, Theorem 4.1]).

(i) *Let $(\mathcal{E}_X, \mathcal{E}'_X)$ be an ordered pair of given subfamilies of $P(X)$ satisfying ($\mathcal{A}$). If a subset A is $(\mathcal{E}_X, \mathcal{E}'_X)$-$\lambda$-closed in (X, τ), then $A = \mathcal{E}_X$-$Ker(A) \cap \mathcal{E}'_X$-$Cl(A)$ holds.*

(ii) *Let $(\mathcal{E}_X, \mathcal{E}'_X)$ be an ordered pair satisfying ($\mathcal{A}$) and the following property:*

$(\mathcal{B})_{\mathcal{E}'}$ *The union of any family of subsets belonging to $\mathcal{E}'_X$ belongs to $\mathcal{E}'_X$.*

Then, the converse of (i) *is true, i.e. a subset A is $(\mathcal{E}_X, \mathcal{E}'_X)$- λ-closed in (X, τ) if and only if $A = \mathcal{E}_X$-$Ker(A) \cap \mathcal{E}'_X$-$Cl(A)$ holds.*

Indeed, the property $(\mathcal{B})_{\mathcal{E}'}$ is equivalent to the property $(\mathcal{C})_{\mathcal{E}'}$ in [7, Theorem 4.1] and so (ii) is obtained from [7, Theorem 4.1].

(iii) *Under the assumptions ($\mathcal{A}$) and $(\mathcal{B})_{\mathcal{E}}$, a subset B is $\mathcal{E}_X$-closed if and only if $\mathcal{E}_X$-$Cl(B) = B$ holds.*

Example 2.1 **(i)** In the above, let $\mathcal{E} = \mathcal{E}' = \tau$, then we have τ-$Ker(A) = Ker(A)$, τ-$Cl(A) = Cl(A)$, (τ, τ)-λ-closed set= λ-closed set. We note that ($\mathcal{A}$) and $(\mathcal{B})_\tau$ are satisfied in (X, τ). **(ii)** Let $\mathcal{E} = \mathcal{E}' = SO(X, \tau)$, then we have $SO(X, \tau)$-$Ker(A) = sKer(A)$, $SO(X, \tau)$-$Cl(A) = sCl(A)$, $(SO(X, \tau), SO(X, \tau))$-$\lambda$-closed set =semi-$\lambda$-closed set. We note that ($\mathcal{A}$) and $(\mathcal{B})_{SO(X,\tau)}$ are satisfied in (X, τ); and so a subset A is semi-λ-closed in (X, τ) if and only if $A = sKer(A) \cap sCl(A)$ holds.

Lemma 2.2 *Let $(\mathcal{E}_X, \mathcal{E}'_X)$ be an ordered pair of given subfamilies of $P(X)$ satisfying ($\mathcal{A}$) and $(\mathcal{B})_{\mathcal{E}'}$:*

($\mathcal{A}$) *$\{\emptyset, X\} \subset \mathcal{E}_X$ and $\{\emptyset, X\} \subset \mathcal{E}'_X$;*

$(\mathcal{B})_{\mathcal{E}'}$ *The union of any family of subsets belonging to $\mathcal{E}'_X$ belongs to $\mathcal{E}'_X$.*

Then, for a space (X, τ) and a subset F of X, the following properties are equivalent:

(1) *For every point $y \notin F$, there exists a set V_y containing F and disjoint from $\{y\}$ such that $V_y \in \mathcal{E}_X$ or $X \setminus V_y \in \mathcal{E}'_X$;*

(2) *The subset F is $(\mathcal{E}_X, \mathcal{E}'_X)$- λ-closed in (X, τ).*

Proof. **(1)⇒(2)** For each point $y \notin F$, there exists a set V_y such that $V_y \cap \{y\} = \emptyset, F \subset V_y$ and $V_y \in \mathcal{E}_X$ or $X \setminus V_y \in \mathcal{E}'_X$. Set $\mathcal{L} := \{V_y | V_y \in \mathcal{E}_X,\ y \notin F\ \}$ and $\mathcal{C} := \{V_y | X \setminus V_y \in \mathcal{E}'_X,\ y \notin F\ \}$. Let $L := \bigcap\{V | V \in \mathcal{L}\}$ if $\mathcal{L} \neq \emptyset$, $L := X$ if $\mathcal{L} = \emptyset$ and $C := \bigcap\{V | V \in \mathcal{C}\}$ if $\mathcal{C} \neq \emptyset$, $C := X$ if $\mathcal{C} = \emptyset$. Then, it is shown that $F = L \cap C$ holds, L is an $\mathcal{E}_X$-Λ-set and $X \setminus C \in \mathcal{E}'_X$ because $(\mathcal{B})_{\mathcal{E}'}$ is assumed. Hence, F is $(\mathcal{E}_X, \mathcal{E}'_X)$-$\lambda$-closed in (X, τ). **(2)⇒(1)** Let y be a point of X such that $y \notin F$. It follows from assumption and (i) above [7, Theorem 4.1] that $F = \mathcal{E}_X$-$Ker(F) \cap \mathcal{E}'_X$-$Cl(F)$.

Case 1. $y \notin \mathcal{E}'_X$-$Cl(F)$: Let $V_y := \mathcal{E}'_X$-$Cl(F)$. Then, by $(\mathcal{B})_{\mathcal{E}'}, X \setminus V_y \in \mathcal{E}'_X, F \subset V_y$ and $V_y \cap \{y\} = \emptyset$ hold.

Case 2. $y \in \mathcal{E}'_X\text{-}Cl(F)$: Since $y \notin \mathcal{E}_X\text{-}Ker(F)$, there exists a subset $U \in \mathcal{E}_X$ such that $y \notin U$ and $F \subset U$. Thus, let $V_y := U$.

Therefore, we show (1) for both cases. □

In Lemma 2.2 above, let $\mathcal{E}_X = \mathcal{E}'_X = SO(X,\tau)$ for a space (X,τ). Then, we finally have the following lemma as a corollary (cf. Example 2.1(ii)).

Corollary 2.3 *For a space (X,τ) and a subset F of X, the followings are equivalent:*

(1) *For every point $y \notin F$, there exists a set V_y containing F and disjoint from $\{y\}$ such that $V_y \in SO(X,\tau)$ or $V_y \in SC(X,\tau)$;*

(2) *F is semi-λ-closed in (X,τ).* □

3 Proofs of Theorem 1.1 and Theorem 1.2 We first recall the following properties:

Theorem 3.1 ([6, Theorem 2.7]) *A subset A of (X,τ) is semi-λ-closed if and only if $X_2 \cap sCl(A) \subset A$, where $X_2 := \{x \in X | \{x\}$ is preopen $\}$.*

The proof of the following theorem was not given in [6] and this fact is used in the present paper. We give a sketch of the proof.

Theorem 3.2 ([6, p.263]) *For any space $(X,\tau), SLO(X,\tau)$ is a topology on X.*

Proof. It is evident that $\emptyset$ and X are semi-λ-closed. We first claim that if A and B are semi-λ-closed then $A \cup B$ is semi-λ-closed. Let $x \in X_2 \cap sCl(A \cup B)$. We have that $x \in IntCl(\{x\}) \subset IntClIntCl(A \cup B) \subset IntCl(A \cup B) \subset IntCl(A) \cup Cl(B)$. We consider the following two cases:

Case 1. $x \in IntCl(A)$: Since $X_2 \cap sCl(A) = X_2 \cap (A \cup IntCl(A)) \subset A$ by Theorem 3.1, we have that $x \in A$.

Case 2. $x \in Cl(B)$: We have that $x \in IntCl(\{x\}) \subset IntClCl(B) = IntCl(B)$. Then, we show that $x \in B$, because $X_2 \cap sCl(B) = X_2 \cap (B \cup IntCl(B)) \subset B$ hold (cf. Theorem 3.1).

Thus we show that $X_2 \cap sCl(A \cup B) \subset A \cup B$ and so $A \cup B$ is semi-λ-closed by Theorem 3.1. Finally, we show that if $A_i, i \in \mathcal{B}$, are semi-λ-closed, then $\bigcap\{A_i | i \in \mathcal{B}\}$ is semi-λ-closed, where the index set $\mathcal{B}$ is not necessarily finite. We have that $X_2 \cap sCl(\bigcap\{A_i | i \in \mathcal{B}\}) \subset X_2 \cap sCl(A_i) \subset A_i$ for any $i \in \mathcal{B}$ and so $X_2 \cap sCl(\bigcap\{A_i | i \in \mathcal{B}\}) \subset \bigcap\{A_i | i \in \mathcal{B}\}$. Therefore, $\bigcap\{A_i | i \in \mathcal{B}\}$ is semi-λ-closed (cf. Theorem 3.1). □

We have the proofs of Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1

(1)⇒(2) Let $x \in X$. It follows from the assumption that, for every point $y \neq x$ there exists a set U_y containing x and disjoint $\{y\}$ such that U_y is semi-open or semi-closed. Let $\mathcal{L} := \{U_y | U_y \in SO(X,\tau),\ y \neq x\}$ and $\mathcal{C} := \{U_y | U_y \in SC(X,\tau),\ y \neq x\}$. We define the following sets L and C: $L := \bigcap\{V | V \in \mathcal{L}\}$ if $\mathcal{L} \neq \emptyset$, $L := X$ if $\mathcal{L} = \emptyset$ and $C := \bigcap\{V | V \in \mathcal{C}\}$ if $\mathcal{C} \neq \emptyset$ and $C := X$ if $\mathcal{C} = \emptyset$. Then, L is a semi-Λ-set and C is a semi-closed set. It is shown that $L \cap C = \{x\}$ and hence $\{x\}$ is semi-λ-closed. **(2)⇒(1)** Let x and y be two different points of X. Since $\{x\}$ is semi-λ-closed, there exist a semi-Λ-set L and a semi-closed sets C such that $\{x\} = L \cap C$.

Case 1. $y \notin C$: The set $X \setminus C$ is a semi-open set containing y such that $x \notin X \setminus C$.

Case 2. $y \in C$: Since $y \notin L$, there exists a semi-open set W_y containing x such that $y \notin W_y$.

Therefore, (X, τ) is semi-T_0. $\square$

For a subset E of a space (X, τ), let $E_{\mathcal{SO}} := \{x | x \in E, \{x\} \in SO(X, \tau)\}$.

Proof of Theorem 1.2

(1)⇒(2) We first recall that a space (X, τ) is semi-$\mathrm{T}_{1/2}$ if and only if for each $x \in X, \{x\}$ is semi-open(=open) or semi-closed ([13, Theorem 4.8]). Let A be a subset of X. Set $B := X \setminus (A \cup (X \setminus A)_{\mathcal{SO}})$. We define the following subsets L and C as follows: $L := \cap\{X \setminus \{x\} | x \in B\}$ and $C := \cap\{X \setminus \{y\} | y \in (X \setminus A)_{\mathcal{SO}}\}$. Then, C is semi-closed. Indeed, for any $y \in (X \setminus A)_{\mathcal{SO}}, X \setminus \{y\}$ is a semi-closed set containing A. And, L is a semi-Λ-set. Indeed, for any $x \in B$, $X \setminus \{x\}$ is a semi-open set containing $A \cup (X \setminus A)_{\mathcal{SO}}$. It is shown that $A = L \cap C$ and hence A is semi-λ-closed.

(2)⇒(1) Assume that $\{x\}$ is not semi-open. We claim that $\{x\}$ is semi-closed. Since $X \setminus \{x\}$ is not semi-closed and it is semi-λ- closed, by [6, Proposition 2.6] we have that $X \setminus \{x\} = sKer(X \setminus \{x\}) \cap sCl(X \setminus \{x\})$ and hence $X \setminus \{x\} = sKer(X \setminus \{x\})$. It is shown that $\{x\}$ is semi-closed. Therefore, (X, τ) is semi-$\mathrm{T}_{1/2}$, by [13, Theorem 4.8]. **(2)⇔(3)** It is evident. $\square$

4 The digital plane is semi-$\mathbf{T}_{1/2}$ In the present paper, *the digital plane* $(\mathbf{Z}^2, \kappa^2)$ is the topological product of two copies of the digital line $(\mathbf{Z}, \kappa)$, where $\mathbf{Z}^2 = \mathbf{Z} \times \mathbf{Z}$ and $\kappa^2 = \kappa \times \kappa$ (eg. [5] [8] [12]; cf. [10, p.10]). The proof of the following proposition is done using Theorem 1.2 (cf. Remark 4.2(i)).

Proposition 4.1 *The digital plane $(\mathbf{Z}^2, \kappa^2)$ is semi-$T_{1/2}$.*

Proof. We first claime that

$(*)$ every singleton $\{x\}$ of the digital plane $(\mathbf{Z}^2, \kappa^2)$ is semi-λ-open, that is, F is semi-λ-closed, where $F := \mathbf{Z}^2 \setminus \{x\}$.

Case 1. $x = (2n, 2m)$, where $n, m \in \mathbf{Z}$: Then, it is shown that $x \in sCl(F)$ and so $\mathbf{Z}^2 = sCl(F)$. Indeed, $sCl(F) = F \cup IntCl(F) = F \cup \mathbf{Z}^2 = \mathbf{Z}^2$. By using Theorem 3.1, the set F is semi-λ-closed. Indeed, $(\mathbf{Z}^2)_2 \cap sCl(F) = (\mathbf{Z}^2)_2 \cap \mathbf{Z}^2 = (\mathbf{Z}^2)_2 \subset F$, because $\{x\}$ is not preopen and so $x \notin (\mathbf{Z}^2)_2$, where $(\mathbf{Z}^2)_2 = \{y \in \mathbf{Z}^2 | \{y\} \subset IntCl(\{y\})\}$.

Case 2. $x = (2n+1, 2m+1)$, where $n, m \in \mathbf{Z}^2$: Since $\{x\}$ is open, F is closed and so it is semi-λ closed.

Case 3. $x = (2n, 2m+1)$, where $n, m \in \mathbf{Z}$: Let U be an open set containing x. Then, $U(x) := \{2n-1, 2n, 2n+1\} \times \{2m+1\} \in \kappa^2$ and $x \in U(x) \subset U$. Because $(2n+1, 2m+1) \in F \cap U$, we have that $Cl(F) = \mathbf{Z}^2; sCl(F) = F \cup IntCl(F) = \mathbf{Z}^2$. By using Theorem 3.1, F is semi-λ-closed because $\{x\}$ is not preopen and so $(\mathbf{Z}^2)_2 \cap sCl(F) \subset F$ holds.

Case 4. $x = (2n+1, 2m)$, where $n, m \in \mathbf{Z}$: By an argument similar to that in Case 3 above, it is shown that F is semi-λ-closed.

Thus, we have that the set F is semi-λ-closed; so every singleton $\{x\}$ is semi-λ-open, that is, $\{x\} \in SLO(\mathbf{Z}^2, \kappa^2)$. We finally have that, for every subset E of $\mathbf{Z}^2, E = \cup\{\{x\} | x \in E\} \in SLO(\mathbf{Z}^2, \kappa^2)$ (cf. Theorem 3.2 and $(*)$ above). We now conclude as follows: $(\mathbf{Z}^2, \kappa^2)$ is semi-$T_{1/2}$ (cf. Theorem 1.2). $\square$

Remark 4.2 (i) We have an alternative proof of Proposition 4.1 using a fact that a topological space is semi-$T_{1/2}$ if and only if every singleton is semi-open or semi-closed ([13, Theorem 4.8]). **(ii)** The digital plane $(\mathbf{Z}^2, \kappa^2)$ is not $\mathrm{T}_{1/4}$ (cf.[1]); so it is not $\mathrm{T}_{1/2}$. Every $\mathrm{T}_{1/2}$-space is semi-$\mathrm{T}_{1/2}$ [3]. The Proposition 4.1 shows that the converse of an implication above is not true.

(iii) Using Theorem 1.1, [2, Theorem 2.6] and definitions, we obtain the following diagram of implications.

$$
\begin{tikzcd}
\textit{semi-}T_{1/2} \arrow[r] & \textit{semi-}T_0 & \\
 & & T_0 \arrow[ul] \\
T_{1/2} \arrow[uu] \arrow[r] & T_{1/4} \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$

All implications are not reversible (cf. (ii) above and well known facts).

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Nobuyuki Araki; Department of Mathematics, Chienkan High School, Saga, 849-0915 Japan

Maximilian Ganster; Department of Mathematics, Graz University of Technology, Steyrergasse 30, A-8010, Graz, Austria

Haruo Maki; Wakagidai 2-10-13, Fukutsu-shi, Fukuoka-ken, 811-3221, Japan
e-mail: makih@pop12.odn.ne.jp

Akihiro Nishi; Department of Mathematics, Faculty of Culture and Education, Saga University, Saga, 840-8502 Japan