ANDO'S THEOREM FOR HADAMARD PRODUCTS AND OPERATOR MEANS

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Abstract. First we see that Ando's inequality $A * B \geq (A\#B) * (A\#B)$ gives a characterization of the geometric operator mean $\#$, where $*$ is the Hadamard product. Extending this, we discuss inequalities for operator means and the Hadamard product. Moreover we show monotone convergence theorems including such inequalities.

1 Introduction. One of the author [5] discussed inequalities for Haramard products for operators from the following viewpoint: For a standard orthonormal basis $\{e_k\}$ of a (separable) Hilbert space, the *Hadamard product* $A * B$ for operators A and B on H is defined as

$$U^*(A \otimes B)U$$

where U is the isometric operator $\sum_k (e_k \otimes e_k) \otimes \overline{e_k}$ from H to $H \otimes H$. Thereby the Hadamard product has the monotone convergence property for selfadjoint operators:

$$A_n \downarrow A \text{ and } B_n \downarrow B \text{ imply } A_n * B_n \downarrow A * B$$

The theory of operator means is started at Ando's lecture note [3] and established as the Kubo-Ando theory [11]. For positive operators on a Hilbert space H, the theory of operator means is defined axiomatically: An (*operator*) *connection* m is a binary operation on positive operators satisfying the following axioms:

monotonicity: $A_1 \leq A_2$ and $B_1 \leq B_2$ imply $A_1 \; m \; B_1 \leq A_2 \; m \; B_2$.

semi-continuity: $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n \; m \; B_n \downarrow A \; m \; B$.

transformer inequality: $T^*(A \; m \; B)T \leq (T^*AT)m(T^*BT)$.

An *operator mean* is a connection m satisfying

normalization: $A \; m \; A = A$.

When we discuss operator means, we may assume that positive operators are invertible by virtue of the above semi-continuity. Moreover, when discussing inequalities of Hadamard products of operator means, we have only to show the case for positive-definite matrices by approximated by the simple functions of them.

It is easy to show that the transformer inequality becomes equality if T is invertible. For an operator mean m, the *representing function* $f_m(x) = 1 \; m \; x$ is *operator monotone*:

$$0 \leq A \leq B \qquad \text{implies} \qquad f_m(A) \leq f_m(B).$$

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This correspondence $m \mapsto f_m$ is bijective. In fact, if f is a continuous nonnegative operator monotone functional on $[0, \infty)$ with $f(1) = 1$, then a binary operation m defined by

$$A\ m\ B = A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}$$

for positive invertible operators A and B induces an operator mean $A\ m\ B$. They also introduced the three operations in operator means; the *transpose* $^\circ$, the *adjoint* * and the *dual* $^\perp$:

$$\begin{aligned} A\ m^\circ B &= B\ m\ A, & f^\circ(x) &= x f\left(\frac{1}{x}\right) \\ A\ m^* B &= (A^{-1} m\ B^{-1})^{-1}, & f^*(x) &= \frac{1}{f(1/x)} \\ A\ m^\perp B &= (B^{-1} m\ A^{-1})^{-1}, & f^\perp(x) &= \frac{x}{f(x)} \end{aligned}$$

An operation in the above is the composition of the other two. Self-transpose means are called *symmetric* and the geometric (operator) mean $\#$ is invariant for all the above operations. The arithmetic and the harmonic ones, other typical symmetric means, are adjoint (or dual) each other.

In this note, we discuss inequalities of Hadamard products for operator means, which is based on the following Ando's theorem [2]:

Theorem (Ando). *If A and B are positive operators, then*

$$A * B \geq (A \# B) * (A \# B).$$

2 Various interpretations.

As in the above, the geometric mean $\#$ is the central one as operator means. So Ando's theorem might be extend in various ways. First we consider the form $(A\ m\ B) * (A\ m\ B)$. It is known that the greatest (resp. smallest) symmetric operator mean is the arithmetic (resp. harmonic) one. Ando's theorem shows

$$A * B \geq (A \# B) * (A \# B) \geq (A\ m\ B) * (A\ m\ B)$$

if $\# \geq m$ like the harmonic mean. Moreover the geometric mean is characterized as the maximum mean satisfying such inequalities:

Theorem 1. *The geometric mean $\#$ is the maximum among the operator means m satisfying*

$$A * B \geq (A\ m\ B) * (A\ m\ B)$$

for all $A, B \geq 0$.

Proof. Let f be the representing function of an operator mean m satisfying the above condition. Then, the scalar inequality

$$x = 1 * x \geq f(x) * f(x) = f(x)^2$$

implies $f(x) \leq \sqrt{x}$ and hence $m \leq \#$. The geometric mean $\#$ itself satisfies it by Ando's theorem (see another proof in Theorem 3). □

Remark 1. The geometric operator mean $\#$ is characterized in various ways: Ando himself characterized it by operator matrices in [3]. A remarkable presentation is in [1]: $A\#B$ is a unique positive solution of the Riccati operator equation $XA^{-1}X = B$. Related several ones are shown in [6].

On the other hand, we showed in [5] the following inequality, which is an extension of the inequality of Aujla and Vasudeva [4], which also extend Ando's theorem:

Theorem F. *If m is an operator mean whose representing function f is supermultiplicative (i.e., $f(xy) \geq f(x)f(y)$), then*

$$(A * B)m(C * D) \geq (AmC) * (BmD)$$

for all operators $A, B, C, D \geq 0$.

Thus, we have the following inequality putting $C = B$ and $D = A$:

Corollary 2. *If m is an operator mean whose representing function f is supermultiplicative, then*

$$A * B \geq (AmB) * (Am^\circ B)$$

for all operators $A, B \geq 0$.

Ando's theorem was slightly extended for other operator means as in [8, Th. 6.6]. So, as a general result along the same line, we show the following extension:

Theorem 3. *If A and B are positive operators, then*

$$A * B \geq (AmB) * (Am^\perp B).$$

Proof. We may assume that operators are invertible and have the following spectral decompositions:

$$X \equiv A^{-1/2}BA^{-1/2} = \sum_k t_k E_k \quad \text{and} \quad C_k \equiv A^{1/2}E_kA^{1/2} \geq 0.$$

Here note that

$$AmB = A^{1/2} f\left(\sum_k t_k E_k\right) A^{1/2} = \sum_k f(t_k)C_k.$$

It follows that

$$\begin{aligned} A * B &= \left(\sum_k C_k\right) * \left(A^{1/2}XA^{1/2}\right) = \sum_{k,j} t_k C_k * C_j \\ &= \sum_k t_k C_k * C_k + \sum_{k<j} (t_k + t_j) C_k * C_j \end{aligned}$$

and $f(x)f^\perp(x) = x$ implies

$$\begin{aligned} (AmB) * (Am^\perp B) &= \sum_{k,j} f(t_k)f^\perp(t_j) C_k * C_j \\ &= \sum_k t_k C_k * C_k + \sum_{k<j} \big(f(t_k)f^\perp(t_j) + f(t_j)f^\perp(t_k)\big) C_k * C_j. \end{aligned}$$

Thus we have

$$\begin{aligned} A * B - (AmB) * (Am^\perp B) &= \sum_{k<j} \Big(t_k + t_j - f(t_k)f^\perp(t_j) - f(t_j)f^\perp(t_k)\Big) C_k * C_j \\ &= \sum_{k<j} (f^\perp(t_k) - f^\perp(t_j))(f(t_k) - f(t_j)) C_k * C_j \geq 0. \end{aligned}$$

□

3 Parametrizations. Here we discuss parametrizations for Theorem 3. First we consider the path

$$A\#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$$

which runs from A to B via the midpoint $A\#B = A\#_{1/2}B$ as t runs from 0 to 1. By $\#_t^\perp = \#_{1-t}$, means $A\#_t^\perp B$ give a reversed path from B to A. If $B \geq A$, then it is shown that $A\#_t B$ (resp. $A\#_t^\perp B$) is nondecreasing (resp. nonincreasing). Though such monotonicity does not hold in general, we have:

Theorem 4. *For $t < 1/2$ (resp. $t > 1/2$) The operators $(A\#_t B) * (A\#_t^\perp B)$ is not greater than*

$$(A\#B) * (A\#^\perp B) = \lim_{t \to 1/2}(A\#_t B) * (A\#_t^\perp B)$$

and converges monotone increasingly to $A * B$ as $t \downarrow 0$ (resp. $t \uparrow 1$).

Proof. By symmetricity, we have only to show the case $t < 1/2$. Let $0 < r < s < 1/2$. Since $\#_t^\perp = \#_{1-t}$, then Theorem 3 implies

$$\begin{aligned}(A\#_r B) * (A\#_r^\perp B) &\geq (A\#_r B)\#_t(A\#_{1-r}B) * (A\#_r B)\#_t^\perp(A\#_{1-r}B)\\ &= (A\#_{(1-t)r+t(1-r)}B) * (A\#_{tr+(1-t)(1-r)}B)\\ &= (A\#_s B) * (A\#_{1-s}B) = (A\#_s B) * (A\#_s^\perp B)\end{aligned}$$

for $t = (s-r)/(1-2r)$. We can also show the case $t > 1/2$. □

But along the above line, we cannot extend it to general operator means. So we take another path: For $0 < p < 1$, if f is operator monotone, then so is $f_{[p]}(x) = f(x)^p$. The mean corresponding to $f_{[p]}$ is denoted by $m_{[p]}$;

$$Am_{[p]}B = A^{1/2}f(A^{-1/2}BA^{-1/2})^p A^{1/2}.$$

For $x > 0$, we have

$$\lim_{p\downarrow 0} f_{[p]}(x) = 1, \qquad \lim_{p\downarrow 0} f_{[p]}^\perp(x) = \lim_{p\downarrow 0} x/f_{[p]}(x) = \lim_{p\downarrow 0} x/f(x)^p = x,$$

which implies

$$\lim_{p\downarrow 0} Am_{[p]}B = A, \qquad \lim_{p\downarrow 0} Am_{[p]}^\perp B = B.$$

Thus this path is a bridge between both sides in Theorem 3:

$$\begin{aligned}A * B &= \lim_{p\downarrow 0}(Am_{[p]}B) * (Am_{[p]}^\perp B) \geq (Am_{[p]}B) * (Am_{[p]}^\perp B)\\ (AmB) * (Am^\perp B) &= \lim_{p\uparrow 1}(Am_{[p]}B) * (Am_{[p]}^\perp B)\end{aligned}$$

Now we have the following monotone property of this path:

Theorem 5. *For $p \leq 1/2$, $(Am_{[p]}B) * (Am_{[p]}^\perp B) \uparrow A * B \quad (p \downarrow 0)$.*

Proof. We may assume that $1 \leq X \equiv A^{-1/2}BA^{-1/2} = \sum_k t_k E_k$ is a spectral decomposition with distinct eigenvalues t_k. Put $C_k = A^{1/2}E_k A^{1/2}$. It follows from $f_{[p]}(x)f_{[p]}^\perp(x) = x$ that

$$\begin{aligned}(Am_{[p]}B) * (Am_{[p]}^\perp B) &= \left(\sum_k f_{[p]}(t_k)C_k\right) * \left(\sum_k f_{[p]}^\perp(t_k)C_k\right)\\ &= \sum_{i,j} f_{[p]}(t_i)f_{[p]}^\perp(t_j)C_i * C_j\\ &= \sum_{i<j}\left(f_{[p]}(t_i)f_{[p]}^\perp(t_j) + f_{[p]}(t_j)f_{[p]}^\perp(t_i)\right)C_i * C_j + \sum_i t_i C_i * C_i.\end{aligned}$$

Thus we have only to show the following function is monotone decreasing for p and fixed t and s:

$$\begin{aligned} H(p) &\equiv f_{[p]}(t) f^{\perp}_{[p]}(s) + f_{[p]}(s) f^{\perp}_{[p]}(t) \\ &= s\left(\frac{f(t)}{f(s)}\right)^p + t\left(\frac{f(s)}{f(t)}\right)^p \end{aligned}$$

Under the condition $t \neq s$, we have

$$\begin{aligned} H'(p) &= s\left(\frac{f(t)}{f(s)}\right)^p \log\frac{f(t)}{f(s)} + t\left(\frac{f(s)}{f(t)}\right)^p \log\frac{f(s)}{f(t)} = \left[s\left(\frac{f(t)}{f(s)}\right)^p - t\left(\frac{f(s)}{f(t)}\right)^p\right]\log\frac{f(t)}{f(s)} \\ &= s\left(\frac{f(t)}{f(s)}\right)^p\left[1-\frac{t}{s}\left(\frac{f(s)}{f(t)}\right)^{2p}\right]\log\frac{f(t)}{f(s)} = s\left(\frac{f(t)}{f(s)}\right)^p\left[1-\frac{t/f(t)^{2p}}{s/f(s)^{2p}}\right]\log\frac{f(t)}{f(s)} \\ &= s\left(\frac{f(t)}{f(s)}\right)^p\log\frac{f(t)}{f(s)}\left[1-\frac{f^{\perp}_{[2p]}(t)}{f^{\perp}_{[2p]}(s)}\right] = \frac{s}{p}\eta\left(\left(\frac{f(t)}{f(s)}\right)^p\right)\left[\frac{f^{\perp}_{[2p]}(t)}{f^{\perp}_{[2p]}(s)}-1\right] \le 0 \end{aligned}$$

where $\eta(x) = -x\log x$ is the entropy function. Thereby $H(p)$ is monotone decreasing. □

Remark 2. The condition $p \le 1/2$ cannot be deleted in the above theorem. In fact, if $f(x) = x^t$, then the above path is increasing for $(t/2, 1]$

4 Addendum. The proofs of Theorem 3 and 5 suggest us another principle to show inequalities for Hadamard products. Here we show the following Fielder inequality and its reverse one [10] (see also [8, Th.6.14]) in such ways, which is originally shown by Kijima's inequality in [9] (see also [8, Lem. 6.13]): If $m \le A \le M$, then

$$1 \le A * A^{-1} \le \frac{(M^2+m^2)}{2mM}.$$

Let $A = \sum_i t_i E_i$ be the spectral decomposition. Then we have the Fiedler inequality since

$$\begin{aligned} A * A^{-1} &= \sum_{i,j}\frac{t_i}{t_j}E_i * E_j = \sum_i E_i * E_i + \sum_{i<j}\left(\frac{t_i}{t_j}+\frac{t_j}{t_i}\right)E_i * E_j \\ &\ge \sum_i E_i * E_i + \sum_{i<j} 2E_i * E_j = \sum_{i,j} E_i * E_j = 1 \end{aligned}$$

by the arithmetic-geometric mean inequality:

$$\frac{\frac{t_i}{t_j}+\frac{t_j}{t_i}}{2} \ge \sqrt{\frac{t_i}{t_j}\frac{t_j}{t_i}} = 1.$$

For $x > 0$, we have

$$(*) \qquad m/M \le x \le M/m \quad \text{implies} \quad x + 1/x \le \frac{M^2+m^2}{mM}.$$

In fact, we have the right hand of (*) as the minimum for k with $x^2 - kx + 1 \le 0$ for $m/M \le x \le M/m$. It holds if the following inequality assures:

$$(M/m)^2 + 1 \le k(M/m), \qquad (m/M)^2 + 1 \le k(m/M).$$

It follows that

$$k^2 \geq \left((M/m)^2+1\right)\left((m/M)^2+1\right) = \frac{(M^2+m\ 2)^2}{m^2M^2},$$

which shows $(*)$. Now $1 \leq (M^2+m^2)/(2mM)$ implies

$$A * A^{-1} \leq \sum_i E_i * E_i + \sum_{i<j} \frac{M^2+m^2}{mM} E_i * E_j \leq \sum_{i,j} \frac{M^2+m^2}{2mM} E_i * E_j = \frac{M^2+m^2}{2mM}.$$

References

[1] W.N.Anderson and G.E.Trapp: *Operator means and electrical networks*, 1980 IEEE Int. Sym. on Cuicuits and Systems, (1980), 523–527.

[2] T.Ando: *Concavity of ceratin maps on positive definite matrices and applications to Hadamard products*, Linear Alg. Appl., **26**(1979), 203–241.

[3] T.Ando: Topics on operator inequalities, Hokkaido Univ. Lecture Note, 1978.

[4] J.S.Aujla and H.L.Vasudeva: *Inequalities involving Hadamard product and operator means.* Math. Japon., **42**(1995), 265–272.

[5] J.I.Fujii: *The Marcus-Khan theorem for Hilbert space operators*, Math. Japon., **41**(1995),531–535.

[6] J.I.Fujii and M.Fujii: *Some remarks on operator means*, Math. Japon., **24**(1979), 335–339.

[7] J.I.Fujii and M.Fujii: *Jensen's Inequalities on any interval for operators*, Proc. of the 3rd Int. Cof. on Nonlinear Analysis and Convex Analysis, (2004), 29–39.

[8] T.Furuta, J.Mićić, J.E.Pečarić and Y.Seo: "Mond-Pečarić Method in Operator Inequalities", Monographs in Inequalities 1, Element, Zagreb, 2005.

[9] Y.Kijima: *On some operator inequalities (in Japanese)*, Proc. 1st Symp. Funct. Anal. held at Tokyo Inst. Tech., **1**(1966), 25–29.

[10] K.Kitamura and Y.Seo: *Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities*, Sci. Math. **1** (1998), 237–241.

[11] F.Kubo and T.Ando: *Means of positive linear operators*, Math. Ann., **248** (1980) 205–224.

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