Scientiae Mathematicae Japonicae Online, **e-2006**, 577–593

A LIMIT THEOREM OF SUPERPROCESSES WITH NON-VANISHING DETERMINISTIC IMMIGRATION

Isamu Dôku

Received March 21, 2006; revised May 3, 2006

Abstract. A class of immigration superprocesses with dependent spatial motion for deterministic immigration rate is considered, and we discuss a convergence problem for the rescaled processes. When the immigration rate converges to a non-vanishing deterministic one, then we can prove that under a suitable scaling, the rescaled immigration superprocesses associated with SDSM converge to a class of immigration superprocesses associated with coalescing spatial motion in the sense of probability distribution on the space of measure-valued continuous paths. This scaled limit not only provides with a new class of superprocesses but also gives a new type of limit theorem.

1. Introduction

Let us begin with the story of a super-Brownian motion (SBM), which is a typical example of measure-valued processes. Roughly speaking, starting from a family of branching Brownian motions, via renormalization procedure (which is sometimes called short time high density limit), the super-Brownian motion can be obtained, in fact, as a measure-valued Markov process, see S. Watanabe [28]. It is often called a Dawson-Watanabe superprocess, too. Various sorts of superprocesses have been studied by many probabilists, and in most cases those superprocesses are obtained as a limit of branching particle systems under variety of settings. For instance, a superprocess with dependent spatial motion (SDSM) is obtained by short time high density limit from a family of interacting branching particle systems, whose branching density depends on its particle location. Such an SDSM with interaction parameter ρ and branching rate σ was originally constructed by H. Wang [27]. There is a function $c(x)$, one of those parameters that play an important role in determination of the SDSM. When $c(x)$ $(\neq 0)$ is bounded, then under a suitable scaling SDSMs converge to super-Brownian motion, see Dawson-Li-Wang [2]. On the other hand, for the same SDSM the situation has changed drastically when $c(x) \equiv 0$. Under the same scaling as in the above model, SDSMs converge this time to a superprocess with coalescing spatial motion (SCSM). This remarkable dichotomy was proved by Dawson-Li-Zhou [3].

Let us now consider a little bit more complicated model with interaction, in which a notion of immigration is taken into account. For instance, such an immigration superprocess associated with SDSM was constructed in Dawson-Li [1]. In [10] we discussed a problem of rescaled limits for a certain class of superprocesses with deterministic immigration in the case of parametrized immigration rate vanishing at infinity. More precisely, we showed that when the immigration superprocess is given, then its rescaled process becomes again an immigration superprocess of the same kind. Furthermore we proved that under a suitable

2000 *Mathematics Subject Classification.* Primary 60B10, 60F99; Secondary 60G57, 60J80, 60K35.
Key word and phrases. Limit theorem, immigration superprocess, scaling, purely atomic state, interacting location process.

scaling, the rescaled immigration superprocesses associated with SDSM converge to a superprocess with coalescing spatial motion (SCSM) in the sense of probability distribution on the space of measure-valued continuous paths. This article is devoted to a generalization of the limit theorem obtained in [10]. As described in §7 (v) of [10], the purpose of this paper is to discuss a convergence problem for rescaled superprocesses with deterministic immigration of the above type, and also to clarify what kind of superprocess emerges as a limiting process when the parametrized immigration rate does not vanish at infinity but converges to some constant. As seen in the proof of the limit theorem in [10], it is expected that some immigration superprocesses associated with coalescing spatial motion appear in the limit. In other words, a new class of superprocesses arises naturally as a rescaled limit. The construction of such a superprocess as well as the uniqueness are discussed in detail in the companion paper [11]. Our goal is to prove the above-mentioned limit theorem. Our result of this paper is not only a simple generalization of the main theorem in [10], but provides also with a new type of limit theorem. Of course, we can consider a more complicated situation where the deterministic immigration rate is replaced by a certain class of function, which means that the immigration rate may depend on the particle location. This challenging problem will be dealt with in the forthcoming paper [12].

After we submitted this paper, the related work done by Li-Wang-Xiong [29] has been pointed out by the referee, where several types of scaling limit theorems of the SDSM were established.

2. Notations and Preliminaries

Let $M_F(\mathbb{R})$ (resp. $M_a(\mathbb{R})$) be the space of all finite (resp. purely-atomic) measures on $\mathbb{R}$ respectively, and we always consider the space $M_F(\mathbb{R})$ endowed with the topology of weak convergence. We denote by $C(\mathbb{R})$ the space of all bounded and continuous functions on $\mathbb{R}$, and $C(\mathbb{R})^+$ denotes a subset of $C(\mathbb{R})$ consisting of all positive menbers. The symbol $\langle f, \mu \rangle$ denotes an integral $\int f \mathrm{d}\mu$ of a measurable function f with respect to a measure μ. For $h \in C^1(\mathbb{R})$ and both $h, h' \in L^2(\mathbb{R})$, we define $\rho(x) = \int h(y-x)h(y)\mathrm{d}y$ for $x \in \mathbb{R}$. For a topological space E, let $B(E)$ denote the totality of all bounded Borel functions on E, and $\mathcal{P}(E)$ denotes the space of all probability measures on E. For $F \in B(M_F(\mathbb{R}))$, we define $\delta F(\mu)/\delta\mu(x)$ as the usual variational derivative of F with respect to $\mu \in M_F(\mathbb{R})$ for $x \in \mathbb{R}$, if the limit exists. Then $\delta^2 F(\mu)/\delta\mu(x)\delta\mu(y)$ is the second variational derivative of F. For simplicity we put $C_M(\mathbb{R}_+) = C([0,\infty),\ M_F(\mathbb{R}))$ for the space of all finite measure-valued continuous paths on $\mathbb{R}_+$, and for the Skorokhod space we use $D_M(\mathbb{R}_+) = D([0,\infty),$ $M_F(\mathbb{R}))$. For the generator $\mathcal{A}$, we say that an $M_F(\mathbb{R})$-valued càdlàg process $X = (X_t)_{t\geq 0}$ is a solution of the $(\mathcal{A}, \mathrm{Dom}(\mathcal{A}))$-martingale problem, if there is a probability measure $\mathbb{P}_\mu$ $\in \mathcal{P}(D_M(\mathbb{R}_+))$ on the space $D([0,\infty),\ M_F(\mathbb{R}))$ such that $\mathbb{P}_\mu(X_0 = \mu) = 1$ and $F(X_t) - F(X_0) - \int_0^t \mathcal{A}F(X_s)\mathrm{d}s$, $t \geq 0$, is a martingale under $\mathbb{P}_\mu$ for each $F \in \mathrm{Dom}(\mathcal{A})$.

2.1. Superprocess with Dependent Spatial Motion

Let σ be a constant. We denote by $\mathcal{D}(\mathcal{L})$ the domain of the generator $\mathcal{L}$, which is a subset of the space $B(M_F(\mathbb{R}))$ of measurable functions on $M_F(\mathbb{R})$. More precisely, let $\mathcal{D}(\mathcal{L})$ be the union of all functions $F(\mu)$ on $M_F(\mathbb{R})$ of the form $F(\mu) = F_{f,\{\phi_i\}}(\mu) = f(\langle \phi_1, \mu \rangle,$ $\ldots, \langle \phi_n, \mu \rangle)$ for $\mu \in M_F(\mathbb{R})$ with $f \in C^2(\mathbb{R}^n)$ and $\{\phi_i\} \subset C^2(\mathbb{R})$ and all functions of the form $F(\mu) = F_{m,f}(\mu) = \langle f, \mu^m \rangle$ for $\mu \in M_F(\mathbb{R})$ with $f \in C^2(\mathbb{R}^m)$ where μ^m is a tensor product of measures $\mu^{\otimes m}$. For any $F \in \mathcal{D}(\mathcal{L})$ we define

$$\begin{aligned}\mathcal{L}F(\mu) = {} & \frac{1}{2}\int_{\mathbb{R}} \rho(0)\frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(\mathrm{d}x) + \frac{1}{2}\int_{\mathbb{R}} \sigma\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(\mathrm{d}x) \\ & + \frac{1}{2}\iint_{\mathbb{R}\times\mathbb{R}} \rho(x-y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(\mathrm{d}x)\mu(\mathrm{d}y).\end{aligned} \tag{1}$$

Here the function ρ in the second line of (1) expresses interaction, and the second term in the first line of (1) expresses the branching mechanism. An $M_F(\mathbb{R})$-valued diffusion process $X = (X_t)$ is called a $\{\rho(0), \rho, \sigma\}$-superprocess with dependent spatial motion (or $\{\rho(0), \rho, \sigma\}$-SDSM) if X solves the $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$-martingale problem, cf. [2] (see also [27]). Moreover, according to Dawson-Li-Wang [2], the above martingale problem permits a unique solution $\mathbb{P}_\mu$, and the system $\{\mathbb{P}_\mu; \mu \in M_F(\mathbb{R})\}$ defines a diffusion process. Actually it is proved that $\{\rho(0), \rho, \sigma\}$-SDSM lies in the space $M_a(\mathbb{R})$ for any initial state $\mu \in M_F(\mathbb{R})$.

Next we shall introduce the explicit representation of SDSM. Now let us consider a general initial state $\mu \in M_F(\mathbb{R})$ with $\langle 1, \mu\rangle > 0$. Suppose that there are a time-space white noise $W(\mathrm{d}s, \mathrm{d}y)$ on $[0,\infty)\times\mathbb{R}$ based on the Lebesgue measure $\mathrm{d}\ell$ and a Poisson random measure $N(\mathrm{d}a, \mathrm{d}w)$ on $\mathbb{R}\times W_0$ with intensity $\mu(\mathrm{d}a)Q_k(\mathrm{d}w)$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Q_k is the excursion law of the β-branching diffusion, and W_0 is a subset of paths $w \in W = C([0,\infty), \mathbb{R}^+)$ such that $w(0) = w(t) = 0$ for $t \geq \tau_0(w) = \inf\{s > 0; w(s) = 0\}$ for $w \in W$. For the details, see §2 of [1]. We also assume that $\{W(\mathrm{d}s, \mathrm{d}y)\}$ and $\{N(\mathrm{d}a, \mathrm{d}w)\}$ are independent. For any $a \in \mathbb{R}$, let $\{x(a,t); t \geq 0\}$ be a unique solution of the equation

$$x(t) = a + \int_0^t \int_{\mathbb{R}} h(y - x(s))W(\mathrm{d}s, \mathrm{d}y), \quad t \geq 0, \tag{2}$$

cf. Lemma 1.3 of [27, p.46] (see also Lemma 3.1 of [2, p.11]). In addition, enumeration of the atoms of $N(\mathrm{d}a, \mathrm{d}w)$ into supp(N) is given by a sequence $\{(a_i, w_i); i = 1, 2, \dots\}$ such that $\tau_0(w_{i+1}) < \tau_0(w_i)$ a.s. for all $i \geq 1$ and $\tau_0(w_i) \to 0$ as $i \to \infty$. For a fixed constant $\beta > 0$ let $\psi(a,t) = \beta^{-1}\int_0^t \sigma(x(a,s))\mathrm{d}s$ for $t \geq 0$, $a \in \mathbb{R}$, and we define $w(a,t) = w(\psi(a,t))$ for $w \in W_0$.

Theorem 1. (Dawson-Li [1, p.48]) *Let $\{X_t; t \geq 0\}$ be defined by $X_0 = \mu$ and*

$$X_t = \sum_{i=1}^\infty w_i(a_i, t)\delta_{x(a_i,t)} = \int_{\mathbb{R}}\int_{W_0} w(a,t)\delta_{x(a,t)}N(\mathrm{d}a, \mathrm{d}w), \quad t > 0. \tag{3}$$

Then $\{X_t\}$ relative to $(\mathcal{G}_t)_{t\geq 0}$ is an SDSM, where $\mathcal{G}_t$ is the σ-algebra generated by all $\mathbb{P}$-null sets and the families of random variables $\{W([0,s]\times B);\ 0 \leqslant s \leqslant t,\ B \in \mathcal{B}(\mathbb{R})\}$ and $\{w_i(a_i, s);\ 0 \leqslant s \leqslant t,\ i = 1, 2, \dots\}$ for $t \geq 0$.

2.2. Superprocess with Coalescing Spatial Motion

An n-dimensional continuous process $\{(y_1(t), \dots, y_n(t)); t \geq 0\}$ is called an n-system of coalescing Brownian motions (n-SCBM) with speed $\rho(0) > 0$ if each $\{y_i(t); t \geq 0\}$ is a Brownian motion with speed $\rho(0)$ and, for $i \neq j$, $\{|y_i(t) - y_j(t)|; t \geq 0\}$ is a Brownian motion with speed $2\rho(0)$ stopped at the origin. The generator of the superprocess with coalescing spatial motion (SCSM) is given by

$$\begin{aligned}\mathcal{L}_c F(\mu) = {} & \frac{1}{2}\int_{\mathbb{R}} \rho(0)\frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(\mathrm{d}x) + \frac{1}{2}\int_{\mathbb{R}} \sigma\frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\mu(\mathrm{d}x) \\ & + \frac{1}{2}\iint_{\varDelta} \rho(0)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(\mathrm{d}x)\mu(\mathrm{d}y),\end{aligned} \tag{4}$$

where $\rho(0)$ and σ are positive constants, and $\varDelta = \{(x, x); x \in \mathbb{R}\}$.

In what follows we consider the case of purely atomic initial state, namely, having a finite number of atoms, for instance, $\mu_0 = \sum_{i=1}^{n} \xi_i \delta_{a_i}$ just for simplicity. Let $\{(\xi_1(t), \ldots, \xi_n(t)); t \geq 0\}$ be a system of independent standard Feller branching diffusions with initial state $(\xi_1, \ldots, \xi_n) \in \mathbb{R}_+^n$. By setting $\psi_i^\sigma(t) = \int_0^t \sigma(y_i(s))\mathrm{d}s$ and $\xi_i^\sigma(t) = \xi_i(\psi_i^\sigma(t))$, we define

$$X_t = \sum_{i=1}^{n} \xi_i^\sigma(t) \delta_{y_i(t)}, \quad t \geq 0, \tag{5}$$

which gives a continuous $M_F(\mathbb{R})$-valued process. For a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathcal{H}_t$ be the σ-algebra generated by the family of $\mathbb{P}$-null sets in $\mathcal{F}$ and the family of random variables $\{(y_1(s), \ldots, y_n(s),\ \xi_1^\sigma(s), \ldots, \xi_n^\sigma(s));\ 0 \leqslant s \leqslant t\}$. The process $\{X_t; t \geq 0\}$ defined by (5) is a diffusion process relative to $(\mathcal{H}_t)_{t \geq 0}$ with state space $M_a(\mathbb{R})$, cf. Theorem 3.1 of [3, p.682]. Let $\mathcal{D}(\mathcal{L}_c)$ be the set of all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m \rangle$ with $\mu \in M_F(\mathbb{R})$. Theorem 3.3 of [3, p.684] provides with the fact that $\{X_t; t \geq 0\}$ solves the $(\mathcal{L}_c, \mathcal{D}(\mathcal{L}_c))$-martingale problem, namely, for each $F_{m,f} \in \mathcal{D}(\mathcal{L}_c)$,

$$F_{m,f}(X_t) - F_{m,f}(X_0) - \int_0^t \mathcal{L}_c F_{m,f}(X_s)\mathrm{d}s, \quad \forall t \geq 0 \tag{6}$$

is a $(\mathcal{H}_t)$-martingale.

The distribution of $\{X_t; t \geq 0\}$ can be characterized in terms of a dual process. Let us consider a non-negative integer-valued càdlàg Markov process $\{M_t; t \geq 0\}$ which is well known as Kingman's coalescent process, [18]. For $1 \leqslant k \leqslant M_0 - 1$, τ_k denotes the k-th jump time of $\{M_t; t \geq 0\}$ with $\tau_0 = 0$ and $\tau_{M_0} = \infty$. Let $\{\Gamma_k\}$ $(1 \leqslant k \leqslant M_0 - 1)$ be a sequence of random operators from $C(\mathbb{R}^m)$ to $C(\mathbb{R}^{m-1})$, satisfying

$$\mathbb{P}\left\{\Gamma_k = \Phi_{ij} | M(\tau_k -) = \ell\right\} = \frac{1}{\ell(\ell - 1)}, \qquad 1 \leqslant i \neq j \leqslant \ell. \tag{7}$$

Let C^* denote the topological union of $\{C(\mathbb{R}^m); m = 1, 2, \ldots\}$, endowed with pointwise convergence on each $C(\mathbb{R}^m)$. By making use of the transition semigroup $(P_t^{(m)})_{t \geq 0}$ of the m-SCBM, $\{Y_t; t \geq 0\}$ taking values from C^* is defined by $Y_t = P_{t-\tau_k}^{(M_{\tau_k})} \Gamma_k P_{\tau_k - \tau_{k-1}}^{(M_{\tau_{k-1}})} \Gamma_{k-1} \cdots P_{\tau_2 - \tau_1}^{(M_{\tau_1})} \Gamma_1 P_{\tau_1}^{(M_0)} Y_0$ for $\tau_k \leqslant t < \tau_{k+1}$, $0 \leqslant k \leqslant M_0 - 1$. Clearly, $\{(M_t, Y_t); t \geq 0\}$ is a Markov process. We denote by $\mathbb{E}_{m,f}^\sigma$ the expectation related to (M_t, Y_t) given $M_0 = m$ and $Y_0 = f \in C(\mathbb{R}^m)$. Let $Q_t(\mu_0, \mathrm{d}\nu)$ denote the distribution of X_t on $M_F(\mathbb{R})$ given $X_0 = \mu_0 \in M_a(\mathbb{R})$.

Theorem 2. (Dawson-Li-Zhou [3, p.685]) *If $\{X_t; t \geq 0\}$ is a continuous $M_F(\mathbb{R})$-valued process such that $\mathbb{E}[\langle 1, X_t \rangle^m]$ is locally bounded in $t \geq 0$ for each $m \geq 1$ and $\{X_t\}$ solves the $(\mathcal{L}_c, \mathcal{D}(\mathcal{L}_c))$-martingale problem with $X_0 = \mu_0$, then*

$$\int_{M_F(\mathbb{R})} \langle f, \nu^m \rangle Q_t(\mu_0, \mathrm{d}\nu) = \mathbb{E}_{m,f}^\sigma \left[\langle Y_t, \mu_0^{M_t} \rangle \exp\left\{ \frac{1}{2} \int_0^t M_s(M_s - 1)\mathrm{d}s \right\} \right] \tag{8}$$

for $t \geq 0$, $m \geq 1$ and $f \in C(\mathbb{R}^m)$. $\square$

A Markov process on $M_F(\mathbb{R})$ with transition semigroup $(Q_t)_{t \geq 0}$ given by (8) is called a superprocess with coalescing spatial motion with speed $\rho(0)$, branching rate σ and initial state $\mu_0 \in M_a(\mathbb{R})$, or shortly a $\{\rho(0), \sigma\}$-SCSM.

Remark 1. It is obvious that $\{(y_1(t),\dots,y_n(t)); t\geq 0\}$ is an n-SCBM with speed $\tilde\rho$ if and only if $\langle y_i,y_j\rangle(t)=\tilde\rho\cdot(t-t\wedge\tau_{ij})$ holds for $1\leqslant i,j\leqslant n$ where $\tau_{ij}=\inf\{t\geq 0; y_i(t)=y_j(t)\}$.

Remark 2. The last term on the right-hand side of (4) shows that interactions in the spatial motion only occur between particles located at the same positions.

2.3. Immigration Superprocess Associated with SDSM

Suppose that $m\in M_F(\mathbb{R})$ satisfies $\langle 1,m\rangle>0$ and $q(\cdot,\cdot)\equiv q$ is a constant. We define

$$\mathcal{I}F(\mu)=\mathcal{L}F(\mu)+\int_{\mathbb{R}} q\frac{\delta F(\mu)}{\delta\mu(x)}m(\mathrm{d}x),\qquad \mu\in M_F(\mathbb{R}),\tag{9}$$

where q is an immigration rate and m is a reference measure related to the immigration. We put $\mathcal{D}(\mathcal{I})=\mathcal{D}(\mathcal{L})$. The $(\mathcal{I},\mathcal{D}(\mathcal{I}))$-martingale problem has a unique solution $\{Y_t\}$. The solution process is a diffusion, and this immigration SDSM started with any initial state actually lives in the space $M_a(\mathbb{R})$. Moreover, a continuous $M_F(\mathbb{R})$-valued process $\{Y_t; t\geq 0\}$ is a solution of the $(\mathcal{I},\mathcal{D}(\mathcal{I}))$-martingale problem if and only if for each $\varphi\in C^2(\mathbb{R})$,

$$M_t(\varphi)=\langle\varphi,Y_t\rangle-\langle\varphi,Y_0\rangle-q\langle\varphi,m\rangle t-\int_0^t\left\langle\frac{\rho(0)}{2}\varphi'',Y_s\right\rangle\mathrm{d}s,\quad t\geq 0,\tag{10}$$

is a martingale with quadratic variation process

$$\langle M(\varphi)\rangle_t=\int_0^t\langle\sigma\varphi^2,Y_s\rangle\mathrm{d}s+\int_0^t\mathrm{d}s\int_{\mathbb{R}}\langle h(z-\cdot)\varphi',Y_s\rangle^2\mathrm{d}z.\tag{11}$$

Next we treat the case with a general interactive immigration rate. The results below are originally discussed in §5 of Dawson-Li [1]. Suppose that there are (i) a white noise $W(\mathrm{d}s,\mathrm{d}y)$ on $[0,\infty)\times\mathbb{R}$ based on $\mathrm{d}\ell$; (ii) a sequence of independent σ-branching diffusions $\{\xi_i(t); t\geq 0\}$ with $\xi_i(0)\geq 0$ $(i=1,2,\dots)$; (iii) a Poisson random measure $N(\mathrm{d}s,\mathrm{d}a,\mathrm{d}u,\mathrm{d}w)$ on $[0,\infty)\times\mathbb{R}\times[0,\infty)\times W_0$ with intensity $\mathrm{d}sm(\mathrm{d}a)\mathrm{d}uQ_k(\mathrm{d}w)$, on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$. In addition we assume that $\sum_{i=1}^\infty\xi_i(0)<\infty$ and that $\{W\}$, $\{\xi_i\}$ and $\{N\}$ are independent of each other. For $t\geq 0$ let $\hat{\mathcal{G}}_t$ be the σ-algebra generated by all $\mathbb{P}$-null sets and the families of random variables $\{W([0,s]\times B),\ \xi_i(s);\ 0\leqslant s\leqslant t,\ B\in\mathcal{B}(\mathbb{R}),\ i=1,2,\dots\}$ and $\{N(J\times A);\ J\in\mathcal{B}([0,s]\times\mathbb{R}\times[0,\infty)),\ A\in\mathcal{B}_{t-s}(W_0),\ 0\leqslant s\leqslant t\}$. Let $q=q(\mu,a)$ be a Borel function on $M_F(\mathbb{R})\times\mathbb{R}$ such that there are some constants $K_1>0$ and $K_2(R)>0$ (for each $R>0$ given) satisfying

$$\langle q(\mu,\cdot),m\rangle\leqslant K_1(1+\|\mu\|)\quad\text{for}\quad \mu\in M_F(\mathbb{R}),$$
$$\text{and}\quad\langle|q(\mu,\cdot)-q(\nu,\cdot)|,m\rangle\leqslant K_2(R)\|\mu-\nu\|$$

for μ and ν in $M_F(\mathbb{R})$ satisfying $\langle 1,\mu\rangle\leqslant R$ and $\langle 1,\nu\rangle\leqslant R$, where the reference measure m is a σ-finite Borel measure on $\mathbb{R}$ and $\|\cdot\|$ denotes the total variation (cf. [1, p.58]). Let us consider a stochastic equation with purely atomic initial state. That is, for any sequence $\{a_i\}\subset\mathbb{R}$, we consider the stochastic equation: for $t\geq 0$,

$$Y_t=\sum_{i=1}^\infty\xi_i(t)\delta_{x(0,a_i,t)}+\int_0^t\int_{\mathbb{R}}\int_0^{q(Y_s,a)}\int_{W_0}w(t-s)\delta_{x(s,a,t)}N(\mathrm{d}s,\mathrm{d}a,\mathrm{d}u,\mathrm{d}w).\tag{12}$$

Then it follows that the equation (12) has a unique continuous solution $\{Y_t; t\geq 0\}$, which is a diffusion process relative to $(\hat{\mathcal{G}}_t)$, cf. Theorem 5.3 of [1, p.59]. Furthermore, for each $\varphi\in C^2(\mathbb{R})$,

$$M_t(\varphi)=\langle\varphi,Y_t\rangle-\langle\varphi,Y_0\rangle-\int_0^t\left\langle\frac{\rho(0)}{2}\varphi'',Y_s\right\rangle\mathrm{d}s-\int_0^t\langle q(Y_s,\cdot)\varphi,m\rangle\mathrm{d}s,\ t\geq 0,\tag{13}$$

is a continuous martingale with respect to the filtration $(\hat{\mathcal{G}}_t)_{t\geq 0}$ and its quadratic variation process is given by

$$\langle M(\varphi)\rangle_t = \int_0^t \langle \sigma\varphi^2, Y_s\rangle \mathrm{d}s + \int_0^t \mathrm{d}s \int_{\mathbb{R}} \langle h(z-\cdot)\varphi', Y_s\rangle^2 \mathrm{d}z, \quad t \geq 0. \tag{14}$$

The solution of (12) can be regarded as immigration processes associated with the SDSM with interactive immigration. Then the generator of the diffusion process $\{Y_t; t \geq 0\}$ is given by

$$\mathcal{I}F(\nu) = \mathcal{L}F(\nu) + \int_{\mathbb{R}} q(\nu,x)\frac{\delta F(\nu)}{\delta\nu(x)} m(\mathrm{d}x), \qquad \nu \in M_F(\mathbb{R}), \tag{15}$$

where $\mathcal{L}$ is defined by (1) and $q(\cdot,\cdot)$ is the interactive immigration rate. We call this process $\{Y_t\}$ an immigration superprocess associated with SDSM or simply a $\{\rho(0),\rho,\sigma,q,m\}$-IMS.

Remark 3. The Markov property of $\{Y_t\}$ was obtained from the uniqueness of solution of (12). This application of the stochastic equation is essential since the uniqueness of solution of the martingale problem given by (13) and (14) still remains open.

3. Main Results

We begin with introducing our object IMS model. Let $Y = \{Y_t; t \geq 0\}$ be a $\{\rho(0),\rho,\sigma,q,m\}$-immigration superprocess (IMS) in the sense of §2.3 with the purely atomic initial state $Y_0 = \mu = \sum_{i=1}^{\infty} \xi_i(0)\delta_{a_i} \in M_a(\mathbb{R})$ for $\{a_i\}_i \subset \mathbb{R}$. Here ρ is a C^2-function defined in the begining of §2, σ is a positive constant, $q(\cdot,\cdot) \equiv q \in \mathbb{R}$ and m is a finite Borel measure on $\mathbb{R}$ such that $0 < \langle 1, m\rangle < \infty$. The generator of Y is given by $(\mathcal{I},\mathcal{D}(\mathcal{I}))$ of (9) in §2.3 with $\mathcal{L}$ of (1) in §2.1. We define $\mathrm{Dom}(\mathcal{I}) = \mathcal{D}(\mathcal{L})$. This Y solves the $(\mathcal{I},\mathcal{D}(\mathcal{I}))$-martingale problem, and as we have seen in §2.3 this martingale problem is well-posed. Let $Y_t^{(\theta)}$ be an immigration superprocess with parameters $\{\rho(0),\rho,\sigma_\theta,q_\theta,K_{1/\theta}m\}$ and initial state $Y_0^{(\theta)} = \theta^2 K_{1/\theta}\mu$. According to the scaling argument in §3.2 of [10], we put $Y_t^\theta := \theta^{-2}K_\theta Y_{\theta^2 t}^{(\theta)}$ with $\theta \geq 1$ for any $t > 0$, where K_θ is an operator on $M_F(\mathbb{R})$ defined by $K_\theta\mu(B) = \mu(\{\theta x;\ x \in B\})$ for any Borel set B in $\mathbb{R}$. Then the rescaled process $\{Y_t^\theta; t \geq 0\}$ has generator

$$\begin{aligned}\mathcal{I}_\theta F(\nu) = {}& \frac{1}{2}\rho(0) f'(\langle\phi,\nu\rangle)\langle\phi'',\nu\rangle \\ &+ \frac{1}{2} f''(\langle\phi,\nu\rangle)\iint_{\mathbb{R}^2} \rho_\theta(x-y)\phi'(x)\phi'(y)\nu(\mathrm{d}x)\nu(\mathrm{d}y) \\ &+ \frac{1}{2}\sigma_\theta f''(\langle\phi,\nu\rangle)\langle\phi^2,\nu\rangle + q_\theta \cdot f'(\langle\phi,\nu\rangle)\langle\phi,m\rangle,\end{aligned} \tag{16}$$

for $F(\nu) = f(\langle\phi,\nu\rangle) \in \mathrm{Dom}(\mathcal{I})$ with $f,\phi \in C^2(\mathbb{R})$, where $\rho_\theta(x) = \rho(\theta x)$, $\{\sigma_\theta\}$ is a sequence of positive numbers and $\{q_\theta\}$ is a sequence of real numbers.

Theorem 3. *The rescaled processes $\{Y_t^\theta; t \geq 0\}_\theta$ lie in the family of $\{\rho(0),\rho_\theta,\sigma_\theta,\ q_\theta, m\}$-IMS with initial state $Y_0^\theta = \mu$. Moreover, the $(\mathcal{I}_\theta, \mathrm{Dom}(\mathcal{I}_\theta))$-martingale problem for $\{Y_t^\theta\}$ has a unique solution.*

Proof. Just see Proposition 4 in §3.3 of [10]. q.e.d.

Suppose that $q(\mu,a) \equiv q(a) \in L^1(\mathbb{R},m)$. Let $D_{q(a)}$ denote the set $\{(s,a,u,w);\ s \geq 0,\ a \in \mathbb{R},\ 0 \leqslant u \leqslant q(a),\ w \in W_0\}$, and set $N_{q(a)} := N \restriction D_{q(a)}$. Then we denote by $\tilde{N}_{q(a)}$ the image of $N_{q(a)}$ under the mapping : $(s, a, u, w) \mapsto (s, a, w)$. Note that $\tilde{N}_{q(a)}$ is a Poisson measure on $[0, \infty) \times \mathbb{R} \times W_0$ with intensity $\mathrm{d}s \cdot q(a)m(\mathrm{d}a)Q_k(\mathrm{d}w)$. When we replace ρ by ρ_θ, then by definition the function h should also be replaced by the scaled function $\sqrt{\theta}h_\theta$. On this account, the interacting flow $\{x^\theta(0, a_i^\theta, t)\}$ for $\{a_i^\theta\}_\theta \subset \mathbb{R}$ for each $i \in \mathbb{N}$ is a unique solution of (2) in §2.1 with a (resp. x, h) replaced by a_i^θ (resp. x^θ, $\sqrt{\theta}h_\theta$) respectively.

Theorem 4. *For each $\theta \geq 1$ we have the following purely atomic representation:*

$$Y_t^\theta = \sum_{i=1}^{\infty} \xi_i^\theta(t)\delta_{x_i^\theta} + \int_0^t \int_{\mathbb{R}} \int_{W_0} w(t-s)\delta_{x_*^\theta} N_\theta(\mathrm{d}s, \mathrm{d}a, \mathrm{d}w), \quad t \geq 0 \tag{17}$$

where we put $\xi_i^\theta(t) = \xi_i(\sigma_\theta t)$, $x_i^\theta = x^\theta(0, a_i^\theta, t)$, $x_*^\theta = x^\theta(s, a^\theta, t)$ and $N_\theta = \tilde{N}_{q_\theta}$ for vrevity's sake.

Proof. See Proposition 5 in §3.3 of [10]. q.e.d.

We assume:

(A.1) $\rho(x) \to 0$ (as $|x| \to \infty$);
(A.2) For $\{\sigma_\theta\}_\theta \subset \mathbb{R}^+$, $\sigma_\theta \to (\exists)\sigma_0 \in \mathbb{R}^+$ (as $\theta \to \infty$);
(A.3) For $\{q_\theta\}_\theta \subset \mathbb{R}^+$, $q_\theta \to (\exists)q_0 \in \mathbb{R}^+$ (as $\theta \to \infty$);
(A.4) For the initial state,

$$\mu_\theta = \sum_{i=1}^{\infty} \xi_i(0)\delta_{a_i^\theta} \to \mu_0 = \sum_{i=1}^{\infty} \xi_i(0)\delta_{b_i} \in M_a(\mathbb{R})$$

(as $\theta \to \infty$).

Lemma 5. *For any $\eta > 0$ and each $T > 0$, we have the following estimate:*

$$\mathbb{P}\left(\sup_{0 \leqslant t \leqslant T} \langle 1, Y_t^\theta \rangle > \eta\right) \leqslant \frac{C_0\{\langle 1, m\rangle + \langle 1, \mu_\theta\rangle\}}{\eta} < \infty \tag{18}$$

where C_0 is some positive constant depending only on T and the parameter σ_θ.

Proof. It is the same as in the proof of Lemma 8 of [10]. So we shall omit the detail. By the discussion similar to Lemma 4.1 of [1], $N_q^*(\mathrm{d}s, \mathrm{d}w)$ is a Poisson random measure on $[0, \infty) \times W_0$ with intensity $\langle 1, m\rangle \mathrm{d}s Q_k(\mathrm{d}w)$, which is obtained by the image of $\tilde{N}_q(\mathrm{d}s, \mathrm{d}a, \mathrm{d}w)$ under the mapping : $(s, a, w) \to (s, w)$. Notice that N_q^* is independent of Feller branching diffusions $\{\xi_i(t); t \geq 0\}$ $i \in \mathbb{N}$. We may take advantage of the pathwise expression (17) to obtain

$$\langle 1, Y_t^\theta \rangle = \sum_{i=1}^{\infty} \xi_i^\theta(t) + \int_0^t \int_{W_0} w(t-s) N_{q_\theta}^*(\mathrm{d}s, \mathrm{d}w), \quad t \geq 0. \tag{19}$$

While, by Theorem 4.1 of Pitman-Yor [24, p.442], $\langle 1, Y_t^\theta \rangle$ is a diffusion process with generator $\frac{1}{2}\sigma_\theta x(d^2/dx^2) + \langle 1, m\rangle(d/dx)$. Hence, the standard theory of diffusion processes [17] yields to that $V_t^\theta := \langle 1, Y_t^\theta \rangle$ satisfies a stochastic differential equation (SDE) : $\mathrm{d}V_t^\theta = \sqrt{\sigma_\theta V_t^\theta}\mathrm{d}B_t + \langle 1, m\rangle \mathrm{d}t$ with $V_0^\theta = \langle 1, \mu_\theta \rangle < \infty$, where $\{B_t\}$ is a one-dimensional standard Brownian motion. The general theory of SDEs guarantees the existence of unique solution

V_t^θ. So that, by employing the Markov inequality an easy calculation together with Doob's martingale inequality leads to

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant T}|V_t^\theta|>\eta\right)\leqslant\frac{2}{\eta}\mathbb{E}\left\{\sup_t|V_0^\theta+\langle 1,m\rangle t|\right\}+\frac{4}{\eta^2}\mathbb{E}\left|\int_0^T\sqrt{\sigma_\theta V_s^\theta}\mathrm{d}B_s\right|^2$$
$$\leqslant\frac{C_0}{\eta}\{\langle 1,m\rangle+\langle 1,\mu_\theta\rangle\},\tag{20}$$

because we made use of Itô's isometry for the stochastic integral term and the Fubini theorem. This completes the proof. q.e.d.

Theorem 6. *The family* $\{Y_t^\theta;t\geq 0\}$, $\theta\geq 1$ *is tight in the space* $C_M(\mathbb{R}_+)$.

Proof. Once one gets the fundamental estimate like (18) in Lemma 5, it is a routine work to verify the tightness. For the detail, see the proof of Proposition 7 in §4 of [10]. Note that $M_F(\mathbb{R})$ is a complete separable metric space by a metric that induces the weak topology. By virtue of Theorem 9.1 of Ethier-Kurtz [14], we have only to check the compact containment condition. Let $\hat{\mathbb{R}}:=\mathbb{R}\cup\{\partial\}$ be the one-point compactification, which we are going to use so as to avoid the difficulty arising from the non-compactness of the space $\mathbb{R}$. Then it follows immediately from Lemma 5 that

$$\inf_\theta\mathbb{P}\left\{\sup_{0\leqslant t\leqslant T}\langle 1,Y_t^\theta\rangle\leqslant\eta\right\}\geq 1-\frac{C_0}{\eta}\{\langle 1,m\rangle+\langle 1,\mu_\theta\rangle\}\tag{21}$$

for $\forall\eta>0$ and each $T>0$. Consequently, the tightness of distributions of $\{Y_t^\theta;t\geq 0\}$ in $C([0,\infty),M_F(\hat{\mathbb{R}}))$ has been attributed to that of $\{F\circ Y_t^\theta\}$ in $C([0,\infty),\hat{\mathbb{R}})$ for each F in the dense subset H of $C(M_F(\mathbb{R}))$ in the topology of uniform convergence on compact sets. If we apply Itô's formula to the function $F(Y_t^\theta)=F_{f,\{\phi_i\}}(Y_t^\theta)=f(\langle\phi_1,Y_t^\theta\rangle,\ \ldots,\ \langle\phi_n,Y_t^\theta\rangle)$ by paying attention to the fact that the relation

$$\mathrm{d}\langle\phi,Y_t^\theta\rangle=\{\rho(0)\langle\phi''/2,Y_t^\theta\rangle+q_\theta\langle\phi,m\rangle\}\mathrm{d}t+\mathrm{d}M_t^\theta(\phi)\tag{22}$$

from (10) reveals the process $\langle\phi,Y_t^\theta\rangle$'s being an Itô process, then a direct computation implies that

$$F_{f,\{\phi_i\}}(Y_t^\theta)-F_{f,\{\phi_i\}}(Y_0^\theta)-\int_0^t\mathcal{I}_\theta F_{f,\{\phi_i\}}(Y_s^\theta)\mathrm{d}s\tag{23}$$

is a $(\hat{\mathcal{G}}_t)$-martingale under the probability measure $\mathbb{P}_{\mu_\theta}$ for which valid is the martingale characterization (in Proposition 5 of [10]) of $\{\rho(0),\rho_\theta,\sigma_\theta\}$-SDSM with deterministic immigration q_θ and reference measure m. Therefore, by Ethier-Kurtz' criterion (cf. Theorem 9.4 of [14]), $\{F_{f,\{\phi_i\}}(Y_t^\theta)\}_\theta$ is relatively compact for each $F_{f,\{\phi_i\}}\in\mathrm{Dom}(\mathcal{I}_\theta)$. After all, the tightness of $\{Y_t^\theta\}$ in $C([0,\infty),M_F(\hat{\mathbb{R}}))$ follows. Since the process Y_t^θ lives indeed in $M_F(\mathbb{R})$, the distributions $\mathbb{Q}_\theta$ of $\{Y_t^\theta\}$ is tight in $C_M(\mathbb{R}_+)$. This finishes the proof. q.e.d.

The generator $\mathcal{A}$ is given by

$$\mathcal{A}F(\mu)=\mathcal{L}_cF(\mu)+\int_{\mathbb{R}}q\frac{\delta F(\mu)}{\delta\mu(x)}m(\mathrm{d}x)\tag{24}$$

for $F\in\mathrm{Dom}(\mathcal{A})$, where $\mathcal{L}_c$ is given by (4) in §2.2, the branching rate σ is a positive constant and q is a deterministic immigration rate. A continuous $M_F(\mathbb{R})$-valued process

$X = \{X_t; t \geq 0\}$ is said to be $\{\rho(0), \sigma, q, m\}$-immigration superprocess associated with coalescing spatial motion, or shortly $\{\rho(0), \sigma, q, m\}$-IM-SCSM, if $\{X_t; t \geq 0\}$ solves the $(\mathcal{A}, \mathrm{Dom}(\mathcal{A}))$-martingale problem. Now we are in a position to state the principal result in this paper, say, the limit theorem for rescaled immigration superprocesses.

Theorem 7. *Assume* (A.1) – (A.4). *For $\{\rho(0), \rho, \sigma_\theta, q_\theta, K_{1/\theta}m\}$-immigration superprocess $Y^{(\theta)} = \{Y_t^{(\theta)}; t \geq 0\}$, put $Y_t^\theta := \theta^{-2} K_\theta Y_{\theta^2 t}^{(\theta)}$ for $\theta \geq 1$. Then we have the followings:*
(a) *There exists a proper version $\hat{Y}_t^\theta$ of Y_t^θ converges a.s. as $\theta \to \infty$ to a process X_t having the purely atomic representation*

$$\sum_{i=1}^{\infty} \xi_i(\sigma_0 t)\delta_{y_i(0,b_i,t)} + \int_0^t \int_{\mathbb{R}} \int_{W_0} w(t-s)\delta_{y(s,b,t)} \tilde{N}_{q_0}(\mathrm{d}s, \mathrm{d}b, \mathrm{d}w) \tag{25}$$

for each $t \geq 0$, where $\{y_i(0, b_i, t)\}$ is a coalescing Brownian motion started at point b_i for each $i \in \mathbb{N}$, and $y(s, b, t)$ denotes Harris' stochastic flow [16] *of coalescing Brownian motion with $y(s, b, s) = b$.*
(b) *The conditional distribution of $\{\rho(0), \rho_\theta, \sigma_\theta, q_\theta, m\}$-immigration superprocess $Y^\theta = \{Y_t^\theta; t \geq 0\}$ given $Y_0^\theta = \mu_\theta$ converges as $\theta \to \infty$ to that of $\{\rho(0), \sigma_0, q_0, m\}$-immigration superprocess associated with coalescing spatial motion $X = \{X_t; t \geq 0\}$ with initial state μ_0 in* (A.4).
(c) *The generator of the limiting process $X = \{X_t\}$ is given by*

$$\begin{aligned}\mathcal{I}_\infty F(\nu) = {} & \frac{1}{2}\int_{\mathbb{R}} \rho(0) \frac{d^2}{dx^2} \frac{\delta F(\nu)}{\delta \nu(x)} \nu(\mathrm{d}x) \\ & + \frac{1}{2} \iint_{\Delta} \rho(0) \frac{d^2}{dxdy} \frac{\delta^2 F(\nu)}{\delta\nu(x)\delta\nu(y)} \nu(\mathrm{d}x)\nu(\mathrm{d}y) \\ & + \frac{1}{2}\int_{\mathbb{R}} \sigma_0 \frac{\delta^2 F(\nu)}{\delta\nu(x)^2}\nu(\mathrm{d}x) + \int_{\mathbb{R}} q_0 \frac{\delta F(\nu)}{\delta\nu(x)} m(\mathrm{d}x).\end{aligned} \tag{26}$$

The proof of this main theorem will be given in the succeeding section.

4. Proof of the Limit Theorem

Since the family of rescaled processes $\{Y_t^\theta; t \geq 0\}_\theta$ is tight in $C_M(\mathbb{R}_+)$ from Theorem 6, we can extract a convergent subsequence of distributions of $\{Y_t^\theta\}$. Choose any sequence $\{\theta_k\}_k \subset \{\theta \geq 1\}$ such that the distributions $\{\mathbb{Q} \circ (Y_\cdot^{\theta_k})^{-1}\}_k$ converge as $k \to \infty$ to some probability measure $\mathbb{Q}_{\mu_0} \in \mathcal{P}(C_M(\mathbb{R}_+))$ on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By virtue of Skorokhod's representation theorem (cf. Theorem 1.4 of [13, p.274]), we can construct $\{Y_t^k; t \geq 0\}$ and $\{Y_t^0; t \geq 0\}$ on a new proper probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ in such a way that (i) (identical distribution) $\mathcal{L}(\{Y_\cdot^{\theta_k}\}) = \mathcal{L}(\{Y_\cdot^k\})$; (ii) the limiting process $\{Y_\cdot^0\}$ has the distribution $\mathbb{Q}_{\mu_0}$; (iii) $\{Y_\cdot^k\}$ converges almost surely as $k \to \infty$ to $\{Y_\cdot^0\}$ in the space $C_M(\mathbb{R})$. Because $\{Y_t^{\theta_k}\}$ is a unique solution of the IMS martingale problem, it is obvious from (i) that $\{Y_t^k\}$ solves the $(\mathcal{I}_{\theta_k}, \mathrm{Dom}(\mathcal{I}_{\theta_k}))$-martingale problem. That is to say:

Lemma 8. *For each k,*

$$F(Y_t^k) - F(Y_0^k) - \int_0^t \mathcal{I}_{\theta_k} F(Y_s^k)\mathrm{d}s, \quad t > 0 \tag{27}$$

is a continuous martingale relative to $(\hat{\mathcal{G}}_t)_{t\geq 0}$.

Our first concern is to show that generator $\mathcal{I}_{\theta_k}$ of the form (16) in §3 converges to the generator $\mathcal{I}_\infty$ of (26). Note that for $F(\mu) = f(\langle \phi, \mu \rangle)$ with $f, \phi \in C^2(\mathbb{R})$ and $\mu \in M_F(\mathbb{R})$, the generator $\mathcal{I}_\infty$ has the form

$$\begin{aligned}\mathcal{I}_\infty F(\mu) &= \frac{1}{2}\rho(0) f'(\langle \phi, \mu \rangle)\langle \phi'', \mu \rangle \\ &+ \frac{1}{2} f''(\langle \phi, \mu \rangle) \iint_{\varDelta} \rho(0)\phi'(x)\phi'(y)\mu(\mathrm{d}x)\mu(\mathrm{d}y) \\ &+ \frac{1}{2}\sigma_0 \cdot f''(\langle \phi, \mu \rangle)\langle \phi^2, \mu \rangle + q_\theta \cdot f'(\langle \phi, \mu \rangle)\langle \phi, m \rangle.\end{aligned} \tag{28}$$

Since for $\forall \phi \in C^2(\mathbb{R})$ and each $T > 0$ we have

$$\sup_{0 \leqslant t \leqslant T} |\langle \phi, Y_t^k \rangle - \langle \phi, Y_t^0 \rangle| \to 0 \quad \text{a.s.} \quad \text{as} \quad k \to \infty,$$

it is obvious that as $k \to \infty$, $F(Y_t^k) \to F(Y_t^0)$ a.s. uniformly in t on compact sets and $F(Y_0^k) \to F(Y_0^0)$ a.s. for any $F \in \mathrm{Dom}(\mathcal{I}_\theta) = \mathrm{Dom}(\mathcal{I})$.

Lemma 9. *For any $t > 0$ we have*

$$\lim_{k \to \infty} \int_0^t \mathbb{E}\left|\sigma_{\theta_k} f''(\langle \phi, Y_s^k \rangle)\langle \phi^2, Y_s^k \rangle - \sigma_0 f''(\langle \phi, Y_s^0 \rangle)\langle \phi^2, Y_s^0 \rangle\right| \mathrm{d}s = 0. \tag{29}$$

Proof. We readily get

$$\begin{aligned}&|\sigma_{\theta_k} f''(\langle \phi, Y_t^k \rangle)\langle \phi^2, Y_t^k \rangle - \sigma_0 f''(\langle \phi, Y_t^0 \rangle)\langle \phi^2, Y_t^0 \rangle| \\ &\leqslant \sigma_{\theta_k} |f''(\phi, Y_t^k \rangle) - f''(\langle \phi, Y_t^0 \rangle)| \cdot |\langle \phi^2, Y_t^k \rangle| \\ &\qquad + \sigma_{\theta_k} |f''(\langle \phi, Y_t^0 \rangle)| \cdot |\langle \phi^2, Y_t^k \rangle - \langle \phi^2, Y_t^0 \rangle| \\ &\qquad + |\sigma_{\theta_k} - \sigma_0| \cdot |f''(\langle \phi, Y_t^0 \rangle) \cdot \langle \phi^2, Y_t^0 \rangle| \\ &\qquad =: A_1^k(t) + A_2^k(t) + A_3^k(t).\end{aligned}$$

Because of almost sure convergence of Y_t^k, we can deduce from continuity of the function that as $k \to \infty$,

$$\int_0^t \mathbb{E}\{A_1^k(s)\}\mathrm{d}s \leqslant t\sigma_{\theta_k}\mathbb{E}\left\{\|f''(\langle \phi, Y_\cdot^k \rangle) - f''(\langle \phi, Y_\cdot^0 \rangle)\|_\infty \|\langle \phi^2, Y_\cdot^k \rangle\|_\infty\right\} \to 0,$$

by employing the Fubini theorem and the Lebesgue convergence theorem. As for the second and third terms $A_i^k(t)$ $(i = 2, 3)$, it goes almost similarly by the same reasons. q.e.d.

Lemma 10. *For any $t > 0$ we have*

$$\lim_{k \to \infty} \int_0^t \mathbb{E}\left|\{q_{\theta_k} f'(\langle \phi, Y_s^k \rangle) - q_0 f'(\langle \phi, Y_s^0 \rangle)\}\langle \phi, m \rangle\right| = 0. \tag{30}$$

Proof. Since $|\langle \phi, m \rangle| \leqslant \|\phi\| \cdot |\langle 1, m \rangle| < \infty$, it is sufficient to show the vanishing result only for the term $\Xi(k, s) := |q_{\theta_k} f'(\langle \phi, Y_s^k \rangle) - q_0 f'(\langle \phi, Y_s^0 \rangle)|$. In fact,

$$\begin{aligned}\Xi(k, s) &\leqslant q_{\theta_k} |f'(\langle \phi, Y_s^k \rangle) - f'(\langle \phi, Y_s^0 \rangle)| + |q_{\theta_k} - q_0| \cdot |f'(\langle \phi, Y_s^0 \rangle)| \\ &=: B_1^k(s) + B_2^k.\end{aligned}$$

We may apply the Fubini and Lebesgue theorems to obtain $\lim_k \int_0^t \mathbb{E}\{B_1^k(s)\}\mathrm{d}s = 0$. On the other hand, the convergence $\int_0^t \mathbb{E}\{B_2^k\}\mathrm{d}s \to 0$ yields simply from (A.3). q.e.d.

Then we can show the following key proposition. The proof will be given in Section 5.

Proposition 11. *For $t > 0$ we have*

(31)
$$\lim_{k\to\infty} \int_0^t \mathbb{E}\bigg| f''(\langle \phi, Y_s^k\rangle) \iint_{\mathbb{R}^2} \rho_{\theta_k}(x-y)\phi'(x)\phi'(y)Y_s^k(\mathrm{d}x)Y_s^k(\mathrm{d}y) - \frac{\rho(0)}{2} f''(\langle \phi, Y_s^0\rangle) \iint_{\Delta} \phi'(x)\phi'(y)Y_s^0(\mathrm{d}x)Y_s^0(\mathrm{d}y)\bigg| \mathrm{d}s = 0.$$

Proposition 12. *For any $t > 0$ we have*

$$\lim_{k\to\infty} \int_0^t \mathbb{E}\left|\mathcal{I}_{\theta_k}F(Y_s^k) - \mathcal{I}_\infty F(Y_s^0)\right| \mathrm{d}s = 0. \tag{32}$$

Proof. The assertion follows directly from Lemmas 9 and 10 and Proposition 11. q.e.d.

Theorem 13. *For $F \in \mathrm{Dom}(\mathcal{I}_\infty) = \mathrm{Dom}(\mathcal{I})$,*

$$F(Y_t^0) - F(Y_0^0) - \int_0^t \mathcal{I}_\infty F(Y_s^0)\mathrm{d}s, \quad t \geq 0 \tag{33}$$

is a martingale.

Proof. By approximation procedure it suffices to verify (33) only for the function $F(\mu) = f(\langle\phi,\mu\rangle)$ with $f, \phi \in C^2(\mathbb{R})$. Suppose that a collection of functions $\{\Phi_i\}_{i=1}^n$ forms a subset of $C(M_F(\mathbb{R}))$. Let $\tilde{\Delta} = \{t_k\}$ be a partition of times such that $0 \leqslant t_k < t_{k+1}$ for any k up to $n+1$. By using the Fubini theorem and the Lebesgue theorem it is easy to see from the remark stated between Lemma 8 and Lemma 9, together with Proposition 12, that

$$\begin{aligned}
&\mathbb{E}\left\{\left(F(Y_{t_{n+1}}^0) - F(Y_{t_n}^0) - \int_{t_n}^{t_{n+1}} \mathcal{I}_\infty F(Y_s^0)\mathrm{d}s\right)\cdot \prod_{i=1}^n \Phi_i(Y_{t_i}^0)\right\} \qquad (34)\\
&= \mathbb{E}\{F(Y_{t_{n+1}}^0)\Phi_n^*(Y^0)\} - \mathbb{E}\{F(Y_{t_n}^0)\Phi_n^*(Y^0)\} - \int_{t_n}^{t_{n+1}} \mathbb{E}\{\mathcal{I}_\infty F(Y_s^0)\Phi_n^*(Y^0)\}\mathrm{d}s\\
&= \lim_{k\to\infty} \mathbb{E}\{F(Y_{t_{n+1}}^k)\Phi_n^*(Y^k)\} - \lim_{k\to\infty} \mathbb{E}\{F(Y_{t_n}^k)\Phi_n^*(Y^k)\} - \lim_{k\to\infty}\int_{t_n}^{t_{n+1}} \mathbb{E}\{\mathcal{I}_{\theta_k}F(Y_s^k)\cdot\Phi_n^*(Y^k)\}\mathrm{d}s\\
&= \lim_{k\to\infty} \mathbb{E}\left\{\left(F(Y_{t_{n+1}}^k) - F(Y_{t_n}^k) - \int_{t_n}^{t_{n+1}} \mathcal{I}_{\theta_k} F(Y_s^k)\mathrm{d}s\right)\cdot \prod_{i=1}^n \Phi_i(Y_{t_i}^k)\right\} = 0,
\end{aligned}$$

where we put $\Phi_n^*(Y^\ell) = \prod_{i=1}^n \Phi_i(Y_{t_i}^\ell)$, ($\ell = k$ or 0) for simplicity. Here the last equality of (34) yields from Lemma 8 because $\{Y_t^k\}$ solves the ($\mathcal{I}_{\theta_k}$, $\mathrm{Dom}(\mathcal{I}_{\theta_k})$)-martingale problem.

Moreover, repeating the same argument we observe that (34) remains valid even for any collection $\{\Phi_i\}$, any $n \in \mathbb{N}$ and any partition $\tilde{\Delta}$. This obviously implies (33). q.e.d.

Theorem 14. *The* $(\mathcal{I}_\infty, \mathrm{Dom}(\mathcal{I}_\infty))$*-martingale problem has a unique solution for the case of purely atomic initial state. Note that* $\mathrm{Dom}(\mathcal{I}_\infty) = \mathrm{Dom}(\mathcal{I}_\theta) = \mathrm{Dom}(\mathcal{I}) = \mathcal{D}(\mathcal{L})$.

Proof. The proof of the uniqueness goes in part similarly as in the proof of Theorem 4.4 of Dawson-Li [1]. For the details, see the companion paper [11]. Hence we omit it here. q.e.d.

So that, the following is an immediate result from Theorem 14.

Theorem 15. *The process* $X = \{X_t; t \geq 0\}$ *given by* (25) *is a solution of the* $(\mathcal{I}_\infty, \mathrm{Dom}(\mathcal{I}_\infty))$*-martingale problem.*

Furthermore we can show:

Lemma 16. *Under the assumptions* (A.1) – (A.4) *the processes* $\{Y_t^{\theta_k}; t \geq 0\}_k$ *of* (17) *in Theorem 4 converge almost surely as* $k \to \infty$ *to the process* $\{Y_t^0; t \geq 0\}$ *of the form* (25).

Proof. Essentially it is due to the convergence in distribution of interacting Brownian motions towards the coalescing Brownian motion. q.e.d.

By Theorem 13, clearly the process $\{Y_t^0; t \geq 0\}$ becomes a solution of the $(\mathcal{I}_\infty$, $\mathrm{Dom}(\mathcal{I}_\infty))$-martingale problem. In addition, Theorems 14 and 15 insist that under the purely atomic initial state $\mu_0 \in M_a(\mathbb{R})$, the distribution of the solution process of $(\mathcal{I}_\infty$, $\mathrm{Dom}(\mathcal{I}_\infty))$-martingale problem is unique, and the limiting process $\{Y_t^0\}$ possesses a purely atomic representation of the form (25), which is nothing but a $\{\rho(0), \sigma_0, q_0, m\}$-superprocess associated with coalescing spatial motion. Since we have $\mathbb{P} \circ \{Y_t^{\theta_k}\}^{-1}$ (with initial state μ_θ) $\Rightarrow \mathbb{Q}_{\mu_0}^k$ (as $k \to \infty$), in fact it turns out to be that $\mathbb{Q}_{\mu_0}^k = \mathbb{Q}_{\mu_0}$ for any subsequence $\{k\}$. Therefore, by virtue of Theorem 13, the $(\mathcal{I}_\theta$, $\mathrm{Dom}(\mathcal{I}_\theta))$-martingale problem induces the $(\mathcal{I}_\infty$, $\mathrm{Dom}(\mathcal{I}_\infty))$-martingale problem. This finishes the proof of (b) of Theorem 7. q.e.d.

5. Proof of Key Proposition

This section is devoted to the proof of Proposition 11. First of all, note that the assertion of Proposition 11 is equivalent to :

Proposition 17. *For* $t > 0$ *we have*

$$\begin{aligned}(35)\qquad \lim_{k\to\infty} \mathbb{E}\bigg|\int_0^t \mathrm{d}s\, f''(\langle\phi, Y_s^k\rangle) \iint_{\mathbb{R}^2} \rho_{\theta_k}(x-y)\phi'(x)\phi'(y) Y_s^k(\mathrm{d}x) Y_s^k(\mathrm{d}y) \\ - \int_0^t \mathrm{d}s f''(\langle\phi, Y_s^0\rangle) \iint_{\Delta} \rho(0)\phi'(x)\phi'(y) Y_s^0(\mathrm{d}x) Y_s^0(\mathrm{d}y)\bigg| = 0.\end{aligned}$$

For simplicity we put $\rho_k = \rho_{\theta_k}$, $D_2 = \mathbb{R}^2$, $f''[Y_s^*] = f''(\langle\phi, Y_s^*\rangle)$, $\mathrm{d}Y_x^* = Y_s^*(\mathrm{d}x)$ with $* = k$ or 0. Because we readily get

$$\begin{aligned}\bigg|\int_0^t \mathrm{d}s\, f''[Y_s^k] \iint_{D_2} \rho_k(x-y)\phi'(x)\phi'(y)\mathrm{d}Y_x^k \mathrm{d}Y_y^k \\ - \int_0^t \delta s\; f''[Y_s^0] \iint_{D_2} \rho_k(x-y)\phi'(x)\phi'(y)\mathrm{d}Y_x^k \mathrm{d}Y_y^k\bigg|\end{aligned}$$

$$\leqslant \|f''[Y_\cdot^k] - f''[Y_\cdot^0]\|_\infty \cdot \int_0^t \left| \iint_{D_2} \rho_k(x-y)\phi'(x)\phi'(y) \mathrm{d}Y_x^k \mathrm{d}Y_y^k \right| \mathrm{d}s$$
$$\to 0 \quad (\text{as} \quad k \to \infty),$$

and also because

$$\lim_{k\to\infty} \mathbb{E} \left| \int_0^t \mathrm{d}s \iint_{\mathbb{R}^2\setminus\Delta} \rho_{\theta_k}(x-y)\phi'(x)\phi'(y) Y_s^k(\mathrm{d}x) Y_s^k(\mathrm{d}y) \right| = 0 \tag{36}$$

for any $t > 0$, by a simple reduction, in order to verify the assertion (35) it suffices to show the following:

Lemma 18. *For $t > 0$*

$$\begin{aligned} \lim_{k\to\infty} \mathbb{E} \Bigg| \int_0^t \mathrm{d} \iint_{\Delta} \rho(0)\phi'(x)\phi'(y) Y_s^k(\mathrm{d}x) Y_s^k(\mathrm{d}y) \\ - \int_0^t \mathrm{d}s \iint_{\Delta} \rho(0)\phi'(x)\phi'(y) Y_s^0(\mathrm{d}x) Y_s^0(\mathrm{d}y) \Bigg| = 0. \end{aligned} \tag{37}$$

Recall here useful purely atomic representations: namely,

$$\begin{aligned} Y_t^k &= \sum_{i=1}^\infty \xi_i^k(t)\delta_{x_i^k} + \int_0^t \int_{\mathbb{R}} \int_{W_0} w(t-s)\delta_{x_*^k} N_k(\mathrm{d}s, \mathrm{d}a, \mathrm{d}w) \\ &=: Y_t^{k,1} + Y_t^{k,2} (\overset{\text{or}}{=} Y_t^1 + Y_t^2), \end{aligned} \tag{38}$$

and as almost sure limit of Y_t^k (see Lemma 16)

$$\begin{aligned} Y_t^0 &= \sum_{i=1}^\infty \xi_i(\sigma_0 t)\delta_{y_i(0,b_i,t)} + \int_0^t \int_{\mathbb{R}} \int_{W_0} w(t-s)\delta_{y(s,b,t)} \tilde{N}_{q_0}(\mathrm{d}s, \mathrm{d}b, \mathrm{d}w), \\ &=: Z_t^1 + Z_t^2, \end{aligned} \tag{39}$$

where we put $\xi_i^k(t) = \xi_i(\sigma_{\theta_k} t)$, $x_i^k = x^{\theta_k}(0, a_i^{\theta_k}, t)$, $x_*^k = x^{\theta_k}(s, a^{\theta_k}, t)$, $N_k := \tilde{N}_{q_{\theta_k}}$, $\delta_{y_i} = \delta_{y_i(0,b_i,t)}$, $\delta_y = \delta_{y(s,b,t)}$ and $N_0 = \tilde{N}_{q_0}$. In addition, we put $\eta(x,y) = \eta_{\rho,\phi}(x,y) = \rho(0)\phi'(x)\phi'(y)$ for simplicity. Then we have

$$\iint_\Delta \eta(x,y) \mathrm{d}Y_x^k \mathrm{d}Y_y^k = \iint_\Delta \eta(x,y)(Y_s^1 + Y_s^2)(\mathrm{d}x) \cdot (Y_s^1 + Y_s^2)(\mathrm{d}y)$$

$$\begin{aligned} = \iint_\Delta \eta(x,y) Y_s^1(\mathrm{d}x) Y_s^1(\mathrm{d}y) + \iint_\Delta \eta(x,y) Y_s^2(\mathrm{d}x) Y_s^1(\mathrm{d}y) \\ + \iint_\Delta \eta(x,y) Y_s^1(\mathrm{d}x) Y_s^2(\mathrm{d}y) + \iint_\Delta \eta(x,y) Y_s^2(\mathrm{d}x) Y_s^2(\mathrm{d}y), \end{aligned}$$

and the second integral term in (37) has also similar decomposition as described above. The integral term $\iint_\Delta \eta \mathrm{d}Y_x^1 \mathrm{d}Y_y^1$ is exactly the same case as discussed in [10], and the assertion

$$\lim_k \mathbb{E} \left| \int_0^t \mathrm{d}s \iint_\Delta \eta \mathrm{d}Y_x^1 \mathrm{d}Y_y^1 - \int_0^t \mathrm{d}s \iint_\Delta \eta \mathrm{d}Z_x^1 \mathrm{d}Z_y^1 \right| = 0 \tag{40}$$

holds for $t > 0$ (see Eq.(105) of Lemma 18 in §6.2 of [10]). Moreover, essentially, the cases $Y_x^2 Y_y^1$ and $Y_x^1 Y_y^2$ are the symmetrically same. So only two cases I_{12} : $Y_x^1 Y_y^2$ and I_{22} : $Y_x^2 Y_y^2$ should be discussed. In consequence, what really we have to show is the following two lemmas.

Lemma 19. (I_{12}-type estimate) *For $t > 0$ we have*

$$\lim_{k\to\infty} \mathbb{E}\left|\int_0^t \mathrm{d}s \iint_\Delta \eta(x,y)\mathrm{d}Y_x^1 \mathrm{d}Y_y^2 - \int_0^t \mathrm{d}s \iint_\Delta \eta(x,y)\mathrm{d}Z_x^1 \mathrm{d}Z_y^2\right| = 0. \tag{41}$$

Lemma 20. (I_{22}-type estimate) *For $t > 0$ we have*

$$\lim_{k\to\infty} \mathbb{E}\left|\int_0^t \mathrm{d}s \iint_\Delta \eta(x,y)\mathrm{d}Y_x^2 \mathrm{d}Y_y^2 - \int_0^t \mathrm{d}s \iint_\Delta \eta(x,y)\mathrm{d}Z_x^2 \mathrm{d}Z_y^2\right| = 0. \tag{42}$$

Proof of Lemma 19. Basically the followings are essential key points when considering the convergence of I_{12}-type : [*1] "the law of m-sytem of interacting Brownian motions starting at point $(a_1^k, \ldots, a_m^k)$ with $a_i^k = a_i^{\theta_k}$ converges as $k \to \infty$ to the law of m-SCBM started at $(b_1, \ldots, b_m)$ under the condition (A.1) if $a_i^k \to b_i$ for each i", e.g. see Theorem 2.4 of [3]. Especially it is easy to see that [*2] "$\delta_{x_i^k} \Rightarrow \delta_{y_i}$ (as $k \to \infty$) for each i"; and also [*3] "$N_{\theta_k} \Rightarrow \tilde{N}_{q_0}$ (as $k \to \infty$)". Hence, for a proper random measurable integrand H, we have [*4] "$\int \iint H \mathrm{d}N_{\theta_k} \to \int \iint H \mathrm{d}\tilde{N}_{q_0}$ (as $k \to \infty$) over the integral region $D_* = [0,t] \times \mathbb{R} \times W_0$".

Lemma 21. *For any $t > 0$ we have*

$$\int \iint_{D_*} H\delta_{x_*^k} \mathrm{d}N_{\theta_k} \to \int \iint_{D_*} H\delta_y \mathrm{d}\tilde{N}_{q_0}, \quad \text{a.s.} \quad \text{as} \quad k \to \infty. \tag{43}$$

Proof of Lemma 21. Here we use some abbreviated notation: e.g. $\int H \cdot N$ instead of $\int \iint_{D_*} H \mathrm{d}N$. Then we readily obtain

$$\begin{aligned}
&|\int (H\delta_{x_*^k}) \cdot N_{\theta_k} - \int (H\delta_y) \cdot \tilde{N}_{q_0}| \\
&\leqslant |\int (H\delta_{x_*^k}) \cdot N_{\theta_k} - \int (H\delta_{x_*^k}) \cdot \tilde{N}_{q_0}| + |\int (H\delta_{x_*^k}) \cdot \tilde{N}_{q_0} - \int (H\delta_y) \cdot \tilde{N}_{q_0}| \\
&\leqslant |\int (H\delta_{x_*^k}) \cdot (N_{\theta_k} - \tilde{N}_{q_0})| + |\int \{H(\delta_{x_*^k} - \delta_y)\} \cdot \tilde{N}_{q_0}| \\
&=: J_1 + J_2.
\end{aligned}$$

Essentially, $J_1 \to 0$ yields from [*3] together with the discussion similar to the proof of Lemma 17 in §6.2 of [10], and $J_2 \to 0$ follows from [*2] and the Lebesgue type theorem. q.e.d.

To go back to the proof of Lemma 19, we have immediately

$$\left|\int_0^t \mathrm{d}s \iint_\Delta \eta(x,y)\mathrm{d}Y_x^1 \mathrm{d}Y_y^2 - \int_0^t \mathrm{d}s \iint_\Delta \eta(x,y)\mathrm{d}Z_x^1 \mathrm{d}Z_y^2\right|$$

$$\begin{aligned}&\leqslant \left|\int_0^t \mathrm{d}s \iint_\Delta \eta \mathrm{d}Y_x^1 \mathrm{d}Y_y^2 - \int_0^t \mathrm{d}s \iint_\Delta \eta \mathrm{d}Y_x^1 \mathrm{d}Z_y^2\right| \\ &\qquad + \left|\int_0^t \mathrm{d}s \iint_\Delta \eta \mathrm{d}Y_x^1 \mathrm{d}Z_y^2 - \int_0^t \mathrm{d}s \iint_\Delta \eta \mathrm{d}Z_x^1 \mathrm{d}Z_y^2\right| \\ &\qquad =: J_1 + J_2.\end{aligned}$$

As to J_1, because we have

$$J_1 \leqslant \int_0^t \mathrm{d}s \iint_\Delta \mathrm{d}Y_x^1 \left|\int (w\delta_{x_*^k}) \cdot N_{\theta_k} - \int (w\delta_y) \cdot \tilde{N}_{q_0}\right|,$$

the assertion $\lim_k \mathbb{E}\{J_1\} = 0$ follows from Lemma 21 and the same discussion as in Lemma 17 of [10]. While, clearly the Fubini theorem and [*2] together with (A.1), (A.2) and (A.4) yields to $\lim_k \mathbb{E}\{J_2\} = 0$ because

$$\begin{aligned}\mathbb{E}\{J_2\} &= \mathbb{E}\left|\int_0^t \mathrm{d}s \iint_\Delta \eta \left\{\int (w\delta_y) \cdot \tilde{N}_{q_0}\right\} \mathrm{d}(Y_x^1 - Z_x^1)\right| \\ &\leqslant t \cdot \|\eta\| \sup_{s\leqslant t} \mathbb{E}\left|\int w \cdot \tilde{N}_{q_0}\right| \cdot \sup_{s\leqslant t} \mathbb{E}|||(Y^1 - Z^1) \otimes \delta_y(\mathrm{d}x, \mathrm{d}y)|||_\Delta\end{aligned}$$

and the almost sure convergnce $Y_x^1 \to Z_x^1$ with the total variation $|||\cdot|||_\Delta$. This completes the proof of Lemma 19. q.e.d.

Proof of Lemma 20. We readily get

$$\left|\int_0^t \mathrm{d}s \iint_\Delta \eta \mathrm{d}Y_x^2 \mathrm{d}Y_y^2 - \int_0^t \mathrm{d}s \iint_\Delta \eta \mathrm{d}Z_x^2 \mathrm{d}Z_y^2\right|$$

$$\begin{aligned}&\leqslant \left|\int_0^t \mathrm{d}s \iint_\Delta \eta \mathrm{d}Y_x^2 \mathrm{d}Y_y^2 - \int_0^t \mathrm{d}s \iint_\Delta \eta \mathrm{d}Z_x^2 \mathrm{d}Y_y^2\right| \\ &\qquad + \left|\int_0^t \mathrm{d}s \iint_\Delta \eta \mathrm{d}Z_x^2 \mathrm{d}Y_y^2 - \int_0^t \iint_\Delta \eta \mathrm{d}Z_x^2 \mathrm{d}Z_y^2\right| \\ &\qquad =: K_1 + K_2.\end{aligned}$$

As to K_1, since we have

$$K_1 \leqslant \left|\int_0^t \mathrm{d}s \iint_\Delta \eta \mathrm{d}Y_y^2 \left\{\int (w\delta_{x_*^k}) \cdot N_{\theta_k} - \int (w\delta_y) \cdot \tilde{N}_{q_0}\right\}\right|,$$

the assertion $\lim_k \mathbb{E}\{K_1\} = 0$ follows immediately from the Lebesgue type theorem and Lemma 21 together with the discussion similar to Lemma 17 of [10]. While, because

$$K_2 \leqslant \left|\int_0^t \mathrm{d}s \iint_\Delta \eta \mathrm{d}Z_x^2 \left\{\int (w\delta_{x_*^k}) \cdot N_{\theta_k} - \int (w\delta_y) \cdot \tilde{N}_{q_0}\right\}\right|,$$

a simple application of Lemma 21 will take care of the convergence $\lim_k \mathbb{E}\{K_2\} = 0$. q.e.d.

Summing up, the key assertion Proposition 11 has been proved.

Acknowledgement

This article is partially based upon the announced results at Spring Meeting on Probability Theory, held in Tokyo Institute of Technology (TIT) during March 16–17, 2005. The author is grateful to the organizer Professor T. Shiga (TIT) for giving him an opportunity to talk about his early results [9] on some convergence theorems for rescaled immigration superprocesses associated with SDSM. The author also expresses his sincere gratitude to Professor Z.-H. Li (Beijing Normal University) for stimulating discussion on superprocesses and for his helpful and valuable comments, especially through Winter Meeting on Probability Theory, held at TIT in January, 2006. In particular, the author would like to thank the anonymous referee for his/or/her useful suggestions and helpful corrections. This work is supported in part by Japanese MEXT Grant-in-Aid for Scientific Research SR(C)(2) 14540101.

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Department of Mathematics, Saitama University, Saitama 338-8570 Japan
E-mail Address: idoku@math.edu.saitama-u.ac.jp