DEGENERATE BOUNDARY-VALUE AND INITIAL BOUNDARY-VALUE PROBLEMS INCLUDING GENERAL WENTZELL BOUNDARY CONDITIONS

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Abstract. We study degenerate boundary-value problems for higher order ordinary differential equations with polynomial spectral parameter in both the equation and boundary conditions. An isomorphism and a coercive solvability of such problems have been established. We also treat initial boundary-value problems for higher order degenerate parabolic equations. Both studies include, in particular, second order equations with the general Wentzell boundary conditions. Moreover, the equation may contain a linear abstract operator and boundary conditions may contain linear functionals and values of an unknown functions and its derivatives in some inside points of the interval of the problem.

1. Introduction

In this paper we study an isomorphism and a coerciveness of degenerate boundary-value problems with the spectral parameter in both the equation and boundary conditions. Moreover, the order of the spectral parameter in boundary conditions can be greater or equal to the order of the spectral parameter in the equation. This allows us to consider, in particular, degenerate differential boundary-value problems with general Wentzell boundary conditions (GWBC). Then we also consider initial boundary-value problems for degenerate parabolic equations with boundary conditions containing differentiation on the time at the same order as the equation. This fact allows us to cover initial boundary-value problems for degenerate parabolic equations with GWBC. The non-degenerate case (both the spectral problem and the parabolic problem) has been studied by S. Yakubov and Ya. Yakubov [YY, Theorem 1, p. 111 and Theorem 1, p. 487].

The heat equation $\frac{\partial u}{\partial t} = Au$ for $t \geq 0$ in a bounded domain Ω in $\mathbb{R}^n$, where $Au = \nabla \cdot (a\nabla u)$, $a \in C^1(\overline{\Omega})$, $a(x) > 0$ in Ω, with GWBC $Au + \beta\frac{\partial u}{\partial n} + \gamma u = 0$ on $\partial\Omega$, where $\beta(x) > 0$, $\gamma(x) \geq 0$ and $\Gamma = \{x \in \partial\Omega \mid a(x) > 0\} \neq \emptyset$, has been studied in the framework of $L_p(\overline{\Omega}, d\mu)$, $1 < p < \infty$, in the paper by A. Favini, G. R. Goldstein, J. A. Goldstein, and S. Romanelli [FGGR2] (see also [FGGR1]) by means of a different at all approach. Here $L_p(\overline{\Omega}, d\mu)$ is a Banach space to be identified with $L_p(\Omega, dx) \times L_p(\Gamma, \frac{adS}{\beta})$. Clearly, the

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spectral estimates for the resolvent involve a complex parameter in the boundary equation and are closely related with what follows. Since we shall confine to one dimension space variable, we shall be able to handle more general boundary conditions in some sense.

GWBC and analytic semigroups on $W_p^1(0,1)$ and on $C[0,1]$ for the non-degenerate operator $\Delta u := u''$ have been considered by A. Favini, G. R. Goldstein, J. A. Goldstein, E. Obrecht, and S. Romanelli [FGGOR]. Here the authors solve the resolvent equation and give directly the best possible bound.

In the present paper we basically adapt the techniques and methods from [YY] to obtain our estimates. We remark that very recently null controllability of degenerate heat equations, similar to some of our, which are related to a Crocco-type equation describing the velocity field of a laminar flow in a flat plate, has been considered by P. Cannarsa, P. Martinez, and J. Vancostenoble [CMV].

To begin with, we introduce some function spaces and recall some lemmas and definitions from [YY].

Let $1 < p, q < \infty,\ \ell = 1, 2, \dots,\ \beta \in \mathbb{R},\ \Omega := [0,1] \times [0,1]$. Consider the following spaces:

1) $L_{q,\beta}(0,1) := \Big\{ u \ \Big|\ y^\beta u(y) \in L_q(0,1)$ with the norm
$$\|u\|_{L_{q,\beta}(0,1)} := \Big(\int_0^1 y^{q\beta} |u(y)|^q dy \Big)^{\frac{1}{q}} \Big\};$$

2) $W^\ell_{q,\beta}(0,1) := \Big\{ u \ \Big|\ u \in L_{q,\beta}(0,1),\ u^{(\ell)} \in L_{q,\beta}(0,1)$ with the norm
$$\|u\|_{W^\ell_{q,\beta}(0,1)} := \|u\|_{L_{q,\beta}(0,1)} + \|u^{(\ell)}\|_{L_{q,\beta}(0,1)} \Big\};$$

3) $W^{[\ell]}_{q,\beta}(0,1) := \Big\{ u \ \Big|\ u \in L_q(0,1),\ (y^\beta D_y)^\ell u \in L_q(0,1)$ with the norm
$$\|u\|_{W^{[\ell]}_{q,\beta}(0,1)} := \|u\|_{L_q(0,1)} + \|(y^\beta D_y)^\ell u\|_{L_q(0,1)} \Big\};$$

4) $C^{[\ell]}_\beta[0,1] := \Big\{ u \ \Big|\ u \in C[0,1],\ (y^\beta D_y)^j u \in C[0,1], j = 1, ..., \ell,$ with the norm
$$\|u\|_{C^{[\ell]}_\beta[0,1]} := \|u\|_{C[0,1]} + \sum_{j=1}^{\ell} \|(y^\beta D_y)^j u\|_{C[0,1]} \Big\};$$

Note that if $\beta = 0$ then we get usual spaces, namely, $L_{q,0}(0,1) = L_q(0,1)$, $W^\ell_{q,0}(0,1) = W^{[\ell]}_{q,0}(0,1) = W^\ell_q(0,1)$, $C^{[\ell]}_0 = C^\ell[0,1]$.

Lemma 1. [YY, p.46] *Let* $\ell \in \mathbb{N},\ \beta < 1,\ q \in (1,\infty)$.

Then, the operator $u(x) \to \tilde{u}(t) := u(t^{\frac{1}{1-\beta}})$ *is an isomorphism from the spaces* $C^{[\ell]}_\beta[0,1]$, $L_q(0,1)$ *and* $W^{[\ell]}_{q,\beta}(0,1)$ *onto the spaces* $C^\ell[0,1]$, $L_{q,\beta_1}(0,1)$ *and* $W^\ell_{q,\beta_1}(0,1)$, *respectively, where* $\beta_1 = \frac{\beta}{q(1-\beta)}$.

We consider the equation
$$L(\lambda)u := \lambda^n u(x) + \sum_{k=1}^{n} \lambda^{n-k} \Big(a_k(x) \big(x^\beta \frac{d}{dx}\big)^{(dk)} u(x) + B_k u|_x \Big) = f(x), \quad x \in (0,1) \quad (1.1)$$

with boundary-functional conditions

$$
\begin{aligned}
L_\nu(\lambda)u := \sum_{k=0}^{n_\nu} \lambda^k \Big(&\alpha_{\nu k} \big(x^\beta \frac{d}{dx}\big)^{(m_\nu - dk)} u(x)|_{x=0} + \beta_{\nu k} \big(x^\beta \frac{d}{dx}\big)^{(m_\nu - dk)} u(x)|_{x=1} \\
&+ \sum_{i=1}^{N_{\nu k}} \delta_{\nu k i} \big(x^\beta \frac{d}{dx}\big)^{(m_\nu - dk)} u(x)|_{x=x_{\nu k i}} + T_{\nu k} u \Big) = f_\nu, \quad \nu = 1, \ldots, m,
\end{aligned} \tag{1.2}
$$

where n, m, n_ν, m_ν are integer numbers, $n \geq 1$, $m \geq 1$, $m_\nu \geq dn_\nu$, $d := \frac{m}{n}$ is a weight of the problem; $\alpha_{\nu k}$, $\beta_{\nu k}$, $\delta_{\nu k i}$, f_ν are complex numbers; $x_{\nu k i} \in (0,1)$; $a_k(x)$ are numerical functions defined on $[0,1]$; $a_k(x) = \alpha_{\nu k} = \beta_{\nu k} = \delta_{\nu k i} = 0$ if dk is not an integer; B_k are operators in $L_q(0,1)$ and $T_{\nu k}$ are functionals in $L_q(0,1)$, $q \in (1,\infty)$; $\beta \in \mathbb{R}$. Here, both the operators B_k and the functionals $T_{\nu k}$ are unbounded. Note that by $\big(x^\beta \frac{d}{dx}\big)^{(m_\nu - dk)} u(x)|_{x=0}$ we mean $\lim\limits_{x\to 0} \big(x^\beta \frac{d}{dx}\big)^{(m_\nu - dk)} u(x)$.

The operator $L(\lambda)$ is of order m and is a polynomial on λ of power n. The operator $L_\nu(\lambda)$ is of order $\leq m_\nu$ and is a polynomial on λ of power n_ν. It is important to define the numbers m_ν correctly. This is done in the following way: first, the forces of all terms of the operator $L_\nu(\lambda)$ are defined, then the largest of them is taken for m_ν. In addition, the force of the member $\lambda^k u^{(j)}$ is equal to $\frac{mk}{n} + j$. For example, for $n = 1$, $m = 2$ and for the boundary condition $L_1(\lambda)u := u'(0) + \lambda u(1) = 0$ we have $m_1 = 2$. With such a definition, m_ν becomes the force of the operator $L_\nu(\lambda)$, but not its order.

A system of functions $\omega_1(x), \ldots, \omega_m(x)$ is called p**-separated** if there exists a straight line P passing through 0 such that no value of the functions $\omega_j(x)$ lies on it, and $\omega_1(x), \ldots, \omega_p(x)$ are on one side of P while $\omega_{p+1}(x), \ldots, \omega_m(x)$ are on the other. The line P does not depend on x.

Boundary-functional condition (1.2) are called p**-regular with respect to a system of functions** $\omega_1(x)$, $\ldots, \omega_m(x)$ if:

a) the system of functions $\omega_1(x), \ldots, \omega_m(x)$ is p-separated and $\theta(0) \neq 0$, $\theta(1) \neq 0$, where

$$
\theta(x) :=
$$

$$
\begin{vmatrix}
\sum\limits_{k=0}^{n_1} \alpha_{1k} \omega_1^{m_1 - dk}(x) & \cdots & \sum\limits_{k=0}^{n_1} \alpha_{1k} \omega_p^{m_1 - dk}(x) & \sum\limits_{k=0}^{n_1} \beta_{1k} \omega_{p+1}^{m_1 - dk}(x) & \cdots \\
\vdots & \cdots & \vdots & \cdots & \\
\sum\limits_{k=0}^{n_m} \alpha_{mk} \omega_1^{m_m - dk}(x) & \cdots & \sum\limits_{k=0}^{n_m} \alpha_{mk} \omega_p^{m_m - dk}(x) & \sum\limits_{k=0}^{n_m} \beta_{mk} \omega_{p+1}^{m_m - dk}(x) & \cdots
\end{vmatrix};
$$

b) $x_{\nu k i} \in (0,1)$ and the functionals $T_{\nu k}$ are continuous in $W_{q,\beta}^{[m_\nu - dk]}(0,1)$ for some $q \in (1,\infty)$ and $\beta \in \mathbb{R}$.

Problem (1.1)–(1.2) is called p**-regular with respect to a system of functions** $\omega_1(x)$, $\ldots, \omega_m(x)$ if:

1) boundary-functional conditions (1.2) are p-regular with respect to a system of functions $\omega_1(x), \ldots, \omega_m(x)$, where $\omega_j(x)$ are the roots of the equation

$$a_n(x)\omega^m + a_{n-1}(x)\omega^{d(n-1)} + \cdots + 1 = 0, \quad x \in [0,1]; \tag{1.3}$$

2) for some $q \in (1,\infty)$ and $\beta \in \mathbb{R}$ the operators B_k from $W^{[dk]}_{q,\beta}(0,1)$ into $L_q(0,1)$ are compact.

The order of the roots of equation (1.3) is important in the definition of p-regularity of problem (1.1)–(1.2).

Here, the case $p = 0$ or $p = m$ is also admitted.

If the principal parts of boundary-functional conditions (1.2) are local, i.e., they are given only in 0 or in 1, then it follows from the p-regularity of the boundary-functional conditions that the number of them in 0 is equal to p, and in 1 is equal to $m - p$.

Note that if problem (1.1)–(1.2) is p-regular with respect to a system of functions $\omega_1(x)$, $\ldots, \omega_m(x)$, then $\omega_j(0) \neq \omega_s(0)$ and $\omega_j(1) \neq \omega_s(1)$ for $j \neq s$.

2. Isomorphism and coerciveness of degenerate boundary value problems for equations with variable coefficients and a weight

Consider a principally boundary value problem (1.1)-(1.2) for ordinary differential equations with variable coefficients, when the spectral parameter appears polynomially in both the equation and the boundary-functional conditions, and the weight $d := \frac{m}{n}$.

Theorem 1. *Let $n \geq 1$, $m \geq 1$, $d := \frac{m}{n}$ is an integer, $m_\nu \geq dn_\nu$, $x_{\nu ki} \in (0,1)$, $a_n \neq 0$, and let the following conditions be satisfied*:

(1) $a_k(\cdot) \in C^{[\ell-m]}_\beta[0,1]$, *where an integer* $\ell \geq \max\{m, m_\nu + 1\}$, $0 \leq \beta < \frac{1}{2}$; $a_n(x) \neq 0$; $a_j(0) = a_j(1)$;[1]

(2) *problem* (1)–(2) *is p-regular with respect to a system of functions* $\omega_1(x), \ldots, \omega_m(x)$;

(3) *for all* $\varepsilon > 0$ *and for some* $q \in [2,\infty)$[2]

$$\|B_k u\|_{L_q(0,1)} \leq \varepsilon \|u\|_{W^{[dk]}_{q,\beta}(0,1)} + C(\varepsilon)\|u\|_{L_q(0,1)}, \quad u \in W^{[dk]}_{q,\beta}(0,1),$$

$$\|B_k u\|_{W^{[\ell-m]}_{q,\beta}(0,1)} \leq \varepsilon \|u\|_{W^{[\ell-m+dk]}_{q,\beta}(0,1)} + C(\varepsilon)\|u\|_{L_q(0,1)}, \quad u \in W^{[\ell-m+dk]}_{q,\beta}(0,1);$$

(4) *the functionals $T_{\nu k}$ in $W^{[m_\nu - dk]}_{q,\beta}(0,1)$ are continuous.*

Then, for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that for all complex numbers λ that satisfy $|\lambda| > R_\varepsilon$ and for some $s = 0, \ldots, n-1$ lying inside the angle

$$\left(\frac{\pi}{2} - \underline{\omega} - 2\pi s\right)d + \varepsilon < \arg\lambda < \left(\frac{3\pi}{2} - \overline{\omega} - 2\pi s\right)d - \varepsilon, \tag{2.1}$$

[1] From $a_j(0) = a_j(1)$ it follows that $\theta(0) = \theta(1)$. If boundary-functional conditions (1.2) are principally local, i.e., for each $\nu = 1, \ldots, m$ or $\alpha_{\nu k} = 0$, $k = 0, \ldots, n_\nu$, or $\beta_{\nu k} = 0$, $k = 0, \ldots, n_\nu$, then the condition $a_j(0) = a_j(1)$ should be omitted.

[2] Here and everywhere $W^{[0]}_{q,\beta}(0,1) := L_q(0,1)$.

where

$$\underline{\omega} := \inf_{x\in[0,1]} \min\Big\{\arg\omega_j(x) : j=1,\dots,p;\ \arg\omega_s(x)+\pi : s=p+1,\dots,m\Big\},$$
$$\overline{\omega} := \sup_{x\in[0,1]} \max\Big\{\arg\omega_j(x) : j=1,\dots,p;\ \arg\omega_s(x)+\pi : s=p+1,\dots,m\Big\},$$

and $\omega_j(x)$ *are roots of equation* (1.3) (*the value* $\arg\omega_j(x)$ *is chosen up to a multiple of* 2π, *so that* $\overline{\omega}-\underline{\omega}<\pi$), *the operator* $\mathbb{L}(\lambda):\ u\to\mathbb{L}(\lambda)u := \Big(L(\lambda)u, L_1(\lambda)u,\dots,\ L_m(\lambda)u\Big)$ *from* $W^{[\ell]}_{q,\beta}(0,1)$ *onto* $W^{[\ell-m]}_{q,\beta}(0,1)\dotplus\mathbb{C}^m$ *is an isomorphism and, for these* λ, *the following estimates hold for a solution of problem* (1.1)–(1.2) :

$$\sum_{k=0}^{\ell}|\lambda|^{d^{-1}(\ell-k)}\|u\|_{W^{[k]}_{q,\beta}(0,1)} \le C(\varepsilon)\Big(\|f\|_{W^{[\ell-m]}_{q,\beta}(0,1)}+|\lambda|^{d^{-1}(\ell-m)}\|f\|_{L_q(0,1)} + \sum_{\nu=1}^{m}|\lambda|^{d^{-1}(\ell-m_\nu-\beta_1-\frac{1}{q})}|f_\nu|\Big), \tag{2.2}$$

and if $\max\{m'_\nu\}-\min\{m_\nu-dn_\nu\}\le m-1$

$$\sum_{k=0}^{m}|\lambda|^{d^{-1}(m-k)}\|u\|_{W^{[k+p]}_{q,\beta}(0,1)} \le C(\varepsilon)\Big(\|f\|_{W^{[p]}_{q,\beta}(0,1)}+\sum_{\nu=1}^{m}|\lambda|^{d^{-1}(m+p-m_\nu-\beta_1-\frac{1}{q})}|f_\nu|\Big),$$

$$\max\{m'_\nu+1-m,0\}\le p\le\min\{m_\nu-dn_\nu,\ell-m\}, \tag{2.3}$$

where m'_ν *denotes the differential order of* $L_\nu(\lambda)u$ *and* $\beta_1=\frac{\beta}{q(1-\beta)}$.

Proof. Let us consider the change of the variable $x=t^{\frac{1}{1-\beta}}$. Then, by virtue of Lemma 1, the operator $u(x)\to\tilde u(t)=u(t^{\frac{1}{1-\beta}})$ is an isomorphism from the spaces $C^{[\ell]}_\beta[0,1]$, $L_q(0,1)$, and $W^{[\ell]}_{q,\beta}(0,1)$ onto the spaces $C^\ell[0,1]$, $L_{q,\beta_1}(0,1)$, and $W^\ell_{q,\beta_1}(0,1)$, respectively, where $\beta_1=\frac{\beta}{q(1-\beta)}$. Moreover, $\big(x^\beta\frac{d}{dx}\big)u(x)=x^\beta\tilde u'(t)t'_x=x^\beta\tilde u'(t)(1-\beta)x^{-\beta}=(1-\beta)\tilde u'(t)$, and, therefore,

$$\big(x^\beta\frac{d}{dx}\big)^{(j)}u(x)=(1-\beta)^j\tilde u^{(j)}(t). \tag{2.4}$$

Then, taking into account (2.4), problem (1.1)-(1.2) is rewritten in the following form for the function $\tilde u(t)$:

$$\tilde L(\lambda)\tilde u := \lambda^n\tilde u(t)+\sum_{k=1}^{n}\lambda^{n-k}\Big(\tilde a_k(t)(1-\beta)^{dk}\tilde u^{(dk)}(t)+\tilde B_k\tilde u|_t\Big)=\tilde f(t), \tag{2.5}$$

$$\tilde L_\nu(\lambda)\tilde u := \sum_{k=0}^{n_\nu}\lambda^k\Big(\alpha_{\nu k}(1-\beta)^{m_\nu-dk}\tilde u^{(m_\nu-dk)}(0)+\beta_{\nu k}(1-\beta)^{m_\nu-dk}\tilde u^{(m_\nu-dk)}(1) + \sum_{i=1}^{N_{\nu k}}\delta_{\nu ki}(1-\beta)^{m_\nu-dk}\tilde u^{(m_\nu-dk)}(x_{\nu ki})+\tilde T_{\nu k}\tilde u\Big)=f_\nu,\quad \nu=1,\dots,m, \tag{2.6}$$

where $\tilde{a}_k(t) = a_k(t^{\frac{1}{1-\beta}})$, $\tilde{B}_k = FB_kF^{-1}$ are operators in $L_{q,\beta_1}(0,1)$, $\tilde{T}_{\nu k} = FT_{\nu k}F^{-1}$ are functionals in $L_{q,\beta_1}(0,1)$ and

$$F : u(x) \to Fu := \tilde{u}(t) := u(t^{\frac{1}{1-\beta}}),$$
$$F^{-1} : \tilde{u}(t) \to F^{-1}\tilde{u} := u(x) := \tilde{u}(x^{1-\beta})$$

are isomorphisms in the corresponding spaces (see the beginning of the proof).

Let us apply Theorem 1 [YY, p.111] with $\gamma = \beta_1 = \frac{\beta}{q(1-\beta)}$ to problem (2.5)-(2.6). Condition (1) of Theorem 1 [YY, p.111] follows from condition (1). Consider the equation

$$\tilde{a}_n(t)(1-\beta)^n\tilde{\omega}^n + \tilde{a}_{n-1}(t)(1-\beta)^{d(n-1)}\tilde{\omega}^{d(n-1)} + \cdots + 1 = 0,\ t \in [0,1] \tag{2.7}$$

and denote its roots by $\tilde{\omega}_j(t)$, $j = 1, \ldots, m$. There is a simple connection between $\tilde{\omega}_j(t)$ and the roots $\omega_j(x)$ of equation (1.3), namely, $(1-\beta)\tilde{\omega}_j(t) = \omega_j(x)$, where $x = t^{\frac{1}{1-\beta}}$. Then, by condition (2), problem (2.5)-(2.6) is p-regular with respect to $\tilde{\omega}_1(t), \ldots, \tilde{\omega}_m(t)$, i. e., condition (2) of Theorem 1 [YY, p.111] is satisfied, too. Further, by virtue of condition (3), for all $\varepsilon > 0$

$$\begin{aligned}
\|\tilde{B}_k\tilde{u}\|_{L_{q,\beta_1}(0,1)} &= \|FB_kF^{-1}\tilde{u}\|_{L_{q,\beta_1}(0,1)} \le C\|B_kF^{-1}\tilde{u}\|_{L_q(0,1)} \\
&\le \varepsilon\|F^{-1}\tilde{u}\|_{W_{q,\beta}^{[dk]}(0,1)} + C(\varepsilon)\|F^{-1}\tilde{u}\|_{L_q(0,1)} \\
&\le \varepsilon\|\tilde{u}\|_{W_{q,\beta_1}^{dk}(0,1)} + C(\varepsilon)\|\tilde{u}\|_{L_{q,\beta_1}(0,1)}, \quad \forall \tilde{u} \in W_{q,\beta_1}^{dk}(0,1).
\end{aligned}$$

So, the first inequality in condition (3) of Theorem 1 [YY, p.111] has been checked. In a similar way one can check the second inequality in condition (3) of Theorem 1 [YY, p.111]. From condition (4) it follows that

$$\begin{aligned}
|\tilde{T}_{\nu k}\tilde{u}| &= |FT_{\nu k}F^{-1}\tilde{u}| = |T_{\nu k}F^{-1}\tilde{u}| \\
&\le C\|F^{-1}\tilde{u}\|_{W_{q,\beta}^{[m_\nu - dk]}} \le C\|\tilde{u}\|_{W_{q,\beta_1}^{m_\nu - dk}},
\end{aligned}$$

i. e., condition (4) of Theorem 1 [YY, p.111] has been checked, too. So, all conditions of Theorem 1 [YY, p.111] have been checked for problem (2.5)-(2.6) and the theorem implies that for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that for all complex numbers λ that satisfy $|\lambda| > R_\varepsilon$ and for some $s = 0, \ldots, n-1$ lying inside the angle

$$\Big(\frac{\pi}{2} - \underline{\tilde{\omega}} - 2\pi s\Big)d + \varepsilon < \arg\lambda < \Big(\frac{3\pi}{2} - \bar{\tilde{\omega}} - 2\pi s\Big)d - \varepsilon, \tag{2.8}$$

where

$$\underline{\tilde{\omega}} := \inf_{t\in[0,1]} \min\Big\{\arg\tilde{\omega}_j(t) : j = 1, \ldots, p;\ \arg\tilde{\omega}_s(t) + \pi : s = p+1, \ldots, m\Big\},$$
$$\bar{\tilde{\omega}} := \sup_{t\in[0,1]} \max\Big\{\arg\tilde{\omega}_j(x) : j = 1, \ldots, p;\ \arg\tilde{\omega}_s(t) + \pi : s = p+1, \ldots, m\Big\},$$

and $\tilde{\omega}_j(t)$ are roots of equation (2.7) (the value $\arg\tilde{\omega}_j(t)$ is chosen up to a multiple of 2π, so that $\tilde{\overline{\omega}} - \underline{\tilde{\omega}} < \pi$), the operator $\tilde{\mathbb{L}}(\lambda) : \tilde{u} \to \tilde{\mathbb{L}}(\lambda)\tilde{u} := \Big(\tilde{L}(\lambda)\tilde{u}, \tilde{L}_1(\lambda)\tilde{u}, \dots, \tilde{L}_m(\lambda)\tilde{u}\Big)$ from $W^\ell_{q,\beta_1}(0,1)$ onto $W^{\ell-m}_{q,\beta_1}(0,1) \dot{+} \mathbb{C}^m$ is an isomorphism, and for these λ, the following estimates hold for a solution of problem (2.5)-(2.6):

$$\sum_{k=0}^{\ell} |\lambda|^{d^{-1}(\ell-k)} \|\tilde{u}\|_{W^k_{q,\beta_1}(0,1)} \le C(\varepsilon)\Big(\|\tilde{f}\|_{W^{\ell-m}_{q,\beta_1}(0,1)} + |\lambda|^{d^{-1}(\ell-m)}\|\tilde{f}\|_{L_{q,\beta_1}(0,1)} + \sum_{\nu=1}^{m} |\lambda|^{d^{-1}(\ell-m_\nu-\beta_1-\frac{1}{q})}|f_\nu|\Big), \tag{2.9}$$

and if $\max\{m'_\nu\} - \min\{m_\nu - dn_\nu\} \le m-1$

$$\sum_{k=0}^{m} |\lambda|^{d^{-1}(m-k)} \|\tilde{u}\|_{W^{k+p}_{q,\beta_1}(0,1)} \le C(\varepsilon)\Big(\|\tilde{f}\|_{W^p_{q,\beta_1}(0,1)} + \sum_{\nu=1}^{m} |\lambda|^{d^{-1}(m+p-m_\nu-\beta_1-\frac{1}{q})}|f_\nu|\Big),$$

$$\max\{m'_\nu + 1 - m, 0\} \le p \le \min\{m_\nu - dn_\nu, \ell - m\}. \tag{2.10}$$

But $\arg\tilde{\omega}_j(t) = \arg\omega_j(x)$ since $(1-\beta)\tilde{\omega}_j(t) = \omega_j(x)$. Therefore, from (2.8) follows (2.1). Using the above arguments of the isomorphisms of F and F^{-1} we conclude that from (2.9) follows (2.2) and from (2.10) follows (2.3). ■

Remark 1. If $\beta = 0$, i.e., the problem is non-degenerate, then Theorem 1 is a particular case of Theorem 1 [YY, p.111].

Remark 2. In fact, Theorem 1 [YY, p.111] can be proved if the weight $d = \frac{m}{n}$ is non-integer, too. Therefore, our Theorem 1 is also true for a non-integer $d = \frac{m}{n}$. One should only to add in the conditions of the theorem that $a_k(x) = \alpha_{\nu k} = \beta_{\nu k} = \delta_{\nu k i} = 0$ if dk is not an integer.

Let us now consider a particular case of problem (1.1)-(1.2) which includes GWBC

$$L(\lambda)u := \lambda u(x) + a(x)\big(x^\beta \frac{d}{dx}\big)^2 u(x) = f(x), \quad x \in (0,1), \tag{2.11}$$

$$\begin{aligned} L_1(\lambda)u :=& \overline{\alpha}_{10} a(0)\big(x^\beta \frac{d}{dx}\big)^2 u(x)|_{x=0} + \overline{\beta}_{10} a(1)\big(x^\beta \frac{d}{dx}\big)^2 u(x)|_{x=1} \\ &+ \gamma_{10}\big(x^\beta \frac{d}{dx}\big)u(x)|_{x=0} + \delta_{10}\big(x^\beta \frac{d}{dx}\big)u(x)|_{x=1} + p_{10}u(0) + s_{10}u(1) \\ &+ \lambda(\alpha_{11}u(0) + \beta_{11}u(1) + \int_0^1 q_1(x)u(x)dx) = f_1, \\ L_2(\lambda)u :=& \overline{\alpha}_{20} a(0)\big(x^\beta \frac{d}{dx}\big)^2 u(x)|_{x=0} + \overline{\beta}_{20} a(1)\big(x^\beta \frac{d}{dx}\big)^2 u(x)|_{x=1} \\ &+ \gamma_{20}\big(x^\beta \frac{d}{dx}\big)u(x)|_{x=0} + \delta_{20}\big(x^\beta \frac{d}{dx}\big)u(x)|_{x=1} + p_{20}u(0) + s_{20}u(1) \\ &+ \lambda(\alpha_{21}u(0) + \beta_{21}u(1) + \int_0^1 q_2(x)u(x)dx) = f_2. \end{aligned} \tag{2.12}$$

Note that in (2.12) by $\left(x^\beta \frac{d}{dx}\right)u(x)|_{x=0}$ and $\left(x^\beta \frac{d}{dx}\right)^2 u(x)|_{x=0}$ we mean $\lim\limits_{x\to 0}\left(x^\beta \frac{d}{dx}\right)u(x)$ and $\lim\limits_{x\to 0}\left(x^\beta \frac{d}{dx}\right)^2 u(x)$, respectively. We correlate with equation (2.11) the following characteristic equation

$$a(x)\omega^2 + 1 = 0. \tag{2.13}$$

The roots of the equation are $\omega_1(x) = \sqrt{-\frac{1}{a(x)}}$, $\omega_2(x) = -\sqrt{-\frac{1}{a(x)}}$. So, for problem (2.11)-(2.12), in comparison with problem (1.1)-(1.2), we have $n=1$, $m=2, d=2$, $\alpha_{\nu 0} = \overline{\alpha}_{\nu 0}a(0)$, $\beta_{\nu 0} = \overline{\beta}_{\nu 0}a(1)$, $m_\nu = 2$, $n_\nu = 1$, $\delta_{\nu k i} = 0$, $T_{\nu 0}u = \gamma_{\nu 0}\left(x^\beta \frac{d}{dx}\right)u(x)|_{x=0} + \delta_{\nu 0}\left(x^\beta \frac{d}{dx}\right)u(x)|_{x=1} + p_{\nu 0}u(0) + s_{\nu 0}u(1)$, $T_{\nu 1}u = \int_0^1 q_\nu(x)u(x)dx$, $B_k = 0$ and, therefore,

$$\theta(x) := \begin{vmatrix} \alpha_{10}\omega_1^2(x) + \alpha_{11} & \beta_{10}\omega_2^2(x) + \beta_{11} \\ \\ \alpha_{20}\omega_1^2(x) + \alpha_{21} & \beta_{20}\omega_2^2(x) + \beta_{21} \end{vmatrix}$$

$$= \begin{vmatrix} -\frac{\overline{\alpha}_{10}a(0)}{a(x)} + \alpha_{11} & -\frac{\overline{\beta}_{10}a(1)}{a(x)} + \beta_{11} \\ -\frac{\overline{\alpha}_{20}a(0)}{a(x)} + \alpha_{21} & -\frac{\overline{\beta}_{20}a(1)}{a(x)} + \beta_{21} \end{vmatrix}.$$

Then the following theorem is a corollary of Theorem 1.

Theorem 2. *Let the following conditions be satisfied*:

(1) $a(\cdot) \in C_\beta^{[\ell-2]}[0,1]$, *where an integer* $\ell \geq 3$, $0 \leq \beta < \frac{1}{2}$, $a(x) \neq 0$, $a(0) = a(1)$;[3]
(2) $\theta(0) \neq 0, \theta(1) \neq 0$;
(3) $q_\nu(\cdot) \in L_{q'}(0,1)$, *where* $\frac{1}{q'} + \frac{1}{q} = 1$ *and* $q \in [2,\infty)$, $\nu = 1,2$.

Then, for any $\varepsilon > 0$ *there exists* $R_\varepsilon > 0$ *such that for all complex numbers* λ *that satisfy* $|\lambda| > R_\varepsilon$ *and lying inside the angle*

$$-\pi - 2\underline{\omega} + \varepsilon < \arg\lambda < \pi - 2\overline{\omega} - \varepsilon,$$

where $\underline{\omega} = \inf\limits_{x\in[0,1]} \arg\sqrt{-\frac{1}{a(x)}}$, $\overline{\omega} = \sup\limits_{x\in[0,1]} \arg\sqrt{-\frac{1}{a(x)}}$, *the operator* $\mathbb{L}(\lambda) : u \to \mathbb{L}(\lambda)u := (L(\lambda)u, L_1(\lambda)u, L_2(\lambda)u)$ *from* $W_{q,\beta}^{[\ell]}(0,1)$ *onto* $W_{q,\beta}^{[\ell-2]}(0,1)\dot{+}\mathbb{C}\dot{+}\mathbb{C}$ *is an isomorphism and, for these* λ, *the following estimate holds for a solution of problem* (2.11)–(2.12)

$$\sum_{k=0}^{\ell} |\lambda|^{\frac{1}{2}(\ell-k)}\|u\|_{W_{q,\beta}^{[k]}(0,1)} \leq C(\varepsilon)\Big(\|f\|_{W_{q,\beta}^{[\ell-2]}(0,1)} + |\lambda|^{\frac{1}{2}(\ell-2)}\|f\|_{L_q(0,1)}$$
$$+ \sum_{\nu=1}^{2} |\lambda|^{\frac{1}{2}(\ell-2-\beta_1-\frac{1}{q})}|f_\nu|\Big). \tag{2.14}$$

Remark 3. We cannot get a solvability and an estimate in Theorem 2 for $f \in W_{q,\beta}^{[0]}(0,1) = L_q(0,1)$. Below we consider such a formulation of problem (1.1)-(1.2) which allows us to get

[3] See the corresponding footnote of Theorem 1.

a solvability and an estimate for $f \in L_q(0,1)$, too. In such formulation we do not formally cover the original form of the boundary value problem with GWBC ($\lambda u - Au = h$ in Ω, $Au + \beta \frac{\partial u}{\partial n} + \gamma u = 0$ on $\partial\Omega$) but only its reduced form ($\lambda u - Au = h$ in Ω, $\beta \frac{\partial u}{\partial n} + (\gamma + \lambda)u = h$ on $\partial\Omega$).

Consider a principally boundary value problem for an ordinary differential equation with a variable coefficient in the case when the spectral parameter appears linearly in the equation and can appear in boundary-functional conditions

$$L(\lambda)u := \lambda u(x) + a(x)\big(x^\beta \frac{d}{dx}\big)^{(m)} u(x) + Bu|_x = f(x), \quad x \in (0,1), \tag{2.15}$$

$$\begin{aligned} L_\nu(\lambda)u :&= \lambda\Big(\alpha_\nu \big(x^\beta \frac{d}{dx}\big)^{(m_\nu)} u(x)|_{x=0} + \beta_\nu \big(x^\beta \frac{d}{dx}\big)^{(m_\nu)} u(x)|_{x=1} \\ &+ \sum_{i=1}^{N_\nu} \delta_{\nu i} \big(x^\beta \frac{d}{dx}\big)^{(m_\nu)} u(x)|_{x=x_{\nu i}} + T_\nu u\Big) + T_{\nu 0} u = g_\nu, \quad \nu = 1, \dots, s, \end{aligned} \tag{2.16}$$

$$\begin{aligned} L_\nu u :&= \alpha_\nu \big(x^\beta \frac{d}{dx}\big)^{(m_\nu)} u(x)|_{x=0} + \beta_\nu \big(x^\beta \frac{d}{dx}\big)^{(m_\nu)} u(x)|_{x=1} \\ &+ \sum_{i=1}^{N_\nu} \delta_{\nu i} \big(x^\beta \frac{d}{dx}\big)^{(m_\nu)} u(x)|_{x=x_{\nu i}} + T_\nu u = 0, \quad \nu = s+1, \dots, m, \end{aligned} \tag{2.17}$$

where $m \geq 1$, $m_\nu \leq m-1$, $x_{\nu i} \in (0,1)$, $0 \leq s \leq m$, B is an operator in $L_q(0,1)$, T_ν and $T_{\nu 0}$ are functionals in $L_q(0,1)$, where $q \in (1,\infty)$. Here, both the operator and the functionals are unbounded.

The results of Theorem 3 (see below) in the case $s = 0$, i.e., without λ in boundary conditions, follow from Theorem 1. But in order to cover the reduced form of the problem with GWBC, we need λ in boundary conditions (see the above Remark 3).

Theorem 3. *Let the following conditions be satisfied*:

(1) $m \geq 1$; $m_\nu \leq m-1$; $0 \leq s \leq m$;

(2) $a \in C[0,1]$; $a(x) \neq 0$; $a(0) = a(1)$[4]; $\sup_{x\in[0,1]} \arg a(x) - \inf_{x\in[0,1]} \arg a(x) < 2\pi$, *if m is even;* $\sup_{x\in[0,1]} \arg a(x) - \inf_{x\in[0,1]} \arg a(x) < \pi$, *if m is odd*;

(3) *for all $\varepsilon > 0$ and for some $q \in [2,\infty)$ and $0 \leq \beta < \frac{1}{2}$*

$$\|Bu\|_{L_q(0,1)} \leq \varepsilon \|u\|_{W^{[m]}_{q,\beta}(0,1)} + C(\varepsilon)\|u\|_{L_q(0,1)}, \quad u \in W^{[m]}_{q,\beta}(0,1);$$

(4) *functionals T_ν in $W^{[m_\nu]}_{q,\beta}(0,1)$ and functionals $T_{\nu 0}$ in $W^{[m-\varepsilon]}_{q,\beta}(0,1)$, for some $\varepsilon > 0$, are continuous;*

(5) *the system*

$$\begin{aligned} C_\nu u :&= \alpha_\nu \big(x^\beta \frac{d}{dx}\big)^{(m_\nu)} u(x)|_{x=0} + \beta_\nu \big(x^\beta \frac{d}{dx}\big)^{(m_\nu)} u(x)|_{x=1} \\ &+ \sum_{i=1}^{N_\nu} \delta_{\nu i} \big(x^\beta \frac{d}{dx}\big)^{(m_\nu)} u(x)|_{x=x_{\nu i}} + T_\nu u, \quad \nu = 1, \dots, m, \end{aligned} \tag{2.18}$$

[4] If boundary-functional conditions (2.16)–(2.17) are principally local, i.e., or $\alpha_\nu = 0$, or $\beta_\nu = 0$ for all $\nu = 1, \dots, m$, then the condition $a(0) = a(1)$ should be omitted.

is p-regular with respect to a system of numbers $\omega_j = \mathrm{e}^{2\pi i \frac{j-1}{m}}, j = 1, \ldots, m$, *i.e.,*

$$\begin{vmatrix} \alpha_1\omega_1^{m_1} & \cdots & \alpha_1\omega_p^{m_1} & \beta_1\omega_{p+1}^{m_1} & \cdots & \beta_1\omega_m^{m_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_m\omega_1^{m_m} & \cdots & \alpha_m\omega_p^{m_m} & \beta_m\omega_{p+1}^{m_m} & \cdots & \beta_m\omega_m^{m_m} \end{vmatrix} \neq 0,$$

where $p = \frac{m}{2}$, *if m is even;* $p = [\frac{m}{2}]$ *or* $p = [\frac{m}{2}] + 1$ *if m is odd.*

Then, for any $\varepsilon > 0$, *there exists* $R_\varepsilon > 0$ *such that for all complex numbers* λ *which satisfy* $|\lambda| > R_\varepsilon$ *and for* $m = 2p$ *lying inside the angle*

$$\frac{\pi m}{2} - \pi + \sup_{x\in[0,1]} \arg a(x) + \varepsilon < \arg\lambda < \frac{\pi m}{2} + \pi + \inf_{x\in[0,1]} \arg a(x) - \varepsilon,$$

for $m = 2p + 1$ *lying inside the angle*

$$\frac{\pi m}{2} + \sup_{x\in[0,1]} \arg a(x) + \varepsilon < \arg\lambda < \frac{\pi m}{2} + \pi + \inf_{x\in[0,1]} \arg a(x) - \varepsilon,$$

and for $m = 2p - 1$ *lying inside the angle*

$$\frac{\pi m}{2} - \pi + \sup_{x\in[0,1]} \arg a(x) + \varepsilon < \arg\lambda < \frac{\pi m}{2} + \inf_{x\in[0,1]} \arg a(x) - \varepsilon,$$

the operator $\mathbb{L}(\lambda) : \ u \to \mathbb{L}(\lambda)u := \Big(L(\lambda)u, L_1(\lambda)u, \ldots, L_s(\lambda)u\Big)$ *from* $W_{q,\beta}^{[m]}((0,1);\, C_\nu u = 0, \nu = s+1, \ldots, m)$ *onto* $L_q(0,1)\dot{+}\mathbb{C}^s$ *is an isomorphism, and for these* λ *for a solution of problem* (2.15)–(2.17) *the estimate*

$$\|u\|_{W_{q,\beta}^{[m]}(0,1)} + |\lambda|\Big(\|u\|_{L_q(0,1)} + \sum_{\nu=1}^{s} |C_\nu u|\Big) \leq C(\varepsilon)\Big(\|f\|_{L_q(0,1)} + \sum_{\nu=1}^{s} |g_\nu|\Big) \tag{2.19}$$

is valid, where C_ν *is defined by* (2.18).

Proof. The idea of the proof is similar to that of Theorem 1. The only thing is that we use Theorem 4 [YY, p.129] instead of Theorem 1 [YY, p.111] to the rewritten problem with $\tilde{u}$. ■

3. Initial boundary value problems for degenerate parabolic equations with boundary conditions containing differentiation on time

A parabolic problem with GWBC is formulated as follows

$$\begin{aligned} &\frac{\partial u}{\partial t} = Au + h(t), \qquad (t,x) \in [0,\infty) \times \Omega, \\ &Au + \beta\frac{\partial u}{\partial n} + \gamma u = 0, \qquad (t,x) \in [0,\infty) \times \partial\Omega, \\ &u(0,x) = f(x), \qquad x \in \Omega, \end{aligned} \tag{3.1}$$

where Ω is a bounded subset of $\mathbb{R}^N$ and $Au := \nabla\cdot(a\nabla u)$, $a\in C^1(\overline{\Omega})$, $a>0$ in Ω, β and γ are some functions in $C^1(\partial\Omega)$, $n = n(x)$ is the unit normal at x. Such problems and the corresponding spectral problems have been studied in the papers by A. Favini, G. R. Goldstein, J. A. Goldstein, and S. Romanelli [FGGR1], [FGGR2]. Another form of the boundary condition in (3.1) is obtained if we change Au by $\frac{\partial u}{\partial t} - h(t)$ in the boundary condition. So, we deal with the following problem

$$\begin{aligned}
&\frac{\partial u}{\partial t} = Au + h(t), \qquad (t,x)\in[0,\infty)\times\Omega,\\
&\frac{\partial u}{\partial t} + \beta\frac{\partial u}{\partial n} + \gamma u = h(t), \qquad (t,x)\in[0,\infty)\times\partial\Omega,\\
&u(0,x) = f(x), \qquad x\in\Omega,
\end{aligned} \tag{3.2}$$

The last boundary condition is a *dynamic* boundary condition involving motion on the boundary. Such conditions arise in the applications and have been studied by H. Amann [A], H. Amann and J. Escher [AE], M. Grobbelaar-Van Dalsen and N. Sauer [GS1], [GS2], T. Hintermann [H], and N. Sauer [S].

The aim of this section is to consider some generalization of problem (3.2) in the case of one-dimensional space variable x.

In the domain $[0,T]\times[0,1]$, consider a principally initial boundary value problem for the parabolic equation

$$L(t,D_t)u := D_t u(t,x) + a(t,x)D_x^{2m}u(t,x) + B(t)u(t,\cdot)\Big|_x = f(t,x), \tag{3.3}$$

$$\begin{aligned}
L_\nu(t,D_t)u &:= D_t\Big(\alpha_\nu D_x^{m_\nu}u(t,0) + \beta_\nu D_x^{m_\nu}u(t,1) + \sum_{i=1}^{N_\nu}\delta_{\nu i}D_x^{m_\nu}u(t,x_{\nu i})\\
&\quad + T_\nu u(t,\cdot)\Big) + T_{\nu 0}(t)u(t,\cdot) = f_\nu(t), \quad t\in[0,T],\ \nu = 1,\dots,s,\\
L_\nu u &:= \alpha_\nu D_x^{m_\nu}u(t,0) + \beta_\nu D_x^{m_\nu}u(t,1) + \sum_{i=1}^{N_\nu}\delta_{\nu i}D_x^{m_\nu}u(t,x_{\nu i}) + T_\nu u(t,\cdot)\\
&\quad = 0, \quad t\in[0,T],\ \nu = s+1,\dots,2m,
\end{aligned} \tag{3.4}$$

$$u(0,x) = u_0(x), \quad x\in[0,1], \tag{3.5}$$

where $m\geq 1$, $m_\nu\leq 2m-1$; $0\leq s\leq 2m$; $\alpha_\nu,\beta_\nu,\delta_{\nu i}$ are complex numbers, $x_{\nu i}\in(0,1)$, $D_t := \frac{\partial}{\partial t}, D_x := \frac{\partial}{\partial x}$, $B(t)$ is an operator in $L_{q,\gamma}(0,1)$, T_ν and $T_{\nu 0}$ are functionals in $L_{q,\gamma}(0,1)$, where $q\in(1,\infty)$, $-\frac{1}{q}<\gamma<\min\{\frac{1}{q},1-\frac{1}{q}\}$. Here, both the operator and the functionals are unbounded.

Denote

$$\begin{aligned}
L_\nu u :&= A_{\nu 0}u := \alpha_\nu D_x^{m_\nu}u(0) + \beta_\nu D_x^{m_\nu}u(1) + \sum_{i=1}^{N_\nu}\delta_{\nu i}D_x^{m_\nu}u(x_{\nu i}) + T_\nu u,\\
&\qquad \nu = 1,\dots,2m,\\
A_{\nu 1}(t) :&= T_{\nu 0}(t), \quad \nu = 1,\dots,s,
\end{aligned}$$

$$L(t,\lambda)u := \lambda u(x) + a(t,x)D_x^{2m}u(x) + B(t)u\Big|_x, \quad x \in [0,1],$$
$$L_\nu(t,\lambda)u := \lambda\Big(\alpha_\nu D_x^{m_\nu}u(0) + \beta_\nu D_x^{m_\nu}u(1) + \sum_{i=1}^{N_\nu}\delta_{\nu i}D_x^{m_\nu}u(x_{\nu i}) + T_\nu u\Big)$$
$$+ T_{\nu 0}(t)u, \quad \nu = 1,\dots,s.$$

Theorem 4. *Let the following conditions be satisfied*:

(1) *for some* $\ell \in (0,1]$ *it holds that* $a \in C^\ell([0,T];C[0,1])$; $a(t,x) \neq 0$ *for* $(t,x) \in [0,T]\times[0,1]$;

(2) $a(t,0) = a(t,1)$[5]; *for some fixed* $\delta > 0$ *we have* $|\arg a(t,x)| \leq \frac{\pi}{2} - \delta$ *if* m *is an even number and* $|\arg a(t,x)| \geq \frac{\pi}{2} + \delta$ *if* m *is odd*;

(3) *boundary-functional conditions* (3.4) *are* m*-regular with respect to a system of numbers* $\omega_1 = 1,\ \omega_2 = \mathrm{e}^{i\frac{\pi}{m}}, \dots, \omega_{2m} = \mathrm{e}^{i\frac{\pi(2m-1)}{m}}$, *i.e.*,

$$\begin{vmatrix} \alpha_1\omega_1^{m_1} & \cdots & \alpha_1\omega_m^{m_1} & \beta_1\omega_{m+1}^{m_1} & \cdots & \beta_1\omega_{2m}^{m_1} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_{2m}\omega_1^{m_{2m}} & \cdots & \alpha_{2m}\omega_m^{m_{2m}} & \beta_{2m}\omega_{m+1}^{m_{2m}} & \cdots & \beta_{2m}\omega_{2m}^{m_{2m}} \end{vmatrix} \neq 0;$$

functionals T_ν *in* $W_{q,\gamma}^{m_\nu}(0,1)$ *and for* $t \in [0,T]$ *functionals* $T_{\nu 0}(t)$ *in* $W_{q,\gamma}^{2m-\varepsilon}(0,1)$, *for some* $\varepsilon > 0$, $q \in (1,\infty)$, *and* $-\frac{1}{q} < \gamma < \min\{\frac{1}{q}, 1-\frac{1}{q}\}$ *are continuous*;

(4) *for* $t \in [0,T]$ *an operator* $B(t)$ *from* $W_{q,\gamma}^{2m}(0,1)$ *into* $L_{q,\gamma}(0,1)$ *is compact; for* $0 \leq t,\tau \leq T$

$$\|(B(t)-B(\tau))u\|_{L_{q,\gamma}(0,1)} \leq C|t-\tau|^\ell\|u\|_{W_{q,\gamma}^{2m}(0,1)}, \quad u \in W_{q,\gamma}^{2m}(0,1),$$
$$\Big|(T_{\nu 0}(t)-T_{\nu 0}(\tau))u\Big| \leq C|t-\tau|^\ell\|u\|_{W_{q,\gamma}^{2m}(0,1)}, \quad u \in W_{q,\gamma}^{2m}(0,1);$$

(5) $f \in C_\mu^\theta((0,T];L_{q,\gamma}(0,1))$, $f_\nu \in C_\mu^\theta(0,T]$ *for some* $\theta \in (0,1]$, $\mu \in [0,1)$;

(6) $u_0 \in W_{q,\gamma}^{2m}((0,1); L_\nu u = 0, \nu = s+1,\dots,2m)$.

Then, there exists a unique solution $u(t,x)$ *of problem* (3.3)–(3.5) *such that the function* $t \to (u(t,x), A_{10}u(t,\cdot),\dots,A_{s0}u(t,\cdot))$ *from* $(0,T)$ *into* $L_{q,\gamma}(0,1)\dot{+}\mathbb{C}^s$ *is continuously differentiable and from* $[0,T]$ *into* $W_{q,\gamma}^{2m}(0,1)\dot{+}\mathbb{C}^s$ *is continuous, and for* $t \in (0,T]$ *the following estimates hold*:

$$\|u(t,\cdot)\|_{L_{q,\gamma}(0,1)} \leq C\Big(\|u_0\|_{W_{q,\gamma}^{2m}(0,1)} + \|f\|_{C_\mu((0,t];L_{q,\gamma}(0,1))} + \sum_{\nu=1}^s \|f_\nu\|_{C_\mu(0,t]}\Big),$$
$$\|D_t u(t,\cdot)\|_{L_{q,\gamma}(0,1)} + \|u(t,\cdot)\|_{W_{q,\gamma}^{2m}(0,1)} \leq C\Big[\|u_0\|_{W_{q,\gamma}^{2m}(0,1)}$$
$$+ t^{-\mu}\Big(\|f\|_{C_\mu^\theta((0,t];L_{q,\gamma}(0,1))} + \sum_{\nu=1}^s \|f_\nu\|_{C_\mu^\theta(0,t]}\Big)\Big].$$

[5]If boundary-functional conditions (3.4) are principally local, i.e., for each $\nu = 1,\dots,m$ or $\alpha_\nu = 0$ or $\beta_\nu = 0$, then the condition $a(t,0) = a(t,1)$ should be omitted.

Proof. Let us apply Theorem 1 [YY, p.422]. For $t \in [0,T]$ denote by $A(t)$ an operator in $E := L_{q,\gamma}(0,1)$ with the domain of definition

$$D(A(t)) := E_1 := W^{2m}_{q,\gamma}((0,1); L_\nu u = 0, \nu = s+1,\dots,2m),$$

independent on $t \in [0,T]$ and with the action law

$$A(t)u := a(t,x)u^{(2m)}(x) + B(t)u|_x.$$

Taking $E^\nu := \mathbb{C}$ problem (1)–(3) can be rewritten in the form

$$\begin{aligned} L(t,D_t)u &:= u'(t) + A(t)u(t) = f(t), \\ L_\nu(t,D_t)u &:= (A_{\nu 0}u(t))' + A_{\nu 1}(t)u(t) = f_\nu(t), \quad \nu = 1,...,s, \\ u(0) &= u_0, \end{aligned} \tag{3.6}$$

where $u(t) := u(t,\cdot)$, $f(t) := f(t,\cdot)$ are functions with values in the Banach space $E := L_{q,\gamma}(0,1)$ and $u_0 := u_0(\cdot)$.

From Theorem 1 [YY, p.169] and Theorem 1 [YY, p.48] it follows that for problem (3.6) conditions (1) and (2) of Theorem 1 [YY, p.422] are satisfied. From Theorem 4 [YY, p.129] it follows that for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that, for all complex numbers λ which satisfy $|\lambda| > R_\varepsilon$ and lying inside the angle

$$\pi m - \pi + \sup_{x\in[0,1]} \arg a(x) + \varepsilon < \arg\lambda < \pi m + \pi + \inf_{x\in[0,1]} \arg a(x) - \varepsilon,$$

the operator $\mathbb{L}(t,\lambda) : u \to \mathbb{L}(t,\lambda)u := \Big(L(t,\lambda)u, L_1(t,\lambda)u,\dots,L_s(t,\lambda)u\Big)$ from $W^{2m}_{q,\gamma}((0,1); L_\nu u = 0, \nu = s+1,\dots,2m)$ onto $L_{q,\gamma}(0,1)\dot{+}\mathbb{C}^s$ is an isomorphism, and for these λ for a solution of problem

$$\begin{aligned} L(t,\lambda)u = f, \quad L_\nu(t,\lambda)u &= f_\nu, \quad \nu = 1,\dots,s, \\ L_\nu u = 0, \quad \nu &= s+1,\dots,2m, \end{aligned}$$

the following estimate holds:

$$\|u\|_{W^{2m}_{q,\gamma}(0,1)} + |\lambda|\Big(\|u\|_{L_{q,\gamma}(0,1)} + \sum_{\nu=1}^{s} |A_{\nu 0}u|\Big) \le C(\varepsilon)\Big(\|f\|_{L_{q,\gamma}(0,1)} + \sum_{\nu=1}^{s} |g_\nu|\Big).$$

Consequently, by virtue of condition (2), for the operator pencil $\mathbb{L}(t,\lambda)$: $u \to \mathbb{L}(t,\lambda)u := \Big(L(t,\lambda)u, L_1(t,\lambda)u,\dots,L_s(t,\lambda)u\Big)$ which acts boundedly from $W^{2m}_{q,\gamma}((0,1); L_\nu u = 0, \nu = s+1,\dots,2m)$ onto $L_{q,\gamma}(0,1)\dot{+}\mathbb{C}^s$ and $|\arg\lambda| \le \frac{\pi}{2}$, $|\lambda| \to \infty$,

$$\begin{aligned} \|\mathbb{L}(t,\lambda)^{-1}\|_{B(L_{q,\gamma}(0,1)\dot{+}\mathbb{C}^s, L_{q,\gamma}(0,1))} &\le C|\lambda|^{-1}, \\ \|A_{\nu 0}\mathbb{L}(t,\lambda)^{-1}\|_{B(L_{q,\gamma}(0,1)\dot{+}\mathbb{C}^s, \mathbb{C})} &\le C|\lambda|^{-1}, \quad \nu = 1,\dots,s, \end{aligned}$$

hold; i.e., condition (3) of Theorem 1 [YY, p.422] is satisfied. Conditions (4)–(6) of Theorem 1 [YY, p.422] are fulfilled in view of conditions (1), (4)–(6). So, for problem (3.6), all conditions of Theorem 1 [YY, p.422] are fulfilled and the statement of Theorem 4 follows. ■

The next step is to consider the corresponding to (3.3)-(3.5) degenerate problem.

In the domain $[0,T]\times[0,1]$, consider a principally initial boundary value problem for the degenerate parabolic equation

$$L(t,D_t)u := D_t u(t,x) + a(t,x)\big(x^\beta D_x\big)^{2m} u(t,x) + B(t)u(t,\cdot)\Big|_x = f(t,x), \tag{3.7}$$

$$\begin{aligned} L_\nu(t,D_t)u := {} & D_t\Big(\alpha_\nu\big(x^\beta D_x\big)^{m_\nu} u(t,0) + \beta_\nu\big(x^\beta D_x\big)^{m_\nu} u(t,1) \\ & + \sum_{i=1}^{N_\nu} \delta_{\nu i}\big(x^\beta D_x\big)^{m_\nu} u(t,x_{\nu i}) + T_\nu u(t,\cdot)\Big) + T_{\nu 0}(t)u(t,\cdot) \\ & = f_\nu(t), \quad t\in[0,T],\ \nu=1,\dots,s, \\ L_\nu u := {} & \alpha_\nu\big(x^\beta D_x\big)^{m_\nu} u(t,0) + \beta_\nu\big(x^\beta D_x\big)^{m_\nu} u(t,1) + \sum_{i=1}^{N_\nu} \delta_{\nu i}\big(x^\beta D_x\big)^{m_\nu} u(t,x_{\nu i}) \\ & + T_\nu u(t,\cdot) = 0, \quad t\in[0,T],\ \nu=s+1,\dots,2m, \end{aligned} \tag{3.8}$$

$$u(0,x) = u_0(x), \quad x\in[0,1], \tag{3.9}$$

where $m\ge 1$, $m_\nu \le 2m-1$; $0\le s\le 2m$; $\alpha_\nu,\beta_\nu,\delta_{\nu i}$ are complex numbers, $x_{\nu i}\in(0,1)$, $D_t := \frac{\partial}{\partial t}, D_x := \frac{\partial}{\partial x}$, $B(t)$ is an operator in $L_q(0,1)$, T_ν and $T_{\nu 0}$ are functionals in $L_q(0,1)$, where $q\in(1,\infty)$. Here, both the operator and the functionals are unbounded.

Denote

$$\begin{aligned} A_{\nu 0}u(t,\cdot) := {} & \alpha_\nu\Big(x^\beta D_x\Big)^{m_\nu} u(t,0) + \beta_\nu\Big(x^\beta D_x\Big)^{m_\nu} u(t,1) + \sum_{i=1}^{N_\nu}\delta_{\nu i}\Big(x^\beta D_x\Big)^{m_\nu} u(t,x_{\nu i}) \\ & + T_\nu u(t,\cdot), \quad \nu=1,\dots,s. \end{aligned}$$

Then using Theorem 4 and the idea of the proof of Theorem 1 one can get the following theorem.

Theorem 5. *Let the following conditions be satisfied*:

(1) *for some* $\ell\in(0,1]$ *it holds that* $a\in C_\beta^{[\ell]}([0,T];C[0,1])$; $0\le\beta<\frac{1}{2}$; $a(t,x)\ne 0$ *for* $(t,x)\in[0,T]\times[0,1]$;

(2) $a(t,0)=a(t,1)$[6]; *for some fixed* $\delta>0$ *we have* $|\arg a(t,x)|\le\frac{\pi}{2}-\delta$ *if* m *is an even number and* $|\arg a(t,x)|\ge\frac{\pi}{2}+\delta$ *if* m *is odd*;

[6] If boundary-functional conditions (3.8) are principally local, i.e., for each $\nu=1,\dots,m$ or $\alpha_\nu=0$ or $\beta_\nu=0$, then the condition $a(t,0)=a(t,1)$ should be omitted.

(3) *boundary-functional conditions* (3.8) *are m-regular with respect to a system of numbers* $\omega_1 = 1,\ \omega_2 = \mathrm{e}^{i\frac{\pi}{m}}, \dots, \omega_{2m} = \mathrm{e}^{i\frac{\pi(2m-1)}{m}}$, *i.e.,*

$$\begin{vmatrix} \alpha_1\omega_1^{m_1} & \cdots & \alpha_1\omega_m^{m_1} & \beta_1\omega_{m+1}^{m_1} & \cdots & \beta_1\omega_{2m}^{m_1} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \alpha_{2m}\omega_1^{m_{2m}} & \cdots & \alpha_{2m}\omega_m^{m_{2m}} & \beta_{2m}\omega_{m+1}^{m_{2m}} & \cdots & \beta_{2m}\omega_{2m}^{m_{2m}} \end{vmatrix} \neq 0;$$

functionals T_ν *in* $W_{q,\beta}^{[m_\nu]}(0,1)$ *and for* $t \in [0,T]$ *functionals* $T_{\nu 0}(t)$ *in* $W_{q,\beta}^{[2m-\varepsilon]}(0,1)$, *for some* $\varepsilon > 0$ *and* $q \in (1,\infty)$, *are continuous*;

(4) *for* $t \in [0,T]$ *an operator* $B(t)$ *from* $W_{q,\beta}^{[2m]}(0,1)$ *into* $L_q(0,1)$ *is compact; for* $0 \leq t,\tau \leq T$

$$\|(B(t) - B(\tau))u\|_{L_q(0,1)} \leq C|t-\tau|^\ell \|u\|_{W_{q,\beta}^{[2m]}(0,1)}, \quad u \in W_{q,\beta}^{[2m]}(0,1),$$
$$\Big|(T_{\nu 0}(t) - T_{\nu 0}(\tau))u\Big| \leq C|t-\tau|^\ell \|u\|_{W_{q,\beta}^{[2m]}(0,1)}, \quad u \in W_{q,\beta}^{[2m]}(0,1);$$

(5) $f \in C_\mu^\theta((0,T]; L_q(0,1))$, $f_\nu \in C_\mu^\theta(0,T]$ *for some* $\theta \in (0,1]$, $\mu \in [0,1)$;
(6) $u_0 \in W_{q,\beta}^{[2m]}((0,1); L_\nu u = 0, \nu = s+1, \dots, 2m)$.

Then, there exists a unique solution $u(t,x)$ *of problem* (3.7)–(3.9) *such that the function* $t \to (u(t,x), A_{10}u(t,\cdot), \dots, A_{s0}u(t,\cdot))$ *from* $(0,T)$ *into* $L_q(0,1)\dot{+}\mathbb{C}^s$ *is continuously differentiable and from* $[0,T]$ *into* $W_{q,\beta}^{[2m]}(0,1)\dot{+}\mathbb{C}^s$ *is continuous, and for* $t \in (0,T]$ *the following estimates hold*:

$$\|u(t,\cdot)\|_{L_q(0,1)} \leq C\Big(\|u_0\|_{W_{q,\beta}^{[2m]}(0,1)} + \|f\|_{C_\mu((0,t];L_q(0,1))} + \sum_{\nu=1}^{s} \|f_\nu\|_{C_\mu(0,t]}\Big),$$
$$\|D_t u(t,\cdot)\|_{L_q(0,1)} + \|u(t,\cdot)\|_{W_{q,\beta}^{[2m]}(0,1)} \leq C\Big[\|u_0\|_{W_{q,\beta}^{[2m]}(0,1)} + t^{-\mu}\Big(\|f\|_{C_\mu^\theta((0,t];L_q(0,1))} + \sum_{\nu=1}^{s} \|f_\nu\|_{C_\mu^\theta(0,t]}\Big)\Big].$$

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