FIXED POINT OF CONTRACTION AND EXPONENTIAL ATTRACTORS

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Abstract. The exponential attractor is known as one of useful notions of invariant attractors in the theory of infinite-dimensional dynamical systems. It is also known that, if the semigroup of a dynamical system satisfies a compact perturbation condition of contraction, then the dynamical system has exponential attractors. In this paper, we clarify the meaning of the compact perturbation condition of contraction and show that the exponential attractor is a natural generalization of the exponentially stable equilibrium.

1 Introduction The study of the long time behavior of systems arising from physics, mechanics and biology is a capital issue, as it is important, for practical purposes, to understand and predict the asymptotic behavior of the system.

For many parabolic and weakly damped wave equations, one can prove the existence of the finite dimensional (in the sense of the Hausdorff or the fractal dimension) global attractor, which is a compact invariant set which attracts uniformly the bounded sets of the phase space (see [12] and [14]). Since it is the smallest set enjoying these properties, it is a suitable set.

Now, the global attractor may present two major defaults for practical purposes. Indeed, the rate of attraction of the trajectories may be small and (consequently) it may be sensible to perturbations.

In order to overcome these difficulties, Foias, Sell and Temam proposed in [7] the notion of inertial manifold, which is a smooth finite dimensional hyperbolic (and thus robust) positively invariant manifold which contains the global attractor and attracts exponentially the trajectories. Unfortunately, all the known constructions of inertial manifolds are based on a restrictive condition, the so-called spectral gap condition. Consequently, the existence of inertial manifolds is not known for many physically important equations. A non-existence result has even been obtained by Mallet-Paret and Sell for a reaction-diffusion equation in higher space dimensions.

Thus, as an intermediate object between the two ideal objects that the global attractor and an inertial manifold are, Eden, Foias, Nicolaenko and Temam proposed in [13] the notion of exponential attractor, which is a compact positively invariant set which contains the global attractor, has a finite fractal dimension and attracts exponentially all the trajectories. So, compared with the global attractor, an exponential attractor is more robust under perturbations and numerical approximations (see [13] and [3, 6] for discussions on this subject). Another motivation for the study of exponential attractors comes from the fact that the global attractor may be trivial (say, reduced to one point) and may thus fail to capture important transient behaviors (see [2, 10]). We note however that, contrarily to the

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global attractor, an exponential attractor is not necessarily unique, so that actual/concrete choice of an exponential attractor is in some sense artificial.

Exponential attractors have been constructed for a large class of equations (see [4, 13] and more recent papers [1, 8, 9, 11]). The known constructions of exponential attractors (see for instance [4] and [13]) make an essential use of orthogonal projectors with finite rank (in order to prove the so-called squeezing property) and are thus valid in Hilbert spaces only. Recently, Efenviev, Miranville and Zelik gave in [5] a construction of exponential attractors that is no longer based on the finite-rank squeezing property but is based on a compact perturbation of the contraction semigroup and that is thus valid in a Banach setting. So, exponential attractors are as general as global attractors.

In this paper, we intend to clarify the meaning of the compact perturbation of contraction semigroup and to show that the notion of exponential attractors is a natural generalization of that of exponentially stable equilibria. So the theory of exponential attractors is considered as a natural generalization of Banach's fixed point theorem for contraction mappings in an infinite-dimensional dynamical system.

2 Basic Concepts Let X be a Banach space with norm $\|\cdot\|_X$. Let $\mathcal{X}$ be a subset of X, $\mathcal{X}$ being a metric space with the distance $d(U,V) = \|U-V\|_X$ $(U, V \in \mathcal{X})$ induced from $\|\cdot\|_X$. A family of nonlinear operators $S(t)$, $0 \le t < \infty$, from $\mathcal{X}$ into itself is called a (nonlinear) semigroup on $\mathcal{X}$ if $S(t)$ enjoys

1. $S(0) = 1$ (the identity mapping on $\mathcal{X}$);
2. $S(t)S(s) = S(t+s),\ 0 \le t,\ s < \infty$ (the semigroup property).

When a semigroup on $\mathcal{X}$ has the property:

$$\text{the mapping } G(t,U) = S(t)U \text{ is continuous from } [0,\infty)\times\mathcal{X} \text{ to } \mathcal{X}, \tag{2.1}$$

$S(t)$ is called a continuous semigroup on $\mathcal{X}$.

Let $S(t)$ be a continuous semigroup on $\mathcal{X}$. For each $U_0 \in \mathcal{X}$, the $\mathcal{X}$-valued continuous function $S(\cdot)U_0$ is called a trajectory starting from U_0. The family of all such trajectories are denoted by $(S(t), \mathcal{X}, X)$, and is called a dynamical system determined by $S(t)$ on the phase space $\mathcal{X}$ in the universal space X.

Let $(S(t), \mathcal{X}, X)$ be a dynamical system. An element $\overline{U} \in \mathcal{X}$ is called an equilibrium of $(S(t), \mathcal{X}, X)$ if

$$S(t)\overline{U} = \overline{U} \qquad \text{for every } t \ge 0. \tag{2.2}$$

More generally, a set $\mathcal{A} \subset \mathcal{X}$ is called an invariant set of $(S(t), \mathcal{X}, X)$ if

$$S(t)\mathcal{A} = \mathcal{A} \qquad \text{for every } t \ge 0,$$

and is called a positively invariant set of $(S(t), \mathcal{X}, X)$ if

$$S(t)\mathcal{A} \subset \mathcal{A} \qquad \text{for every } t \ge 0. \tag{2.3}$$

A set $\mathcal{A}$ is said to absorb a set B if there is a time t_B such that $S(t)B$ enters to $\mathcal{A}$ after the time t_B, namely

$$S(t)B \subset \mathcal{A} \quad \text{for every } t \ge t_B.$$

A set $\mathcal{A}$ is said to attract a set B if for any $\varepsilon > 0$, the ε neighborhood $W_\varepsilon(\mathcal{A})$ of $\mathcal{A}$ absorbs B. Using the Hausdorff pseudo distance $h(\cdot,\cdot)$ defined by

$$h(B_1, B_2) = \sup_{U_1 \in B_1} \inf_{U_2 \in B_2} d(U_1, U_2),$$

this definition can be described as

$$\lim_{t\to\infty} h(S(t)B, \mathcal{A}) = 0. \tag{2.4}$$

Indeed if (2.4) holds, then for any $\varepsilon > 0$, there exists some $t_\varepsilon > 0$ such that $h(S(t)B, \mathcal{A}) \le \frac{\varepsilon}{2}$ for $t \ge t_\varepsilon$; therefore, for any $U \in S(t)B$, $d(U, \mathcal{A}) \le \frac{\varepsilon}{2}$; hence, $U \in W_\varepsilon(\mathcal{A})$, that is

$$S(t)B \subset W_\varepsilon(\mathcal{A}) \quad \text{for every } \ t \ge t_\varepsilon.$$

An equilibrium $\overline{U}$ of $S(t)$ is said to be asymptotically stable if some open neighborhood W of $\overline{U}$ is attracted by $\overline{U}$, namely,

$$\lim_{t\to\infty} h(S(t)W, \overline{U}) = 0,$$

where the set $\{\overline{U}\}$ is denoted simply by $\overline{U}$. Similarly, an invariant set $\mathcal{A}$ of $S(t)$ is called an attractor if some open neighborhood W of $\mathcal{A}$ is attracted by $\mathcal{A}$, namely

$$\lim_{t\to\infty} h(S(t)W, \mathcal{A}) = 0.$$

We call a set $\mathcal{A}$ an absorbing set of $(S(t), \mathcal{X}, X)$ if $\mathcal{A}$ absorbs every bounded set of $\mathcal{X}$, namely for any bounded set $B \subset \mathcal{X}$, there is a time t_B (depending on B) such that

$$S(t)B \subset \mathcal{A} \quad \text{for every } \ t \ge t_B.$$

We call an attractor $\mathcal{A}$ of $(S(t), \mathcal{X}, X)$ a global attractor if

1. $\mathcal{A}$ is a compact subset of X;
2. $\mathcal{A}$ attracts every bounded set of $\mathcal{X}$, namely, for any bounded set $B \subset \mathcal{X}$,

$$\lim_{t\to\infty} h(S(t)B, \mathcal{A}) = 0.$$

When the phase space $\mathcal{X}$ is a compact set of X, the set given by

$$\mathcal{A} = \bigcap_{t\ge 0} \overline{\bigcup_{s\ge t} S(s)\mathcal{X}} \tag{2.5}$$

is shown to be a global attractor of $(S(t), \mathcal{X}, X)$, see [12] or [14] .

3 Exponentially Stable Equilibria and Exponential Attractors It is often very important to seek limit sets which attract other sets exponentially.

An equilibrium $\overline{U}$ of $(S(t), \mathcal{X}, X)$ is said to be exponentially stable if there exists some neighborhood W of $\overline{U}$ which $\overline{U}$ attracts exponentially in the sense that

$$h(S(t)W, \overline{U}) \le C_0 e^{-kt}, \qquad 0 \le t < \infty \tag{3.1}$$

with some constant $C_0 > 0$ and some exponent $k > 0$.

In a neighborhood of the exponentially stable equilibrium the behavior of trajectories can be described by a smaller dynamical system.

Theorem 3.1. *Let $\overline{U}$ be an exponentially stable equilibrium of $(S(t), \mathcal{X}, X)$. Then, there exists some neighborhood $\mathcal{W}$ of $\overline{U}$ such that $\mathcal{W}$ is a positively invariant set (therefore $(S(t), \mathcal{W}, X)$ defines a new dynamical system the phase space of which contains the point $\overline{U}$) and $\mathcal{W}$ is attracted exponentially by $\overline{U}$ in the sense* (3.1).

Proof. There is some open ball $W = B(\overline{U}; r)$, $r > 0$ for which (3.1) holds. Then take a time $T > 0$ such that $C_0 e^{-kT} = \frac{r}{2}$. Then, for every $t \geq T$, $S(t)W \subset W$. So if we set $\mathcal{W} = \bigcup_{0 \leq t < \infty} S(t)W = \bigcup_{0 \leq t \leq T} S(t)W$, then $S(t)$ transforms $\mathcal{W}$ into itself for every $0 \leq t < \infty$; namely, $\mathcal{W}$ is positively invariant.

When $0 \leq \tau \leq T$, $S(\tau)\mathcal{W} \subset \mathcal{W} = \bigcup_{0 \leq t \leq T} S(t)W$; therefore, it holds that $h(S(\tau)\mathcal{W}, \overline{U}) \leq \sup_{0 \leq t \leq T} C_0 e^{-kt} = C_0$. Meanwhile, when $\tau \geq T$, $S(\tau)\mathcal{W} = \bigcup_{0 \leq t \leq T} S(t+\tau)W \subset S(\tau - T) \bigcup_{0 \leq t \leq T} S(t+T)W \subset S(\tau - T)W$; therefore, $h(S(\tau)\mathcal{W}, \overline{U}) \leq C_0 e^{kT} e^{-k\tau}$. These show that $\overline{U}$ attracts $\mathcal{W}$ exponentially. □

We now assume that the phase space $\mathcal{X}$ is a compact subset of X. As noticed above, $(S(t), \mathcal{X}, X)$ has the global attractor $\mathcal{A}$.

A subset $\mathcal{M} \subset \mathcal{X}$ is called an exponential attractor of $(S(t), \mathcal{X}, X)$ if $\mathcal{M}$ enjoys the following properties:

1. $\mathcal{M}$ is a compact set of X such that $\mathcal{A} \subset \mathcal{M} \subset \mathcal{X}$ with finite fractal dimension $d_F(\mathcal{M})$ in X (e.g., see [14, p. 366]).
2. $\mathcal{M}$ is a positively invariant set of $S(t)$, i.e., $S(t)\mathcal{M} \subset \mathcal{M}$, $0 \leq t < \infty$.
3. $\mathcal{M}$ attracts $\mathcal{X}$ exponentially in the sense that

(3.2) $$h(S(t)\mathcal{X}, \mathcal{M}) \leq C_0 e^{-kt}, \qquad 0 \leq t < \infty$$

with some constant $C_0 > 0$ and some exponent $k > 0$.

By some abuse of terminology, $\mathcal{M}$ is called an exponential attractor, but $\mathcal{M}$ may not be an invariant set, and hence may not be an attractor in the precise sense defined above.

4 Contraction semigroups Consider a dynamical system $(S(t), \mathcal{X}, X)$, where $\mathcal{X}$ is a closed bounded subset of a universal space X. We assume that, for some $t^* > 0$, $S(t^*)$ is a contraction mapping from $\mathcal{X}$ into itself, namely

(4.1) $$\|S(t^*)U - S(t^*)V\| \leq \delta \|U - V\|, \qquad U, V \in \mathcal{X}$$

with some constant $0 < \delta < 1$. When a semigroup $S(t)$ satisfies this condition, the semigroup is called a contraction semigroup.

By the fixed point theorem of Banach, there exists a unique fixed point $\overline{U}$ of $S = S(t^*)$ in $\mathcal{X}$. Furthermore, it holds that

$$\|S^n U - \overline{U}\| = \|S^n U - S^n \overline{U}\| \leq \delta^n \|U - \overline{U}\| \leq R\delta^n, \quad U \in \mathcal{X}$$

for all integers $n \geq 0$, where $R = \delta(\mathcal{X})$ is the diameter of $\mathcal{X}$.

We then verify that $\overline{U}$ is an equilibrium of $S(t)$. Indeed, for any $t > 0$, $SS(t)\overline{U} = S(t^* + t)\overline{U} = S(t)S\overline{U} = S(t)\overline{U}$; that is, $S(t)\overline{U}$ is also a fixed point of S; therefore, $S(t)\overline{U} = \overline{U}$. Furthermore, let $t = nt^* + \tau$ with integer $n \geq 0$ and $0 \leq \tau < t^*$. Then,

$$\|S(t)U - \overline{U}\| = \|S^n S(\tau)U - \overline{U}\| \leq R\delta^{\frac{t-\tau}{t^*}} \leq R\delta^{\frac{t-t^*}{t^*}} = R\delta^{-1} e^{-(\frac{1}{t^*}\log \delta^{-1})t}, \quad U \in \mathcal{X}.$$

Thus we have proved the following result.

Theorem 4.1. *Let the phase space $\mathcal{X}$ be closed and bounded in X. Let, for some $t^* > 0$, (4.1) be satisfied with $0 < \delta < 1$. Then, $(S(t), \mathcal{X}, X)$ has a unique equilibrium $\overline{U}$ which attracts $\mathcal{X}$ exponentially.*

5 Compact Perturbation of Contraction Semigroup Consider now a dynamical system $(S(t), \mathcal{X}, X)$, where the phase space $\mathcal{X}$ is a compact set of X. We assume that, for some $t^* > 0$, $S(t^*)$ is decomposed as

$$S(t^*) = S = S_0 + K. \tag{5.1}$$

Here, K is a mapping from $\mathcal{X}$ into a second Banach Z which is compactly embedded in X and satisfies the Lipschitz condition

$$\|KU - KV\|_Z \leq L\|U - V\|_X, \quad U, V \in \mathcal{X}. \tag{5.2}$$

The operator S_0 is a contraction mapping such that

$$\|S_0U - S_0V\|_X \leq \delta\|U - V\|_X, \qquad U, V \in \mathcal{X} \tag{5.3}$$

with $0 \leq \delta < \frac{1}{2}$. The conditions (5.2) and (5.3) are called the compact squeezing property of $S(t^*)$.

Then, the compact squeezing property implies the existence of exponential attractors for the discrete dynamical system $(S^n, \mathcal{X}, X)$.

Theorem 5.1. *Let (5.2) and (5.3) be satisfied with $0 \leq \delta < \frac{1}{2}$. Let θ be any exponent such that $0 < \theta < \frac{1-2\delta}{2L}$. Then there exists an exponential attractor $\mathcal{M}_\theta^*$, $\mathcal{A}^* \subset \mathcal{M}_\theta^* \subset \mathcal{X}$, with the following properties:*

1. *$\mathcal{M}_\theta^*$ is a compact subset of X with finite fractal dimension*
$$d_F(\mathcal{M}_\theta^*) \leq \frac{\log K_\theta}{\log \frac{1}{a_\theta}};$$
2. *$S\mathcal{M}_\theta^* \subset \mathcal{M}_\theta^*$;*
3. *$h(S^n\mathcal{X}, \mathcal{M}_\theta^*) \leq Ra_\theta^n$ for all integers $n \geq 0$.*

Here, $R = \delta(\mathcal{X})$ is a diameter of $\mathcal{X}$, $0 < a_\theta < 1$ is an exponent given by $a_\theta = 2\{\delta + L\theta\}$ and K_θ is the minimal number of balls with radii θ which cover the unit ball $\overline{B}^Z(0; 1)$ in X.

Proof. For the proof of this theorem, we can argue as in [5] (cf. also [13, Theorem 2.1]).

For $n = 0, 1, 2, \ldots$, let us define inductively a finite covering of $S^n\mathcal{X}$ such that

$$S^n\mathcal{X} \subset \bigcup_{i=1}^{K_\theta^n} \overline{B}(W_{n,i};\, Ra^n), \quad \text{where } a = a_\theta, \tag{5.4}$$

with centers $W_{n,i} \in S^n\mathcal{X}$, $1 \leq i \leq K_\theta^n$.

For $n = 0$, it is clear that

$$S^0\mathcal{X} = \mathcal{X} \subset \overline{B}(W_0; R)$$

with an arbitrarily fixed $W_0 \in \mathcal{X}$.

Assume that the covering (5.4) is defined for $n-1$. Then we have

$$S^n\mathcal{X} = S(S^{n-1}\mathcal{X}) \subset \bigcup_{i=1}^{K_\theta^{n-1}} S\big(\overline{B}(W_{n-1,i};\, Ra^{n-1}) \cap S^{n-1}\mathcal{X}\big).$$

So it suffices to cover each $S\big(\overline{B}(W_{n-1,i}; Ra^{n-1}) \cap S^{n-1}\mathcal{X}\big)$ by K_θ-closed balls with a radius Ra^n.

¿From (5.2) we see that

$$K\big(\overline{B}(W_{n-1,i};\, Ra^{n-1}) \cap S^{n-1}\mathcal{X}\big) \subset \overline{B}^Z(KW_{n-1,i};\, LRa^{n-1}).$$

Moreover, by the compactness of closed bounded balls of Z in X, the last ball can be covered by K_θ-closed balls of X in such a way that

$$\overline{B}^Z(KW_{n-1,i};\, LRa^{n-1}) \subset \bigcup_{j=1}^{K_\theta} \overline{B}(\widetilde{W}_{n-1,i,j};\, \theta LRa^{n-1})$$

with centers $\widetilde{W}_{n-1,i,j} \in X$, $1 \le j \le K_\theta$ and a radius θLRa^{n-1}. Therefore we obtain that

$$K\big(\overline{B}(W_{n-1,i};\, Ra^{n-1}) \cap S^{n-1}\mathcal{X}\big) \subset \bigcup_{j=1}^{K_\theta} \overline{B}(\widetilde{W}_{n-1,i,j};\, \theta LRa^{n-1}). \tag{5.5}$$

Here we are allowed to assume that

$$K\big(\overline{B}(W_{n-1,i};\, Ra^{n-1}) \cap S^{n-1}\mathcal{X}\big) \cap \overline{B}(\widetilde{W}_{n-1,i,j};\, \theta LRa^{n-1}) \neq \emptyset$$

for every j, since, if not for some j's, we can exclude balls centered at those points from the covering. Hence, there exist K_θ-vectors $W_{n-1,i,j}$ such that

$$W_{n-1,i,j} \in \overline{B}(W_{n-1,i}; Ra^{n-1}) \cap S^{n-1}\mathcal{X},$$
$$KW_{n-1,i,j} \in \overline{B}(\widetilde{W}_{n-1,i,j};\, \theta LRa^{n-1}).$$

In particular,

$$\overline{B}(\widetilde{W}_{n-1,i,j};\, \theta LRa^{n-1}) \subset \overline{B}(KW_{n-1,i,j};\, 2\theta LRa^{n-1}). \tag{5.6}$$

Let us now verify that

$$S\big(\overline{B}(W_{n-1,i};\, Ra^{n-1}) \cap S^{n-1}\mathcal{X}\big) \subset \bigcup_{j=1}^{K_\theta} \overline{B}(SW_{n-1,i,j};\, Ra^n). \tag{5.7}$$

In fact, if $U \in \overline{B}(W_{n-1,i};\, Ra^{n-1}) \cap S^{n-1}\mathcal{X}$, then from (5.5) and (5.6), there is some j such that $KU \in \overline{B}(KW_{n-1,i,j}; 2\theta LRa^{n-1})$. From (5.1)-(5.3) it is verified that

$$\begin{aligned}\|SU - SW_{n-1,i,j}\|_X &\le \delta\|U - W_{n-1,i,j}\|_X + \|KU - KW_{n-1,i,j}\|_X \\ &\le 2\delta Ra^{n-1} + 2L\theta Ra^{n-1} = Ra^n.\end{aligned}$$

Hence (5.7) has been shown.

Since the covering (5.7) can be constructed for each $1 \le i \le K_\theta^{n-1}$, the desired covering (5.4) for n is obtained by locating central points as

$$\{W_{n,i};\, 1 \le i \le K_\theta^n\} = \{SW_{n-1,i,j};\, 1 \le i \le K_\theta^{n-1},\, 1 \le j \le K_\theta\} \subset S^n\mathcal{X}.$$

Let

$$\mathcal{P} = \{W_{n,i};\, 0 \le n < \infty,\, 1 \le i \le K_\theta^n\}$$

be a collection of all central points of the covering (5.4). And set $\mathcal{M}_\theta^*$ in such a way that

$$\mathcal{M}_\theta^* = \overline{\bigcup_{n=0}^{\infty} S^n\mathcal{P}}.$$

Then $\mathcal{M}_\theta^*$ is shown to be an exponential attractor for $(S^n, \mathcal{X}, X)$. As the proof is the same as that of [13, Theorem 2.1], we may omit it. □

Let us return to the continuous dynamical system $(S(t), \mathcal{X}, X)$. For proving the existence of exponential attractors, we need to assume in addition to (5.2) and (5.3) the Lipschitz condition

$$\|S(t)U - S(s)V\|_X \le C(|t-s| + \|U-V\|_X), \quad t,\, s \in [0, t^*],\;\; U,\, V \in \mathcal{X} \tag{5.8}$$

with some constant $C > 0$. Then by the similar argument as in [13, Theorem 3.1], we can immediately conclude the following result.

Theorem 5.2. *Let* (5.1)-(5.3) *be satisfied with* $0 \le \delta < \frac{1}{2}$, *and let* (5.8) *be satisfied. Then, for any* $0 < \theta < \frac{1-2\delta}{2L}$, *there exists an exponential attractor* $\mathcal{M}_\theta$, $\mathcal{A} \subset \mathcal{M}_\theta \subset \mathcal{X}$, *of fractal dimension* $d_F(\mathcal{M}_\theta) \le d_F(\mathcal{M}_\theta^*) + 1$. *And* $\mathcal{M}_\theta$ *attracts* $S(t)\mathcal{X}$ *at the rate*

$$h(S(t)\mathcal{X}, \mathcal{M}_\theta) \le R a_\theta^{-1} e^{-(\frac{1}{t^*}\log a_\theta^{-1})t}, \qquad 0 \le t < \infty.$$

Here, $R = \delta(\mathcal{X})$ *is a diameter of* $\mathcal{X}$, $\mathcal{M}_\theta^*$ *is the exponential attractor constructed in Theorem 5.1 for the discrete dynamical system and* $a_\theta = 2\{\delta + L\theta\}$.

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