# TANGENT AND NORMAL VECTORS TO FEASIBLE REGIONS WITH GEOMETRICALLY DERIVABLE SETS

Yoshiyuki Sekiguchi and Wataru Takahashi

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Abstract. We give exact formulae for tangent cones and regular normal cones to feasible sets. We impose geometrical derivability on the sets used in constraint systems, which is weaker than Clarke regularity. Necessary optimality conditions with regular normal vectors can be stronger than those with general normal vectors. An example of the situation is given.

**1 Introduction** For constrained optimization problems, tangent and normal vectors to feasible sets play a crucial role to obtain necessary optimality conditions. Let us consider the following problem: minimize $f(x)$ subject to $x \in \mathcal{D}_0$, where $\mathcal{D}_0 = \{x \in X : F(x) = 0\}$, $X$, $Y$ are Banach spaces and $F\colon X \to Y$ is a continuously Fréchet differentiable mapping.

Liusternik [14] showed that if $\bar{x}$ is a local minimum and the regularity condition that $\nabla F(\bar{x})$ is surjective is satisfied, a multiplier rule holds; there exists $y^* \in Y^*$ such that $\nabla f(\bar{x}) = \nabla F(\bar{x})^* y^*$, where $\nabla F(\bar{x})^*$ is the adjoint operator of $\nabla F(\bar{x})$. This is a consequence from the fact that (i) $\nabla f(\bar{x})u \geq 0$ for all tangent vectors $u$ to $\mathcal{D}_0$ at $\bar{x}$ ($\nabla f(\bar{x})$ is orthogonal to them in this case) and (ii) the whole of tangent vectors coincides with the kernel of $\nabla F(\bar{x})$. For a set $C \subset X$, a vector $u \in X$ is said to be *tangent* to $C$ at $\bar{x} \in C$ if there exist $t_n \to +0$ and $c_n \in C$ such that $u = \lim_{n\to\infty} t_n^{-1}(c_n - \bar{x})$. The set $T_C(\bar{x})$ of all such vectors forms a cone which is called the *tangent cone* to $C$ at $\bar{x}$. In the above case, $T_{\mathcal{D}_0}(\bar{x})$ happens to be a subspace. Generally, representations of tangent vectors to feasible sets depend on their structures.

We often encounter more complicated constraint systems, like ones with inequality constraints. Such systems can be written in the form $\mathcal{D} = \{x \in A : F(x) \in B\}$, where $A \subset X$ and $B \subset Y$ are closed sets [8], [12]. For $\bar{x} \in \mathcal{D}$, we can represent a *subset* of $T_{\mathcal{D}}(\bar{x})$ explicitly if some regularity condition on a set-valued mapping associated with $\mathcal{D}$ is satisfied. It is called *metric regularity* and corresponds to a stability property of the constraint system under perturbations on the set $B$; see e.g. [7]. Metric regularity has its root in the Banach open mapping theorem as Liusternik's theorem does, and is equivalent to the Robinson's condition, $0 \in \operatorname{core}[F(\bar{x}) - \nabla F(\bar{x})(A - \bar{x}) - B]$, which is also equivalent to the Mangasarian-Fromovitz constraint qualification in certain cases [17], [1], [4].

We obtain a necessary condition for optimization problems on this set $\mathcal{D}$ by the fact (i) and a representation of tangent vectors. More precisely we give a description of $T_{\mathcal{D}}(\bar{x})$, more information we have on local minima. So, full descriptions of $T_{\mathcal{D}}(\bar{x})$ are desired and have been obtained imposing some condition on the set $A$ and $B$, which is called the *Clarke regularity* and is automatically satisfied by convex sets [1]. However we can easily find sets which are not Clarke regular as seen below.

In this paper, we obtain a full description of $T_{\mathcal{D}}(\bar{x})$ under a milder assumption, *geometrical derivability* of sets, instead of the Clarke regularity. There, we do not assume

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even differentiability of constraint operators. In addition, we give a sufficient condition for metric regularity of a set-valued mapping associated with such nonsmooth constraints by using tangent cones to $A$ and $B$.

By the full description of tangent cones, we characterize the set of *regular normal vectors* to $\mathcal{D}$ in the case that the spaces are finite dimensional and $A$, $B$ are geometrically derivable. It is smaller than the set of normal vectors in general sense if $\mathcal{D}$ is not Clarke regular. Necessary optimality conditions with regular normal vectors are given, which can be stronger than those with general normal vectors. We give an example of this situation.

**2 Preliminaries** Let $X$, $Y$ be real Banach spaces, $B_X$ and $B_Y$ the closed unit balls in $X$ and $Y$, respectively, $C$ a closed subset of $X$ and $\bar{x}$ a point in it. In the following we use $t \searrow 0$ and $x \xrightarrow{C} \bar{x}$ to mean that $t$ converges to 0 with $t > 0$ and $x$ converges to $\bar{x}$ with $x \in C$, respectively. For a mapping $S$ from $X$ into $2^Y$, we define two kinds of limits by

$$y \in \limsup_{x \to \bar{x}} S(x) \Leftrightarrow \text{ there exist } x_n \to \bar{x}, y_n \to y \text{ with } y_n \in S(x_n);$$
$$y \in \liminf_{x \to \bar{x}} S(x) \Leftrightarrow \text{ for all } x_n \to \bar{x}, \text{ there exists } y_n \to y \text{ with } y_n \in S(x_n).$$

Considering $S(t) = t^{-1}(C - \bar{x})$, we can see the *tangent cone* $T_C(\bar{x})$ to $C$ at $\bar{x}$ given in the introduction is written by

$$T_C(\bar{x}) = \limsup_{t \searrow 0} \frac{C - \bar{x}}{t}.$$

We also define the *derivable cone* by

$$\widetilde{T}_C(\bar{x}) = \liminf_{t \searrow 0} \frac{C - \bar{x}}{t}.$$

Similarly considering $S(t, x) = t^{-1}(C - x)$, the *regular tangent cone* is defined by

$$\widehat{T}_C(\bar{x}) = \liminf_{\substack{t \searrow 0 \\ x \xrightarrow{C} \bar{x}}} \frac{C - x}{t}.$$

For detailed discussion on the definition, we refer the reader to [20]. We observe by the definition that

$$\widehat{T}_C(\bar{x}) \subset \widetilde{T}_C(\bar{x}) \subset T_C(\bar{x}).$$

We say $C$ is *geometrically derivable* at $\bar{x}$, when $\widetilde{T}_C(\bar{x}) = T_C(\bar{x})$ and $C$ is *Clarke regular* (*regular*) at $\bar{x}$, when $\widehat{T}_C(\bar{x}) = T_C(\bar{x})$. Obviously if $C$ is regular at $\bar{x}$, $C$ is geometrically derivable there. In addition every convex set is regular at all its points [2].

Note geometrical derivability is properly weaker than regularity. Consider $X = R^3$ and $C = \{(x, y, z) : x \leq 0, -1 \leq y \leq 1, z = 0\} \cup \{(x, y, z) : x \geq 0, -1 \leq y \leq 1, z \leq 0\}$. Then each cone at $\bar{u} = (0, 0, 0)$ is as follows:

$$T_C(\bar{u}) = \widetilde{T}_C(\bar{u}) = \{(x, y, z) : x \leq 0, z = 0\} \cup \{(x, y, z) : x \geq 0, z \leq 0\};$$
$$\widehat{T}_C(\bar{u}) = \{(x, y, z) : z = 0\}.$$

Since $\widehat{T}_C(\bar{x}) \neq T_C(\bar{x})$, this set is not regular but geometrically derivable at $\bar{u}$.

We now define generalized derivatives of extended real valued functions. For a lower semicontinuous function $f : X \to \bar{R} := [-\infty, \infty]$ and a point $\bar{x}$ with $-\infty < f(\bar{x}) < \infty$ , the *subderivative* $df(\bar{x}) : X \to \bar{R}$ is defined by

$$df(\bar{x})(\bar{w}) = \liminf_{\substack{\tau \searrow 0 \\ w \to \bar{w}}} \frac{f(\bar{x} + \tau w) - f(\bar{x})}{\tau}$$

and the *regular subderivative* is

$$\widehat{d}f(\bar{x})(\bar{w}) = \lim_{\delta \searrow 0} \left( \limsup_{\substack{x \xrightarrow{f} \bar{x} \\ \tau \searrow 0}} \left[ \inf_{\|w - \bar{w}\| \le \delta} \frac{f(x + \tau w) - f(x)}{\tau} \right] \right).$$

They have the following geometrical descriptions:

$$\operatorname{epi} df(\bar{x}) = T_{\operatorname{epi} f}(\bar{x}, f(\bar{x})),$$
$$\operatorname{epi} \widehat{d}f(\bar{x}) = \widehat{T}_{\operatorname{epi} f}(\bar{x}, f(\bar{x})).$$

Detailed discussion is given in [18], [20].

We can define derivatives of set-valued mappings; see e.g. [20] . For a mapping $S$ from $X$ into $2^Y$ and $(\bar{x}, \bar{y}) \in \operatorname{gph} S$, the *derivative* $DS(\bar{x}|\bar{y})$ of $S$ at $\bar{x}$ for $\bar{y}$ is a mapping from $X$ into $2^Y$ defined by

$$DS(\bar{x}|\bar{y})(w) = \{v \in Y : (w, v) \in T_{\operatorname{gph} S}(\bar{x}, \bar{y})\},$$

the *proto-derivative* $\widetilde{D}S(\bar{x}|\bar{y})$ is

$$\widetilde{D}S(\bar{x}|\bar{y})(w) = \{v \in Y : (w, v) \in \widetilde{T}_{\operatorname{gph} S}(\bar{x}, \bar{y})\}$$

and the *regular derivative* $\widehat{D}S(\bar{x}|\bar{y})$ is

$$\widehat{D}S(\bar{x}|\bar{y})(w) = \{v \in Y : (w, v) \in \widehat{T}_{\operatorname{gph} S}(\bar{x}, \bar{y})\}.$$

The proto-derivative was defined in [19]. If the proto-derivative coincides with the derivative for all $w \in X$, we say $S$ is *proto-differentiable* at $\bar{x}$ for $\bar{y}$. When $S$ is single-valued, we use $DS(\bar{x})$ for $DS(\bar{x}|S(\bar{x}))$. Other derivatives have the same notation.

In particular a function is proto-differentiable at $\bar{x}$ and the derivative is single-valued if that is locally Lipschitz and directionally differentiable at $\bar{x}$;

$$\lim_{t \searrow 0} \frac{F(\bar{x} + tw) - F(\bar{x})}{t} = dF(\bar{x})(w)$$

exists for all $w$ in $X$. Here we do not assume linearity of $dF(\bar{x})$.

**3 Tangent vectors to feasible sets** We investigate the description of tangent cones to feasible sets with operator constraint systems.

As explained in the introduction, Liusternik showed that in infinite dimensional spaces, the tangent cone to a level set of an operator is equal to the kernel of the derivative at the point if the derivative is surjective [14]. This is proved by using the Banach open mapping theorem and an iterative procedure [6], [7], [21].

The Banach open mapping theorem guarantees that the following assertions are equivalent for a bounded linear operator $L$ from $X$ into $Y$:

(1) $L$ is surjective;

(2) there exists $k > 0$ such that

$$d(x, L^{-1}(y)) \leq kd(y, Lx),$$

for any $(x, y) \in X \times Y$;

(3) there exists $k > 0$ such that

$$B_Y \subset L(kB_X).$$

Their essential properties have been understood in many ways. In our discussion, *metric regularity* is the key property. Here we also present an equivalent property which is a version of open mapping theorems by Graves; see e.g. [6], [7], [20].

**Definition.** Let $S$ be a mapping from $X$ into $2^Y$ and $(\bar{x}, \bar{y}) \in \operatorname{gph} S$.

(1) $S$ is said to be *metrically regular* at $\bar{x}$ for $\bar{y}$ with constant $k$ if there exists $k > 0$ such that

$$d(x, S^{-1}(y)) \leq kd(y, S(x)),$$

for $(x, y)$ close to $(\bar{x}, \bar{y})$. The infimum of such $k$ is called the *rate of metric regularity* and denoted by $\operatorname{reg} S(\bar{x}|\bar{y})$;

(2) $S$ is said to be *linearly open* at $\bar{x}$ for $\bar{y}$ with a rate $k$ if there exist $k > 0$ and a neighborhood $O$ of $\bar{y}$ such that

$$[S(x) + \operatorname{int} tB_Y] \cap O \subset S(x + \operatorname{int} ktB_X),$$

for $x$ close to $\bar{x}$ and any $t > 0$.

The infimum of $k$ in each definition is equal.

Note $\operatorname{reg} S(\bar{x}|\bar{y})$ is finite if and only if $S$ is metrically regular at $\bar{x}$ for $\bar{y}$. Suppose $S$ is single-valued. Then the first property means that the distance between $x_0$ and the solution set to $S(x) = y_0$ can be estimated by looking at the gap between $S(x_0)$ and $y_0$. Moreover the rate is uniform around $(\bar{x}, \bar{y})$. With the second, we can find a solution to $S(x) = y_0$ whenever $y_0$ lies near $\bar{y}$ and the solutions stay in a ball centered at $x_0$ with radius proportional to the distance between $y_0$ and $S(x_0)$. Similarly the rate is uniform.

By using these properties, the Liusternik-Graves theorem can be stated as follows:

**Theorem 1 ([6]).** *Let $F$ be a mapping from $X$ into $Y$ which is strictly differentiable at $\bar{x} \in X$ with the derivative $\nabla F(\bar{x})$. Then*

$$\operatorname{reg} F(\bar{x}|F(\bar{x})) = \operatorname{reg} \nabla F(\bar{x})(0|0).$$

*Thus $F$ is metrically regular at $\bar{x}$ for $F(\bar{x})$ if and only if $DF(\bar{x})$ is surjective.*

If the derivative is surjective at $\bar{x}$, then $\operatorname{reg} F(\bar{x}|F(\bar{x}))$ is finite, by the Banach open mapping theorem and the theorem above. In this case metrical regularity of $F$ is assured and plays the role of an inverse function theorem. Linear openness, an equivalent property, represents an open mapping theorem for nonlinear mappings. In addition the surjectivity of the derivative is a minimum requirement for them. For detailed discussion we refer the reader to [5], [6], [7], [15], [20].

Let $F$ be an operator from $X$ into $Y$ and $A \subset X$, $B \subset Y$ closed sets. We consider the following mathematical programming:

$$\text{minimize } f(x) \text{ subject to } x \in \mathcal{D},$$

where

$$\mathcal{D} = \{x \in A : F(x) \in B\}. \tag{1}$$

We construct a set-valued mapping $S$ to study properties of $\mathcal{D}$ by

$$S(x) = \begin{cases} F(x) - B, & x \in A, \\ \emptyset, & x \notin A. \end{cases} \tag{2}$$

Then $\mathcal{D} = S^{-1}(0)$. If $0 \in S(\bar{x})$ and $S$ is metrically regular at $\bar{x}$ for 0, there exists $k > 0$ such that

$$d(x, S^{-1}(u)) \le kd(F(x), B + u),$$

whenever $x \in A$ and $(x, u)$ close to $(\bar{x}, 0)$.

Several sufficient conditions for this property have been obtained. If $S$ is strictly differentiable and $A$, $B$ are convex, the constraint qualification by Robinson [17] is a necessary and sufficient condition.

**Theorem 2 ([4], [17]).** *Suppose $F$ is a strictly differentiable mapping at $\bar{x} \in \mathcal{D}$ with the derivative $\nabla F(\bar{x})$, where $A$, $B$ are closed convex sets and $\mathcal{D}$, $S$ are defined in* (1), (2). *Then* $0 \in \operatorname{core}[\nabla F(\bar{x})(A - \bar{x}) - (B - F(\bar{x}))]$ *if and only if $S$ is metrically regular at $\bar{x}$ for* 0.

The condition above is reduced to the surjectivity of $\nabla F(\bar{x})$, if $A = X$ and $B = \{F(\bar{x})\}$. In this section, we assume $F$ is locally Lipschitz. If $X$ is reflexive and $B$ is epi-Lipschitzian, Jourani and Thibault [11] gave a sufficient condition using Fréchet subdifferentials. Other sufficient conditions are investigated, for example in [1], [6], [7], [9], [15].

Here we give a sufficient condition with regular tangent cones. The proof uses Theorem 2.3. in [11] and in this paper, Fréchet subdifferentials in infinite dimensional spaces appear only here and their constructions need a lot of discussions. So, we refer the reader to [11], [16] for the definition and give only a simple proof. We mention that reflexivity of the spaces is necessary to ensure that some calculus rules of Fréchet subdifferentials hold. For a set $D \subset X$ and $\bar{x} \in D$, we say $D$ is *epi-Lipschitz* at $\bar{x}$ if there exist $u \in X$ and $\delta > 0$ such that

$$D \cap (\bar{x} + \delta B_X) + t(u + \delta B_X) \subset D$$

for all $t \in (0, \delta)$; see [20]. The polar $C^\circ$ to a cone $C \subset X$ is a weak* closed convex cone in $X^*$ defined by $x^* \in C^\circ \Leftrightarrow \langle x^*, c \rangle \le 0$ for all $c \in C$.

**Proposition 3.** *Let $A \subset X$, $B \subset Y$ be closed sets, $\mathcal{D}$, $S$ defined in* (1), (2) *and $\bar{x} \in \mathcal{D}$. Suppose $X$, $Y$ are reflexive, $F$ is locally Lipschitz around $\bar{x}$, $B$ is epi-Lipschitz at $F(\bar{x})$. Then the condition that*

$$\widehat{D}F(\bar{x})\widehat{T}_A(\bar{x}) - \widehat{T}_B(F(\bar{x})) = Y$$

*is sufficient for metric regularity of $S$ at $\bar{x}$ for* 0.

*Proof.* We are to show if $y^* \in \partial d(F(\bar{x}), B)$ and $0 \in D^*F(\bar{x})(y^*) + K\partial d(\bar{x}, A)$, we have $y^* = 0$. Here $\partial$ means the Fréchet subdifferential and $D^*$ means the coderivative generated by the Fréchet normal cone and $K$ is a Lipschitz constant of $F$ around $\bar{x}$. Now for all $y \in Y$, there exist $w \in \widehat{T}_A(\bar{x})$, $b \in \widehat{T}_B(F(\bar{x}))$ and $v \in \widehat{D}F(\bar{x})(w)$ such that $y = v - b$. By the definition of the derivative, this means $(w, y + b) \in \widehat{T}_{\mathrm{gph}\,F}(\bar{x}, F(\bar{x}))$. Let $y^* \in \partial d(F(\bar{x}), B)$ with $0 \in D^*F(\bar{x})(y^*) + K\partial d(\bar{x}, A)$. Since $\partial d(F(\bar{x}), B) \subset \widehat{T}_B(F(\bar{x}))^\circ$ and $\partial d(\bar{x}, A) \subset \widehat{T}_A(\bar{x})^\circ$, we have $y^* \in \widehat{T}_B(F(\bar{x}))^\circ$ and $0 \in D^*F(\bar{x})(y^*) + x^*$ for some $x^* \in \widehat{T}_A(\bar{x})^\circ$. Hence this implies $(-x^*, -y^*) \in \widehat{T}_{\mathrm{gph}\,F}((\bar{x}, F(\bar{x})))^\circ$. Thus we have

$$0 \leq \langle (x^*, y^*), (w, y + b) \rangle = \langle x^*, w \rangle + \langle y^*, y \rangle + \langle y^*, b \rangle \leq \langle y^*, y \rangle.$$

Since $y$ is arbitrary, we have $y^* = 0$ and conclude $S$ is metrically regular at $\bar{x}$ for 0. □

Liusternik used an inverse function theorem under surjectivity condition and derived a formula for tangent cones to feasible sets with equality constraints and then obtained a multiplier rule from polar relations. For more complicated sets, Borwein [1] obtained the following full description of tangent cones:

**Theorem 4 ([1]).** *Let $A \subset X$ and $B \subset Y$ closed sets and $\mathcal{D}$, $S$ defined in* (1), (2) *and $\bar{x} \in \mathcal{D}$. Suppose $F$ is strictly differentiable at $\bar{x}$ and $S$ is metrically regular at $\bar{x}$ for 0 and $A$, $B$ are Clarke regular at $\bar{x}$ and $F(\bar{x})$ respectively. Then $\mathcal{D}$ is Clarke regular at $\bar{x}$ and*

$$T_{\mathcal{D}}(\bar{x}) = \{w \in T_A(\bar{x}) : \nabla F(\bar{x})w \in T_B(F(\bar{x}))\}.$$

We now give a full description of tangent cones with geometrical derivability, which is weaker than Clarke regularity. The proof is based on the argument in [1].

**Theorem 5.** *Let $A$, $B$ be closed sets and $\mathcal{D}$, $S$ be defined in* (1), (2). *Suppose $F$ is locally Lipschitz continuous at $\bar{x} \in \mathcal{D}$. Then*

$$T_{\mathcal{D}}(\bar{x}) \subset \{w \in T_A(\bar{x}) : \widetilde{D}F(\bar{x})(w) \subset T_B(F(\bar{x}))\}.$$

*If $S$ is metrically regular at $\bar{x}$ for* 0*,*

$$T_{\mathcal{D}}(\bar{x}) \supset \{w \in \widetilde{T}_A(\bar{x}) : DF(\bar{x})(w) \subset \widetilde{T}_B(F(\bar{x}))\}.$$

*Moreover if $A$ and $B$ are geometrically derivable at $\bar{x}$ and $F(\bar{x})$ respectively and $F$ is proto-differentiable at $\bar{x}$, then $\mathcal{D}$ is geometrically derivable at $\bar{x}$ and*

$$T_{\mathcal{D}}(\bar{x}) = \{w \in T_A(\bar{x}) : DF(\bar{x})(w) \subset T_B(F(\bar{x}))\}.$$

*Proof.* We show the first inclusion. Let $w \in T_{\mathcal{D}}(\bar{x})$. We can find $t_n \searrow 0$ and $\{c_n\} \subset \mathcal{D}$ such that $t_n^{-1}(c_n - \bar{x}) \to w$. Since $c_n$ also belongs to $A$, we have $w \in T_A(\bar{x})$. Let $v \in \widetilde{D}F(\bar{x})(w)$. Then for above $\{t_n\}$ there exists $\{u_n\} \subset X$ such that

$$\frac{(u_n, F(u_n)) - (\bar{x}, F(\bar{x}))}{t_n} \to (w, v).$$

Here we obtain

$$\frac{c_n - u_n}{t_n} = \frac{c_n - \bar{x}}{t_n} + \frac{\bar{x} - u_n}{t_n} \to w - w = 0.$$

Let $l$ be a Lipschitz constant of $F$ around $\bar{x}$. Since $c_n$ and $u_n$ converge to $\bar{x}$,

$$\begin{aligned}\|\frac{F(c_n)-F(\bar{x})}{t_n}-v\| &\le \|\frac{F(c_n)-F(u_n)}{t_n}\|+\|\frac{F(u_n)-F(\bar{x})}{t_n}-v\| \\ &\le l\cdot\frac{\|c_n-u_n\|}{t_n}+\|\frac{F(u_n)-F(\bar{x})}{t_n}-v\|\to 0.\end{aligned}$$

Since $F(c_n)\in B$, this means $v\in T_B(F(\bar{x}))$.

Next we assume $S$ is metrically regular at $\bar{x}$ for 0. Let $w$ satisfy that $w\in\widetilde{T}_A(\bar{x})$ and $DF(\bar{x})(w)\subset\widetilde{T}_B(F(\bar{x}))$. For $v\in DF(\bar{x})(w)$, the definition ensures the existence of $t_n\searrow 0$ and $\{u_n\}\subset X$ such that

$$\frac{(u_n,F(u_n))-(\bar{x},F(\bar{x}))}{t_n}\to(w,v).$$

Since $w\in\widetilde{T}_A(\bar{x})$, there exists $\{a_n\}\subset A$ such that $t_n^{-1}(a_n-\bar{x})\to w$. Here we have $t_n^{-1}(u_n-a_n)\to 0$. In addition, since $v\in\widetilde{T}_B(F(\bar{x}))$, there exists $\{b_n\}\subset B$ such that $t_n^{-1}(b_n-F(\bar{x}))\to v$, and hence $t_n^{-1}(b_n-F(u_n))\to 0$. Metric regularity of $S$ implies that for some $k>0$ and a Lipschitz constant $l$ of $F$,

$$\begin{aligned}d(a_n,S^{-1}(0)) &\le kd(F(a_n),B)\le k\|F(a_n)-b_n\| \\ &\le k\|F(a_n)-F(u_n)\|+k\|F(u_n)-b_n\| \\ &< k(l+1)\|a_n-u_n\|+k\|F(u_n)-b_n\|.\end{aligned}$$

Thus we obtain $\{c_n\}\subset\mathcal{D}$ such that

$$\frac{\|a_n-c_n\|}{t_n}\| < k(l+1)\cdot\frac{\|a_n-u_n\|}{t_n}+k\cdot\frac{\|F(u_n)-b_n\|}{t_n}\to 0.$$

Therefore we have

$$\frac{c_n-\bar{x}}{t_n}=\frac{c_n-a_n}{t_n}+\frac{a_n-\bar{x}}{t_n}\to 0+w.$$

This means $w\in T_{\mathcal{D}}(\bar{x})$.

The equality is obvious. The geometrical derivability of $\mathcal{D}$ is shown in the same way as the second inclusion. ☐

**Corollary 6.** *Let $A$, $B$ be closed convex sets and $\mathcal{D}$, $S$ be defined in* (1), (2). *Suppose $F$ is locally Lipschitz continuous and directionally differentiable at $\bar{x}\in\mathcal{D}$. If $S$ is metrically regular at $\bar{x}$ for* 0*, then*

$$T_{\mathcal{D}}(\bar{x})=\{w\in T_A(\bar{x}):dF(\bar{x})(w)\in T_B(F(\bar{x}))\}.$$

Note that we have shown tangent relations for a nonsmooth operator. In [1], it is assumed that $F$ is strictly differentiable but $A$ and $B$ are merely closed. In addition the equality relations are found in [4] if $A$, $B$ are convex and $F$ is strictly differentiable.

We now present a necessary condition for optimization problems with operator constraints. If a function $f$ is locally Lipschitz, the regular derivatives are equal to Clarke generalized derivatives [18]. Here we give a direct proof for a necessary condition using regular derivatives for abstract constraint problems. We define the indicator function $I_C$ of a set $C$ by

$$I_C(x)=\begin{cases}0, & x\in C,\\ \infty, & x\notin C.\end{cases}$$

**Proposition 7.** *Let $C$ be a closed set in $X$, $g$ be a locally Lipschitz function from $X$ into $R$. Suppose $\bar{x}$ is a local minimum of $g$ on $C$. Then*

$$\widehat{d}g(\bar{x})(w) \geq 0$$

*for all $w \in T_C(\bar{x})$.*

*Proof.* Let $f = g + I_C$. Then $f(\bar{x})$ is finite and we have $df(\bar{x})(w) \geq 0$ for all $w \in X$, by the analytical description of the subderivative. It is sufficient to show $df(\bar{x})(w) \leq \widehat{d}g(\bar{x})(w)$ whenever $w \in T_C(\bar{x})$. This is equivalent to the assertion that if $w \in T_C(\bar{x})$,

$$(w, \widehat{d}g(\bar{x})(w)) \in \operatorname{epi} df(\bar{x}) = \limsup_{t \searrow 0} \frac{\operatorname{epi} f - (\bar{x}, f(\bar{x}))}{t}.$$

Since $w \in T_C(\bar{x})$, there exist $t_n \searrow 0$ and $\{c_n\} \subset C$ such that $t_n^{-1}(c_n - \bar{x}) \to w$. For this $t_n$, there exists $\{(x_n, \alpha_n)\} \subset \operatorname{epi} g$ such that

$$\frac{(x_n, \alpha_n) - (\bar{x}, g(\bar{x}))}{t_n} \to (w, \widehat{d}g(\bar{x})(w)).$$

It is implied that $c_n \to \bar{x}$ and $x_n \to \bar{x}$. For these sequences, we have

$$\frac{c_n - x_n}{t_n} = \frac{c_n - \bar{x}}{t_n} + \frac{\bar{x} - x_n}{t_n} \to w - w = 0.$$

Let $l$ be a Lipschitz constant around $\bar{x}$, then

$$g(c_n) - \alpha_n \leq g(c_n) - g(x_n) \leq l\|c_n - x_n\|,$$

and hence $g(c_n) \leq \alpha_n + l\|c_n - x_n\|$. Let $\beta_n$ be the right hand side of the above inequality. Since $c_n \in C$, it follows that $(c_n, \beta_n) \in \operatorname{epi} f$. In addition we have

$$\frac{\beta_n - g(\bar{x})}{t_n} = \frac{\alpha_n - g(\bar{x})}{t_n} + l \cdot \frac{\|c_n - x_n\|}{t_n} \to \widehat{d}g(\bar{x})(w).$$

Therefore

$$\frac{(x_n, \beta_n) - (\bar{x}, f(\bar{x}))}{t_n} = \frac{(x_n, \beta_n) - (\bar{x}, g(\bar{x}))}{t_n} \to (w, \widehat{d}g(\bar{x})(w)).$$

□

**Corollary 8.** *Let $A \subset X$, $B \subset Y$ be closed sets and $\mathcal{D}$, $S$ defined in (1) and (2). Suppose $f$ is a locally Lipschitz function from $X$ into $R$, $F$ is locally Lipschitz at $\bar{x} \in \mathcal{D}$ and $S$ is metrically regular at $\bar{x}$ for 0. If $\bar{x}$ is a local optimal for the following minimization problem:*

$$\text{minimize } f(x) \text{ subject to } x \in \mathcal{D}.$$

*Then*

$$\widehat{d}f(\bar{x})(w) \geq 0,$$

*for all $w \in \widetilde{T}_A(\bar{x}) \cap DF(\bar{x})^{-1}\widetilde{T}_B(F(\bar{x}))$.*

**4 Normal vectors to feasible sets** In this section, we assume $X$ and $Y$ are Euclidean spaces. For constructions of normal cones in infinite dimensional spaces, we refer the reader to [7], [16]. Our notation follows [20].

Let $C$ be a closed subset of $X$ and $\bar{x} \in C$. The *regular normal cone* $\widehat{N}_C(\bar{x})$ to $C$ at $\bar{x}$ is

$$\widehat{N}_C(\bar{x}) = \left\{ x^* \in X : \limsup_{\substack{x \xrightarrow{C} \bar{x} \\ x \neq \bar{x}}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}.$$

The *normal cone* $N_C(\bar{x})$ to $C$ at $\bar{x}$ is defined by

$$N_C(\bar{x}) = \limsup_{x \xrightarrow{C} \bar{x}} \widehat{N}_C(x) = \{x^* \in X : \exists x_k \xrightarrow{C} \bar{x}, x_k^* \to x^* \text{ with } x_k^* \in \widehat{N}_C(x_k)\}.$$

These objects have nice relationships with tangent cones:

$$\widehat{T}_C(\bar{x}) = N_C(\bar{x})^\circ, \quad \widehat{N}_C(\bar{x}) = T_C(\bar{x})^\circ.$$

In finite dimensional cases, it is known that $\nabla F(\bar{x})\widehat{T}_A(\bar{x}) - \widehat{T}_B(F(\bar{x})) = Y$ is sufficient for metric regularity of $S$ defined in (2) if $F$ is strictly differentiable mapping and $A$, $B$ are closed. Note that we do not need the epi-Lipschitz property of $B$ here. For any set $C$, we denote the closure of the convex hull of $C$ by $\overline{\operatorname{conv}}\, C$.

**Theorem 9.** *Let $A \subset X$, $B \subset Y$ be closed sets and $\mathcal{D}$ defined in (1) and $\bar{x} \in \mathcal{D}$. Suppose $A$, $B$ are geometrically derivable at $\bar{x}$ and $F(\bar{x})$ respectively, $F$ is strictly differentiable at $\bar{x}$, $\nabla F(\bar{x})\widehat{T}_A(\bar{x}) - \widehat{T}_B(F(\bar{x})) = Y$ and the condition $\overline{\operatorname{conv}}[T_A(\bar{x}) \cap \nabla F(\bar{x})^{-1} T_B(F(\bar{x}))] = \overline{\operatorname{conv}}\, T_A(\bar{x}) \cap \nabla F(\bar{x})^{-1}\, \overline{\operatorname{conv}}\, T_B(F(\bar{x}))$ holds. Then*

$$\widehat{N}_{\mathcal{D}}(\bar{x}) = \{\nabla F(\bar{x})^* y^* + z^* : y^* \in \widehat{N}_B(F(\bar{x})), z^* \in \widehat{N}_A(\bar{x})\}.$$

*Proof.* By Theorem 5, we have $T_{\mathcal{D}}(\bar{x}) = T_A(\bar{x}) \cap \nabla F(\bar{x})^{-1} T_B(F(\bar{x}))$. We know $\widehat{N}_{\mathcal{D}}(\bar{x}) = T_{\mathcal{D}}(\bar{x})^\circ = (\overline{\operatorname{conv}}\, T_{\mathcal{D}}(\bar{x}))^\circ$. $\widehat{N}_{\mathcal{D}}(\bar{x}) = (\overline{\operatorname{conv}}\, T_A(\bar{x}) \cap \nabla F(\bar{x})^{-1}\, \overline{\operatorname{conv}}\, T_B(F(\bar{x})))^\circ = \nabla F(\bar{x})^* \,(\overline{\operatorname{conv}}\, T_B(F(\bar{x})))^\circ + (\overline{\operatorname{conv}}\, T_A(\bar{x}))^\circ$. This gives the result. □

The following corollary is exact sum rules for tangent and regular normal cones, which have been given in the case that the sets are assumed to be Clarke regular; see e.g. [20] .

**Corollary 10.** *Let $C = C_1 \cap C_2$ for closed sets $C_1, C_2 \subset X$ and $\bar{x} \in C$. Suppose $C_1$, $C_2$ are geometrically derivable at $\bar{x}$ and $\widehat{T}_{C_1}(\bar{x}) - \widehat{T}_{C_2}(\bar{x}) = X$. Then we have*

$$T_C(\bar{x}) = T_{C_1}(\bar{x}) \cap T_{C_2}(\bar{x}),$$

*moreover if* $\overline{\operatorname{conv}}[T_{C_1}(\bar{x}) \cap T_{C_2}(\bar{x})] = \overline{\operatorname{conv}}\, T_{C_1}(\bar{x}) \cap \overline{\operatorname{conv}}\, T_{C_2}(\bar{x})$,

$$\widehat{N}_C(\bar{x}) = \widehat{N}_{C_1}(\bar{x}) + \widehat{N}_{C_2}(\bar{x}).$$

*Proof.* We obtain the results immediately by substituting $F$ in Theorem 5 and Theorem 9 with the identity mapping. □

**Remark.** Let $X = R^2$, $C_1 = \{(x, y) : x \geq 0, 0 \leq y \leq x\}$, $C_2 = \{(x, y) : x \geq 0, 1/4x \leq y \leq 1/2x\} \cup \{(x, y) : x \leq 0, y = 1/2x\}$ and $\bar{u} = (0, 0)$. Then $\widehat{T}_{C_1}(\bar{u}) = C_1$, $\widehat{T}_{C_2}(\bar{u}) = \{(x, y) : x \geq 0, y = 1/2x\}$, and hence $\widehat{T}_{C_1}(\bar{u}) - \widehat{T}_{C_2}(\bar{u}) = R^2$ but the condition on the convex hulls is not satisfied. Now we have $\widehat{N}_{C_1}(\bar{u}) = \{(x, y) : x \leq 0, y \leq -x\}$, $\widehat{N}_{C_2}(\bar{u}) = \{(x, y) : x \leq 0, y = -2x\}$ and $\widehat{N}_{C_1 \cap C_2}(\bar{u}) = \{(x, y) : y + 4x \leq 0, y + 2x \leq 0\}$. However $\widehat{N}_{C_1 \cap C_2}(\bar{u}) \supsetneq \widehat{N}_{C_1}(\bar{u}) + \widehat{N}_{C_2}(\bar{u})$.

**Corollary 11.** *Let $A \subset X$, $B \subset Y$ be closed sets and $\mathcal{D}$, defined in* (1) *and $\bar{x} \in \mathcal{D}$. Suppose $f$ is a function from $X$ into $R$ which is strictly differentiable at $\bar{x}$, $A$, $B$ are geometrically derivable at $\bar{x}$ and $F(\bar{x})$ respectively and $F$ is strictly differentiable at $\bar{x}$. If $\bar{x}$ is a local optimal for the following minimization problem:*

$$\textit{minimize} \;\; f(x) \;\; \textit{subject to} \; x \in \mathcal{D}$$

*and $\nabla F(\bar{x})\widehat{T}_A(\bar{x}) - \widehat{T}_B(F(\bar{x})) = Y$ and $\overline{\mathrm{conv}}[T_A(\bar{x}) \cup \nabla F(\bar{x})^{-1} T_B(\bar{x})] = \overline{\mathrm{conv}}\, T_A(\bar{x}) \cup \nabla F(\bar{x})^{-1}\, \overline{\mathrm{conv}}\, T_B(F(\bar{x}))$, we have*

$$-\nabla F(\bar{x}) \in \{\nabla F(\bar{x})^* y^* + z^* : y^* \in \widehat{N}_B(F(\bar{x})), z^* \in \widehat{N}_A(\bar{x})\}.$$

General normal cones are often used to obtain necessary optimality conditions. However regular normal vectors can give us more precise information on local minima. For example, we consider the following minimization problem:

$$\begin{aligned} &\text{minimize } f(x,y,z) = x^4 - 2x^2 + x + z \\ &\text{subject to } F(x,y,z) = (y, z(x+z)) \in [-1,1] \times \{0\}; \\ &\qquad (x,y,z) \in \{(x,y,z) : -\frac{1}{2} \le x \le 0, z = 0\} \\ &\qquad\qquad \cup \{(x,y,z) : x \ge 0, z \le 0\}. \end{aligned}$$

The feasible set $\mathcal{D}$ is equal to $\{(x,y,z) : x \ge 0, -1 \le y \le 1, z = -x\} \cup \{(x,y,z) : x \ge -1/2, -1 \le y \le 1, z = 0\}$. Now we have $\nabla f(x,y,z) = (4x^3 - 4x + 1, 2y, 1)$ and

$$\begin{aligned} N_{\mathcal{D}}(0,0,0) &= \{t(0,0,1) : t \in R\} \cup \{t(1,0,1) : t \in R\}; \\ \widehat{N}_{\mathcal{D}}(0,0,0) &= \{t(0,0,1) : t \ge 0\}. \end{aligned}$$

Then it can be seen that

$$-\nabla f(0,0,0) = (-1,0,-1) \in N_{\mathcal{D}}(0,0,0).$$

However $-\nabla f(0,0,0) \notin \widehat{N}_{\mathcal{D}}(0,0,0)$ and thus $(0,0,0)$ is not a local minimum.

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(Y. Sekiguchi) Department of Mathematical and Computing Sciences, Graduate School of Information Sciences and Engineering, Tokyo Institute of Technology 2-12-1, Oh-okayama, Meguro, Tokyo 152-8552, Japan

*E-mail address:* `Yoshiyuki.Sekiguchi@is.titech.ac.jp`

(W. Takahashi) Department of Mathematical and Computing Sciences, Graduate School of Information Sciences and Engineering, Tokyo Institute of Technology 2-12-1, Oh-okayama, Meguro, Tokyo 152-8552, Japan

*E-mail address:* `wataru@is.titech.ac.jp`