ON POWERS OF P-POSINORMAL OPERATORS

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Abstract. Let $p > 0$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-posinormal* if $(TT^*)^p \leq \mu(T^*T)^p$ for some $\mu > 1$. In this paper, we prove that if T is p-posinormal then T^n is also p-posinormal for all positive integer n. Moreover, we prove that if $T = U|T|$ is p-posinormal for $0 < p < 1$, the Aluthge transform $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is $(p+\frac{1}{2})$-posinormal.

1. Introduction. Let $\mathcal{H}$ be a separable complex Hilbert space and $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on $\mathcal{H}$. An operator T can be decomposed into $T = U|T|$ where U is partial isometry and $|T|$ is the square root of T^*T with $N(U) = N(|T|)$, and this kernel condition $N(U) = N(|T|)$ uniquely determines U and $|T|$ in the polar decomposition of T. In this paper, $T = U|T|$ denotes the polar decomposition satisfying the kernel condition $N(U) = N(|T|)$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is *positive*, $T \geq 0$, if $(Tx, x) \geq 0$ for all $x \in \mathcal{H}$, and *posinormal* if there exists a positive $P \in \mathcal{L}(\mathcal{H})$ such that $TT^* = T^*PT$. Here, P is called an *interrupter* of T. Let $p > 0$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if

$$(TT^*)^p \leq (T^*T)^p,$$

and *p-posinormal* if

$$(TT^*)^p \leq \mu(T^*T)^p$$

for some $\mu > 1$. It is clear that 1-hyponormal and 1-posinormal are hyponormal and posinormal, respectively. It is well known that a p-posinormal(resp. p-hyponormal) operator is a q-posinormal(resp. q-hyponormal) operator for $0 < q \leq p$ by Löwner-Heinz Inequality. But the converse is not true in general(see [1],[3] and [8]).

Hyponormal and p-hyponormal operators have been studied many authors and it is known that hyponormal operators have many interesting properties similar to those normal operators([1],[2],[6],[7] and [11]). In [10], Rhaly studied spectral properties of posinormal operators and gave many useful examples. In particular, Itoh [8] introduced p-posinormality and proved new characterizations of p-posinormal operators.

In this paper, we consider new properties as an extension of p-hyponormal operators using the generalized Aluthge transform. In this paper, we prove that if T is p-posinormal then T^n is also p-posinormal for all positive integer n. Moreover, we prove that if $T = U|T|$ is p-posinormal for $0 < p < 1$, the Aluthge transform $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is $(p + \frac{1}{2})$-posinormal. We should note that our main tool is Furuta Inequality.

2. Main results. The classes of p-hyponormal has been defined as an extension of hyponormal, and it has been studied by many authors([1],[2] and [6]). For an operator $T = U|T|$, defines $\tilde{T}$ as follows:

$$\tilde{T_{s,t}} = |T|^s U|T|^t$$

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for $s, t > 0$ which is called the *generalized Aluthge transform* of T. Especially, $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is called the *Aluthge transform*. The generalized Aluthge transform is very useful tool in the study of p-hyponormal oprerators. In this section we will study p-posinormal operators using their generalized Aluthge transform.

We have to state the order-preserving operator inequality because it is a base of our discussion in the below.

Furuta Inequality. [5] Let $A \geq B \geq 0$. Then for all $r > 0$,

(1) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

(2) $(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

for $p \geq 0, q \geq 1$ with $(1+r)q \geq p+r$.

Theorem 1. *Let $T = U|T|$ be the polar decomposition of a p-posinormal operator for $0 < p \leq 1$. Then the following assertions hold:*

(1) $\tilde{T}_{s,t} = |T|^sU|T|^t$ is $\frac{p+\min\{s,t\}}{s+t}$-posinormal for $s, t > 0$ such that $\max\{s, t\} \geq p$.

(2) $\tilde{T}_{s,t}$ is posinormal for $0 < s, t \leq p$.

Proof. Suppose that

$$|T^*|^{2p} \leq \mu|T|^{2p} \tag{2.1}$$

for some $\mu > 1$.

(1) Let $A = \mu|T|^{2p}$ and $B = |T^*|^{2p}$. Then

$$\begin{aligned}(\tilde{T}^*_{s,t}\tilde{T}_{s,t})^{\frac{p+\min\{s,t\}}{s+t}} &= (|T|^tU^*|T|^{2s}U|T|^t)^{\frac{p+\min\{s,t\}}{s+t}} \\ &= U^*(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{p+\min\{s,t\}}{s+t}}U \\ &= \mu^{-\frac{s}{p}\frac{p+\min\{s,t\}}{s+t}}U^*(B^{\frac{t}{2p}}A^{\frac{s}{p}}B^{\frac{t}{2p}})^{\frac{p+\min\{s,t\}}{s+t}}U \\ &\geq \mu^{-\frac{s}{p}\frac{p+\min\{s,t\}}{s+t}}U^*B^{\frac{p+\min\{s,t\}}{p}}U \text{ by Furuta Inequality} \\ &= \mu^{-\frac{s}{p}\frac{p+\min\{s,t\}}{s+t}}|T|^{2(p+\min\{s,t\})}\end{aligned} \tag{2.2}$$

since $\frac{s+t}{p+\min\{s,t\}} \geq 1$ and $(1+\frac{t}{p})\frac{s+t}{p+\min\{s,t\}} \geq \frac{s}{p} + \frac{t}{p}$. And

$$\begin{aligned}(\tilde{T}_{s,t}\tilde{T}^*_{s,t})^{\frac{p+\min\{s,t\}}{s+t}} &= (|T|^sU|T|^{2t}U^*|T|^s)^{\frac{p+\min\{s,t\}}{s+t}} \\ &= (|T|^s|T^*|^{2t}|T|^s)^{\frac{p+\min\{s,t\}}{s+t}} \\ &= \mu^{-\frac{s}{p}\frac{p+\min\{s,t\}}{s+t}}(A^{\frac{s}{2p}}B^{\frac{t}{p}}A^{\frac{s}{2p}})^{\frac{p+\min\{s,t\}}{s+t}} \\ &\leq \mu^{\frac{t-s}{p}\frac{p+\min\{s,t\}}{s+t}}A^{\frac{p+\min\{s,t\}}{p}} \text{ by Furuta Inequality} \\ &= \mu^{\frac{t-s}{p}\frac{p+\min\{s,t\}}{s+t}}|T|^{2(p+\min\{s,t\})}\end{aligned} \tag{2.3}$$

since $\frac{s+t}{p+\min\{s,t\}} \geq 1$ and $(1+\frac{s}{p})\frac{s+t}{p+\min\{s,t\}} \geq \frac{t}{p} + \frac{s}{p}$. From (2.2) and (2.3), we have

$$(\tilde{T}_{s,t}\tilde{T}^*_{s,t})^{\frac{p+\min\{s,t\}}{s+t}} \leq \mu^{\frac{t}{p}\frac{p+\min\{s,t\})}{s+t}}(\tilde{T}_{s,t}\tilde{T}^*_{s,t})^{\frac{p+\min\{s,t\}}{s+t}},$$

that is, $\tilde{T}_{s,t}$ is $\frac{p+\min\{s,t\}}{s+t}$ -posinormal for $s, t > 0$ such that $\max\{s, t\} \geq p$.

(2) Applying Löwner-Heinz Inequality to (2.1),

$$|T^*|^{2s} \leq \mu^{\frac{s}{p}}|T|^{2s} \tag{2.4}$$

and

$$|T^*|^{2t} \le \mu^{\frac{t}{p}}|T|^{2t} \tag{2.5}$$

hold for any $0 < s, t \le p$. From (2.4) and (2.5), we have

$$\begin{aligned} \tilde{T}_{s,t}^*\tilde{T}_{s,t} &= |T|^t U^* |T|^{2s} U |T|^t \\ &\ge \mu^{-\frac{s}{p}}|T|^t U^*|T^*|^{2s}U|T|^t \\ &= \mu^{-\frac{s}{p}}|T|^{2(s+t)} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} \tilde{T}_{s,t}\tilde{T}_{s,t}^* &= |T|^s U |T|^{2t} U^* |T|^s \\ &\le |T|^s \mu^{\frac{t}{p}}|T|^{2t}|T|^s \\ &= \mu^{\frac{t}{p}}|T|^{2(s+t)}. \end{aligned} \tag{2.7}$$

So (2.6) and (2.7) ensure

$$\tilde{T}_{s,t}\tilde{T}_{s,t}^* \le \mu^{\frac{s+t}{p}}\tilde{T}_{s,t}^*\tilde{T}_{s,t},$$

and hence $\tilde{T}_{s,t}$ is posinormal.

We note that Theorem 1 yeilds the next result by putting $s = t = \frac{1}{2}$.

Corollary 2. Let $T = U|T|$ be p-posinormal operator for $0 < p < 1$. Then
(1) $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is $(p+\frac{1}{2})$-posinormal for $0 < p < \frac{1}{2}$.
(2) $\tilde{T}$ is posinormal for $\frac{1}{2} \le p < 1$.

Aluthge-Wang [2], Furuta-Yanagida [6] proved that if T is p-hyponormal, then T^n is $\frac{p}{n}$-hyponormal for all positive integers n, respectively. The next result is a version for p-posinormal operators of their result [6; Theorem 1] and Ito [7].

Theorem 3. *Let T be a p-posinormal for some $0 < p < 1$, that is $(TT^*)^p \le \mu^2(T^*T)^p$ for some $\mu > 1$. Then*
(1) $(T^{n*}T^n)^{\frac{p+1}{n}} \ge \mu^{-\frac{p+1}{p}(n-1)}(T^*T)^{p+1}$, and
(2) $(TT^*)^{p+1} \ge \mu^{-\frac{p+1}{p}(n-1)}(T^nT^{n*})^{\frac{p+1}{n}}$
hold for all positive integer n.

Proof. Put

$$A_n = (T^{n*}T^n)^{\frac{p}{n}} = |T^n|^{\frac{2p}{n}}$$

and

$$B_n = (T^nT^{n*})^{\frac{p}{n}} = |T^{n*}|^{\frac{2p}{n}}$$

for all positive integer n.

(1) We will use induction to establish the inequality

$$(T^{n*}T^n)^{\frac{p+1}{n}} \ge \mu^{-\frac{p+1}{p}(n-1)}(T^*T)^{p+1} \text{ holds for } n = k. \tag{2.8}$$

(2.8) is clear for $n = 1$. Assume that (2.8) holds for $k = n$. Since

$$A_k = (T^{*k}T^k)^{\frac{p}{k}} \ge \mu^{-(k-1)}(T^*T)^p \ge \mu^{-(k+1)}B_1, \tag{2.9}$$

it follows that

$$(2.10)\qquad \begin{aligned}({T^{k+1}}^*T^{k+1})^{\frac{p+1}{k+1}} &= (U^*|T^*|{T^k}^*T^k|T^*|U)^{\frac{p+1}{k+1}} \\ &= U^*(|T^*|{T^k}^*T^k|T^*|)^{\frac{p+1}{k+1}}U \\ &= U^*(B_1^{\frac{1}{2p}}A_k^{\frac{k}{p}}B_1^{\frac{1}{2p}})^{\frac{p+1}{k+1}}U \\ &\geq \mu^{-\frac{k}{p}\frac{p+1}{k+1}(k+1)}U^*(B_1^{\frac{1}{2p}}B_1^{\frac{k}{p}}B_1^{\frac{1}{2p}})^{\frac{p+1}{k+1}}U \text{ by Furuta Inequality} \\ &= \mu^{-\frac{p+1}{p}k}(T^*T)^{p+1}.\end{aligned}$$

Whence the proof of (1) is complete.

(2) Similarly to (1), (2) is clear for $n = 1$. Assume that

$$(TT^*)^{p+1} \geq \mu^{-\frac{p+1}{p}(k-1)}(T^k{T^k}^*)^{\frac{p+1}{k}}$$

holds. Then

$$(2.11)\qquad A_1 = (T^*T)^p \geq \mu^{-2}(TT^*)^p \geq \mu^{-(k+1)}(T^k{T^k}^*)^{\frac{p}{k}} = \mu^{-(k+1)}B_k.$$

Hence we have

$$(2.12)\qquad \begin{aligned}(T^{k+1}{T^{k+1}}^*)^{\frac{p+1}{k+1}} &= U(|T|T^k{T^k}^*|T|)^{\frac{p+1}{k+1}}U^* \\ &= U(A_1^{\frac{1}{2p}}B_k^{\frac{k}{p}}A_1^{\frac{1}{2p}})^{\frac{p+1}{k+1}}U^* \\ &\leq \mu^{\frac{k}{p}\frac{p+1}{k+1}(k+1)}U(A_1^{\frac{1}{2p}}A_1^{\frac{k}{p}}A_1^{\frac{1}{2p}})^{\frac{p+1}{k+1}}U^* \text{ by Furuta Inequality} \\ &= \mu^{\frac{p+1}{p}k}|T^*|^{2(p+1)}.\end{aligned}$$

So, $(TT^*)^{p+1} \geq \mu^{-\frac{p+1}{p}n}(T^nT^{n*})^{\frac{p+1}{n+1}}$ holds for all positive integer n.

From Theorem 3, we have the next result.

Corollary 4. *If T is p-posinormal, then T^n is $\frac{p}{n}$-posinormal for all positive integer n.*

Proof. Let $(TT^*)^p \leq \mu^2(T^*T)^p$ for some $\mu > 1$. Then, by Theorem 3,

$$(T^{n*}T^n)^{\frac{p}{n}} \geq \mu^{\frac{p}{p+1}}(T^*T)^p \geq \mu^{\frac{p}{p+1}}\mu^{-2}(TT^*)^p \geq \mu^{-2n}(T^nT^{n*})^{\frac{p}{n}}.$$

So, T^n is $\frac{p}{n}$-posinormal.

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