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WEAK AND STRONG SOLUTIONS OF ISOTROPIC VISCOELASTIC EQUATIONS WITH LONG NONLINEAR MEMORY

J-S. Hwang and S. Nakagiri[1]

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Abstract. The equations of isotropic viscoelastic materials with long nonlinear memory are studied. The model equations are described by second order Volterra integro-differential equation having nonlinear kernels. Based on the variational methods, we have proved the fundamental results on existence, uniqueness and regularity of weak and strong solutions for the equations.

1 Introduction In this paper, we study the equations of isotropic viscoelastic materials with long memory. The term of long memory is represented by Volterra integral terms. Let Ω be a domain in $\mathbf{R}^n$ with smooth boundary Γ and let us denote $Q = (0,T) \times \Omega,\ \Sigma = (0,T) \times \Gamma$ for $T > 0$. The linearized vibrating equation of isotropic viscoelastic materials occupying a domain Ω is given by

$$\frac{\partial^2 y}{\partial t^2} - \alpha \Delta y - \int_0^t k(t-s)\Delta y(s) ds = f \quad \text{in} \quad Q, \tag{1.1}$$

where k is a scalar fading memory kernel (see, e.g., Dafermos [2], Dautray and Lions [3, pp. 660-662]). A large number of authors has studied the linear and nonlinear versions of viscoelastic equations such as

$$\frac{\partial^2 y}{\partial t^2} - \alpha \Delta y - \int_0^t k(t-s)\Delta y(s) ds + f(y) + g(\frac{\partial y}{\partial t}) = 0 \quad \text{in} \quad Q, \tag{1.2}$$

where f and g are nonlinear functions (cf. Dafermos and Nohel [5], Renardy, Hrusa and Nohel [10], Cavalcanti and Oquendo [1], Rivera, Naso and Vegni [11] and others). In the above references the asymptotic behaviour and the existence of global attractors for semilinear equations are studied extensively. Further, the quasilinear viscoelastic model equations such as

$$\frac{\partial^2 y}{\partial t^2} - \mathrm{div}(g(\nabla y)) + \int_0^t k(t-s)\mathrm{div}(h(\nabla y(s))) ds = f \quad \text{in} \quad Q, \tag{1.3}$$

are studied in Engler [4] and Qin and Ni [9] among others, where g and h are sufficiently smooth nonlinear functions.

In this paper we study the following partially linearized Volterra integro-differential equation for the nonlinear isotropic viscoelastic equations

$$\frac{\partial^2 y}{\partial t^2} - \alpha \Delta y - \int_0^t k(t-s)\mathrm{div}\Big(\frac{\nabla y(s)}{\sqrt{1+|\nabla y(s)|^2}}\Big) ds = f \quad \text{in} \quad Q, \tag{1.4}$$

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under the Dirichlet boundary condition and the initial conditions

$$y = 0 \quad \text{on} \;\; \Sigma, \tag{1.5}$$

$$y(0,x) = y_0(x), \qquad \frac{\partial y}{\partial t}(0,x) = y_1(x) \qquad \text{in} \;\; \Omega. \tag{1.6}$$

Partially linearized equation (1.4) can be considered when the derivation of y is sufficiently small in the instantaneous elasticities, but need not sufficiently small in the memory effects. In this point of view, the Laplacian in (1.4) is taken into account of the approximation of the original quasilinear diffusion term which is acting in the memory part. We note that the quasilinear equations studied in [4] and [9] and others do not cover the equation (1.4). For the problem (1.4)-(1.6) we prove the fundamental results on existence, uniqueness and regularity of weak solutions as obtained in Engler [4] by using variational method. In addition, to study the related control and identification problems, it needs more regular solutions than weak solutions. For the purpose we shall prove stronger results on existence, uniqueness and more improved regularity of solutions, called strong solutions, corresponding to more regular data. The well-posedness result for strong solutions is not obtained in [4]. Especially to prove the regularity, we have used the double regularization method orginally given in Lions and Magenes [8], but in the advanced procedure for the Volterra equation (1.4).

2 Main results Throughout this paper we suppose that Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary Γ. We denote $Q = (0,T) \times \Omega$, $\Sigma = (0,T) \times \Gamma$ for $T > 0$. We study the following Dirichlet boundary value problem for the viscoelastic equations with long nonlinear memory:

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} - \alpha \Delta y - \displaystyle\int_0^t k(t-s) \operatorname{div}\Big(\frac{\nabla y(s)}{\sqrt{1+|\nabla y(s)|^2}} \Big) ds = f \quad \text{in} \;\; Q, \\ y = 0 \quad \text{on} \;\; \Sigma, \\ y(0,x) = y_0(x), \quad \dfrac{\partial y}{\partial t}(0,x) = y_1(x) \qquad \text{in} \;\; \Omega, \end{cases} \tag{2.1}$$

where $\alpha > 0$, k is a scalar kernel function, f is an external forcing term and y_0, y_1 are given initial functions. We shall give the notations used throughout this paper. The scalar product and norm on $[L^2(\Omega)]^n$ are also denoted by (ϕ,ψ) and $|\phi|$. Then the scalar product $(\phi,\psi)_{H_0^1(\Omega)}$ and the norm $\|\phi\|$ of $H_0^1(\Omega)$ are given by $(\nabla\phi,\nabla\psi)$ and $\|\phi\| = (\nabla\phi,\nabla\phi)^{\frac{1}{2}}$, respectively. Let $D(\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$. The scalar product and norm on $D(\Delta)$ are denoted by $(\phi,\psi)_{D(\Delta)} = (\Delta\phi,\Delta\psi)$ and $\|\phi\|_{D(\Delta)} = |\Delta\phi|$, respectively. The duality pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$ is denoted by $\langle\phi,\psi\rangle$. Related to the nonlinear term in (2.1), we define the function $G : \mathbf{R}^n \to \mathbf{R}^n$ by $G(x) = \dfrac{x}{\sqrt{1+|x|^2}}, x \in \mathbf{R}^n$. Then it is easily verified that

$$|G(x) - G(y)| \le 2|x-y|, \quad \forall x, y \in \mathbf{R}^n. \tag{2.2}$$

The nonlinear operator $G(\nabla\cdot) : H_0^1(\Omega) \to [L^2(\Omega)]^n$ is defined by

$$G(\nabla\phi)(x) = \frac{\nabla\phi(x)}{\sqrt{1+|\nabla\phi(x)|^2}}, \qquad \text{a.e.} \;\; x \in \Omega, \qquad \forall \phi \in H_0^1(\Omega). \tag{2.3}$$

By the definition of $G(\nabla\cdot)$ in (2.3), we have the following useful property on $G(\nabla\cdot)$:

$$|G(\nabla\phi)| \le |\nabla\phi|, \quad |G(\nabla\phi) - G(\nabla\psi)| \le 2|\nabla\phi - \nabla\psi|, \qquad \forall\phi,\psi \in H_0^1(\Omega). \tag{2.4}$$

The solution space $W(0,T)$ for weak solutions of (2.1) is defined by

$$W(0,T) = \{g|\ g \in L^2(0,T;H_0^1(\Omega)),\ g' \in L^2(0,T;L^2(\Omega)),\ g'' \in L^2(0,T;H^{-1}(\Omega))\}$$

endowed with the norm

$$\|g\|_{W(0,T)} = \left(\|g\|^2_{L^2(0,T;H_0^1(\Omega))} + \|g'\|^2_{L^2(0,T;L^2(\Omega))} + \|g''\|^2_{L^2(0,T;H^{-1}(\Omega))}\right)^{\frac{1}{2}},$$

where g' and g'' denote the first and second order distributive derivatives of g.

Definition 2.1. *A function y is said to be a weak solution of (2.1) if $y \in W(0,T)$ and y satisfies*

$$\begin{cases} \langle y''(\cdot),\phi\rangle + \alpha(\nabla y(\cdot),\nabla\phi) + \displaystyle\int_0^{\cdot} k(\cdot - s)(G(\nabla y(\cdot)),\nabla\phi)ds = (f(\cdot),\phi) \\ \qquad\qquad\qquad\qquad \text{for all } \ \phi \in H_0^1(\Omega) \text{ in the sense of } \mathcal{D}'(0,T), \\ y(0) = y_0 \in H_0^1(\Omega), \qquad y'(0) = y_1 \in L^2(\Omega). \end{cases} \tag{2.5}$$

The following theorem gives the fundamental results on existence, uniqueness and regularity of weak solutions of (2.1).

Theorem 2.1. *Assume that*

$$y_0 \in H_0^1(\Omega), \quad y_1 \in L^2(\Omega), \quad f \in L^2(0,T;L^2(\Omega)), \quad k(\cdot) \in C^1[0,T]. \tag{2.6}$$

Then the problem (2.1) has a unique weak solution $y \in W(0,T) \cap C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))$. Moreover, y has the following estimate

$$|\nabla y(t)|^2 + |y'(t)|^2 \le C(\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,T;L^2(\Omega))}), \quad \forall t \in [0,T],$$

where C is a constant depending only on α and $\|k\|_{C^1[0,T]}$.

Next we introduce the solution space $\tilde{W}(0,T)$ for strong solutions of (2.1) defined by

$$\tilde{W}(0,T) = \{g|\ g \in L^2(0,T;D(\Delta)),\ g' \in L^2(0,T;H_0^1(\Omega)),\ g'' \in L^2(0,T;L^2(\Omega))\}.$$

Definition 2.2. *A function y is said to be a strong solution of (2.1) if $y \in \tilde{W}(0,T)$,* $\mathrm{div}G(\nabla y) \in L^2(0,T;L^2(\Omega))$ *and y satisfies*

$$\begin{cases} y''(t) - \alpha\Delta y(t) - \displaystyle\int_0^t k(t-s)\mathrm{div}\ G(\nabla y(s))ds = f(t), \quad a.e.\ t \in [0,T], \\ y(0) = y_0 \in D(\Delta), \qquad y'(0) = y_1 \in H_0^1(\Omega). \end{cases} \tag{2.7}$$

The next theorem gives a well-posedness result for strong solutions of (2.1).

Theorem 2.2. *Assume that*

$$y_0 \in D(\Delta), \quad y_1 \in H_0^1(\Omega), \quad f \in H^1(0,T;L^2(\Omega)), \quad k(\cdot) \in C^1[0,T]. \tag{2.8}$$

Then the problem (2.1) has a unique strong solution $y \in \tilde{W}(0,T)$ which satisfies

$$y \in C([0,T];D(\Delta)) \cap C^1([0,T];H_0^1(\Omega)) \cap C^2([0,T];L^2(\Omega))$$

and the estimates

$$|\Delta y(t)|^2 + |\nabla y'(t)|^2 + |y''(t)|^2 \le C(\|y_0\|^2_{D(\Delta)} + \|y_1\|^2 + \|f\|^2_{H^1(0,T;L^2(\Omega))}), \quad \forall t \in [0,T],$$

where C is a constant depending only on α and $\|k\|_{C^1[0,T]}$.

3 Proof of main results Proof of Theorem 2.1. The existence of a weak solution y can be proved by applying the Galerkin method. Let $\{y_m\}_{m=1}^{\infty} \subset C^2([0,T]; H_0^1(\Omega)) \cap W(0,T)$ be a sequence of approximate solutions of (2.1) by the Galerkin's procedure as in Dautray and Lions [3] such as $y_m(0) \to y_0$ in $H_0^1(\Omega))$ and $y'_m(0) \to y_1$ in $L^2(\Omega))$ as $m \to \infty$. We shall derive a priori estimates of $y_m(t)$. It is easy to see that $y_m(t)$ satisfies

$$(y''_m(t), y'_m(t)) + \alpha(\nabla y_m(t), \nabla y'_m(t)) + (k * G(\nabla y_m)(t), \nabla y'_m(t)) = (f(t), y'_m(t)), \tag{3.1}$$

where $*$ denotes the convolution operation. Since $k \in C^1[0,T]$ by (2.6), by using

$$\begin{aligned}(k * G(\nabla y_m)(t), \nabla y'_m(t)) &= \frac{d}{dt}(k * G(\nabla y_m)(t), \nabla y_m(t)) \\ &-k(0)(G(\nabla y_m(t)), \nabla y_m(t)) - (k' * G(\nabla y_m)(t), \nabla y_m(t)),\end{aligned}$$

we see that (3.1) can be written as

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big[\alpha(\nabla y_m(t), \nabla y_m(t)) + |y'_m(t)|^2 + 2(k * G(\nabla y_m)(t), \nabla y_m(t))\Big] \\ &= (f(t), y'_m(t)) + k(0)(G(\nabla y_m(t)), \nabla y_m(t)) + (k' * G(\nabla y_m)(t), \nabla y_m(t)).\end{aligned} \tag{3.2}$$

Let $\epsilon > 0$ be a positive real number. We set $k_0 = \|k\|_{C[0,T]}$ and $k_1 = \|k'\|_{C[0,T]}$. First, by (2.4) and Schwarz inequality, we have

$$\begin{aligned}\Big|(2k * G(\nabla y_m)(t), \nabla y_m(t))\Big| &\leq 2k_0|\nabla y_m(t)|\int_0^t |\nabla y_m(s)|ds \\ &\leq \epsilon|\nabla y_m(t)|^2 + c(\epsilon)\int_0^t |\nabla y_m(s)|^2 ds\end{aligned} \tag{3.3}$$

for some $c(\epsilon) > 0$. In what follows, for simplicity of notations, we will omit the integral variables. For example, we shall write $\int_0^t |\nabla y_m|^2 ds$ instead of $\int_0^t |\nabla y_m(s)|^2 ds$. We also have by Schwarz inequality that

$$\begin{cases} \Big|\int_0^t 2k(0)(G(\nabla y_m), \nabla y_m)ds\Big| \leq 2k_0\int_0^t |\nabla y_m|^2 ds, \\ \Big|\int_0^t (2k' * G(\nabla y_m), \nabla y_m)ds\Big| \leq 2k_1\Big(\int_0^t |\nabla y_m|ds\Big)^2 \leq 2k_1 T\int_0^t |\nabla y_m|^2 ds, \\ \Big|\int_0^t 2(f, y'_m)ds\Big| \leq \|f\|^2_{L^2(0,T;L^2(\Omega))} + \int_0^t |y'_m|^2 ds. \end{cases} \tag{3.4}$$

By integrating (3.2) on $[0,t]$ and using the estimates (3.3) and (3.4), we can obtain the following inequality

$$\begin{aligned}&|y'_m(t)|^2 + \alpha|\nabla y_m(t)|^2 \\ \leq\ & |y'_m(0)|^2 + \alpha|\nabla y_m(0)|^2 + \epsilon|\nabla y_m(t)|^2 + \|f\|^2_{L^2(0,T;L^2(\Omega))} \\ +\ & (2k_0 + c(\epsilon) + 2k_1 T)\int_0^t |\nabla y_m|^2 ds + \int_0^t |y'_m|^2 ds.\end{aligned} \tag{3.5}$$

Here in (3.5), we note that $|y'_m(0)|^2 \leq c_1|y_1|^2$ and $|\nabla y_m(0)|^2 \leq c_2\|y_0\|^2$ for some $c_1, c_2 > 0$. If we choose $\epsilon = \frac{\alpha}{2}$ and set $C' = \max\Big\{(2k_0 + c(\frac{\alpha}{2}) + 2k_1 T), 1\Big\}$, then from (3.5) we arrive at

$$\begin{aligned}&|y'_m(t)|^2 + \frac{\alpha}{2}|\nabla y_m(t)|^2 \\ \leq\ & K(\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,T;L^2(\Omega))}) + C'\int_0^t (|\nabla y_m|^2 + |y'_m|^2)ds,\end{aligned} \tag{3.6}$$

where K is some positive constant. Thus it follows by applying the Bellman-Gronwall's inequality to (3.6) that

$$|\nabla y_m(t)|^2 + |y_m'(t)|^2 \le C(\|y_0\|^2 + |y_1|^2 + \|f\|^2_{L^2(0,T;L^2(\Omega))}), \quad \forall t \in [0,T] \tag{3.7}$$

for some $C > 0$. By (3.7) it is easily verified that $G(\nabla y_{m_k})$ is bounded in $L^\infty(0,T;[L^2(\Omega)]^n)$. Hence, we can extract a subsequence $\{y_{m_k}\}$ of $\{y_m\}$ and find $z \in W(0,T) \cap L^\infty(0,T;H_0^1(\Omega))$ and $F \in L^2(0,T;[L^2(\Omega)]^n)$ such that

$$\left\{\begin{array}{ll} \text{i)} & y_{m_k} \to z \ \text{ weakly-star in } \ L^\infty(0,T;H_0^1(\Omega)), \\ & \qquad\qquad \text{and weakly in } W(0,T), \\ \text{ii)} & G(\nabla y_{m_k}) \to F(\cdot) \text{ weakly-star in } \ L^\infty(0,t;[L^2(\Omega)]^n), \\ & \quad \text{and weakly in } \ L^2(0,t;[L^2(\Omega)]^n) \quad \text{for each } t \in [0,T], \end{array}\right. \tag{3.8}$$

as $k \to \infty$. Then by the similar manipulation given in the Dautray and Lions [3, pp. 564-566], it can be verified that the limit z of $\{y_{m_k}\}$ satisfies (2.1) in which the nonlinear term $G(\nabla y(\cdot))$ is replaced by $F(\cdot)$. To prove $F(\cdot) = G(\nabla z)$, we shall show $y_m(t) \to z(t)$ strongly in $H_0^1(\Omega)$ for all $t \in [0,T]$. Integrating (3.2) on $[0,t]$, we obtain

$$\begin{aligned} &\alpha|\nabla y_m(t)|^2 + |y_m'(t)|^2 \\ &= \alpha|\nabla y_m(0)|^2 + |y_m'(0)|^2 - 2(k * G(\nabla y_m)(t), \nabla y_m(t)) + 2\int_0^t (f, y_m')ds \\ &\quad + 2\int_0^t (k(0)G(\nabla y_m) + k' * G(\nabla y_m), \nabla y_m)ds. \end{aligned} \tag{3.9}$$

As shown in Hwang and Nakagiri [7, Proposition 2.1], for the limit z of $\{y_m\}$ we can derive the following energy equality

$$\begin{aligned} \alpha|\nabla z(t)|^2 + |z'(t)|^2 &= \alpha|\nabla y_0|^2 + |y_1|^2 - 2(k * F(t), \nabla z(t)) \\ &+ 2\int_0^t (k(0)F + k' * F, \nabla z)ds + 2\int_0^t (f, z')ds. \end{aligned} \tag{3.10}$$

From (3.9) and (3.10), via the same method in the proof of [6, Theorem 2.1], we can deduce that

$$y_m(t) \ \to \ z(t) \quad \text{strongly in } H_0^1(\Omega) \ \text{ for all } \ t \in [0,T]. \tag{3.11}$$

Therefore by the Lipschitz continuity of G in (2.4) and (3.11), it follows that

$$F(\cdot) = G(\nabla z). \tag{3.12}$$

Thus we have proved the existence of a weak solution z of (2.1). The uniqueness can be shown as follows. Let y_1 and y_2 be the weak solutions of (2.1). Set $\varphi = y_1 - y_2$. Then φ satisfies

$$\left\{\begin{array}{l} \varphi''(t) - \alpha\Delta\varphi(t) - \displaystyle\int_0^t k(t-s)\mathrm{div}\,(G(\nabla y_1(s) - G(\nabla y_2(s))ds = 0 \ \text{ on } \ [0,T], \\ y(0) = 0, \quad y'(0) = 0 \end{array}\right. \tag{3.13}$$

in the weak sense of (2.1). Since $|G(\nabla y_1) - G(\nabla y_2)| \le 2|\nabla\varphi|$ by (2.4), we can repeat the same calculations as in deriving the estimates (3.7) to have that $|\varphi'(t)|^2 + |\nabla\varphi(t)|^2 = 0$, $\forall t \in [0,T]$

and hence $|\varphi(t)| = 0,\ \forall t \in [0,T]$ by the imbedding inequality $|\phi| \le C|\nabla\phi|,\ \forall\phi \in H_0^1(\Omega)$. The proof of the regularity

$$z \in C([0,T]; H_0^1(\Omega)) \text{ and } z' \in C([0,T]; L^2(\Omega))$$

is quite similar to that given in Lions and Magenes [8, p. 279].

Proof of Theorem 2.2. We can proceed the proof as in the proof of Theorem 2.1, but it requires much more complicated calculations. Thus we divide the proof into four steps.

Step 1. Approximate solutions and a priori estimates.
Let $\{y_m\}_{m=1}^{\infty} \subset C^3([0,T]; D(\Delta)) \cap \tilde{W}(0,T)$ be a sequence of approximate solutions of (2.1), by $y_0 \in D(\Delta)$ and $y_1 \in H_0^1(\Omega)$), such as $y_m(0) \to y_0$ in $D(\Delta)$ and $y_m'(0) \to y_1$ in $H_0^1(\Omega)$ as $m \to \infty$. We shall derive a priori estimates of $y_m(t)$ in this case. Under the condition (2.8), we can differentiate the equations for approximate solutions $y_m(t)$ and take inner product with $y_m''(t)$ in $L^2(\Omega)$ to have

$$\begin{aligned}(3.14)\qquad &(y_m'''(t), y_m''(t)) + \alpha(\nabla y_m'(t), \nabla y_m''(t)) - (k(0)\mathrm{div}\,(G(\nabla y_m(t)), y_m''(t)) \\ &\quad - (k' * \mathrm{div}\, G(\nabla y_m)(t), y_m''(t)) = (f'(t), y_m''(t)).\end{aligned}$$

The equality (3.14) can be rewritten as

$$\begin{aligned}(3.15)\quad \frac{1}{2}\frac{d}{dt}\Big[\alpha|\nabla y_m'(t)|^2 + |y_m''(t)|^2\Big] &= (f'(t), y_m''(t)) + (k(0)\mathrm{div}\, G(\nabla y_m(t)), y_m''(t)) \\ &\quad + (k' * \mathrm{div}\, G(\nabla y_m)(t), y_m''(t)).\end{aligned}$$

We integrate (3.15) on $[0,t]$, and use the conditions on y_0, y_1 and f in (2.8). Then we have

$$\begin{aligned}(3.16)\qquad &\alpha|\nabla y_m'(t)|^2 + |y_m''(t)|^2 \\ =\ &\alpha|\nabla y_m'(0)|^2 + |\alpha\Delta y_m(0) + f(0)|^2 \\ &+ 2\int_0^t (k(0)\mathrm{div}\, G(\nabla y_m) + k' * \mathrm{div}\, G(\nabla y_m) + f', y_m'')ds.\end{aligned}$$

On the other hand, we take inner product of the equations for approximate solutions $y_m(t)$ with $\Delta y_m'(t)$ in $L^2(\Omega)$ to have

$$\begin{aligned}(3.17)\qquad &(y_m''(t), \Delta y_m'(t)) - \alpha(\Delta y_m(t), \Delta y_m'(t)) \\ &- (k * \mathrm{div}\, G(\nabla y_m)(t), \Delta y_m'(t)) = (f(t), \Delta y_m'(t)).\end{aligned}$$

Since

$$(y_m''(t), \Delta y_m'(t)) = -\frac{1}{2}\frac{d}{dt}|\nabla y_m'(t)|^2, \quad \alpha(\Delta y_m(t), \Delta y_m'(t)) = \frac{1}{2}\frac{d}{dt}\alpha|\Delta y_m(t)|^2$$

and

$$\begin{aligned}\frac{d}{dt}(k * \mathrm{div}\, G(\nabla y_m)(t), \Delta y_m(t)) =\ &(k * \mathrm{div}\, G(\nabla y_m)(t), \Delta y_m'(t)) \\ &+ (k(0)\mathrm{div}\, G(\nabla y_m(t)) + k' * \mathrm{div}\, G(\nabla y_m)(t), \Delta y_m(t)), \\ \frac{d}{dt}(f(t), \Delta y_m(t)) = (f'(t), &\Delta y_m(t)) + (f(t), \Delta y_m'(t)),\end{aligned}$$

the equality (3.17) can be rewritten as

$$\begin{aligned}(3.18)\qquad &-\frac{1}{2}\frac{d}{dt}\Big[\alpha|\Delta y_m(t)|^2 + |\nabla y_m'(t)|^2 + 2(k * \mathrm{div}\, G(\nabla y_m)(t), \Delta y_m(t))\Big] \\ &= \frac{d}{dt}(f(t), \Delta y_m(t)) - (f'(t), \Delta y_m(t)) \\ &\quad - (k(0)\mathrm{div}\, G(\nabla y_m(t)) + k' * \mathrm{div}\, G(\nabla y_m)(t), \Delta y_m(t)).\end{aligned}$$

In what follows we set

$$W_m(t) = k(0)\mathrm{div}\, G(\nabla y_m(t)) + k' * \mathrm{div}\, G(\nabla y_m)(t)$$

for notational simplicity. We integrate (3.18) on $[0,t]$ to have

$$\begin{aligned}(3.19)\qquad & \alpha|\Delta y_m(t)|^2 + |\nabla y_m'(t)|^2 \\ = \;& \alpha|\Delta y_m(0)|^2 + |\nabla y_m(0)|^2 - 2(f(t),\Delta y_m(t)) + 2(f(0),\Delta y_m(0)) \\ & + 2\int_0^t (f',\Delta y_m)ds + 2\int_0^t (W_m,\Delta y_m)ds \\ & - 2(k * \mathrm{div}\, G(\nabla y_m)(t),\Delta y_m(t)).\end{aligned}$$

Finally, if we sum (3.16) and (3.19), then we have

$$\begin{aligned}(3.20)\qquad & (\alpha+1)|\nabla y_m'(t)|^2 + |y_m''(t)|^2 + \alpha|\Delta y_m(t)|^2 \\ = \;& (\alpha+1)|\nabla y_m'(0)|^2 + |\alpha\Delta y_m(0) + f(0)|^2 + \alpha|\Delta y_m(0)|^2 \\ & - 2(f(t),\Delta y_m(t)) + 2(f(0),\Delta y_m(0)) \\ & + 2\int_0^t (f', y_m'' + \Delta y_m)ds + 2\int_0^t (W_m, y_m'' + \Delta y_m)ds \\ & - 2(k * \mathrm{div}\, G(\nabla y_m)(t),\Delta y_m(t)).\end{aligned}$$

Here in (3.20), we note also that $|\Delta y_m(0)|^2 \le c_3\|y_0\|^2_{D(\Delta)}$ and $|\nabla y_m'(0)|^2 \le c_4\|y_1\|^2_{H_0^1(\Omega)}$ for some $c_3, c_4 > 0$ and that $|f(0)|^2,\ |f(t)|^2 \le \|f\|^2_{C([0,T];L^2(\Omega))} \le c_5\|f\|^2_{H^1(0,T;L^2(\Omega))}$ for some c_5 by imbedding.

To estimate the terms in (3.20), we use the following inequality

$$\begin{aligned}(3.21)\qquad |\mathrm{div}\, G(\nabla\phi)| &= \Big|\frac{\Delta\phi}{\sqrt{1+|\nabla\phi|^2}} - \sum_{i,j=1}^n \frac{\phi_{x_i}\phi_{x_j}\phi_{x_ix_j}}{(1+|\nabla\phi|^2)^{\frac{3}{2}}}\Big| \\ &\le |\Delta\phi| + c_6\sum_{i,j=1}^n |\phi_{x_ix_j}| \le |\Delta\phi| + c_6\|\phi\|_{H^2(\Omega)} \\ &\le (1+c_6c_7)|\Delta\phi| \le C|\Delta\phi| \qquad \forall\phi\in D(\Delta),\end{aligned}$$

where $c_6 > 0$ and c_7 is a constant such that $\|\phi\|_{H^2(\Omega)} \le c_7|\Delta\phi|,\ \forall\phi\in H^2(\Omega)\cap H_0^1(\Omega)$. Let $\epsilon > 0$ be arbitrarily fixed. By Schwartz inequality, we have

$$(3.22)\qquad \Big|2\int_0^t (f', y_m'' + \Delta y_m)ds\Big| \le 2\|f'\|^2_{L^2(0,T;L^2(\Omega))} + \int_0^t |y_m''|^2 ds + \int_0^t |\Delta y_m|^2 ds,$$

$$(3.23)\qquad |2(f(t),\Delta y_m(t))| \le \frac{4c_5^2}{\epsilon}\|f\|^2_{H^1(0,T;L^2(\Omega))} + \epsilon|\Delta y_m(t)|^2.$$

By using the inequality (3.21) and Schwartz inequality, we have as in (3.4)

$$(3.24)\qquad \Big|2\int_0^t (W_m, y_m'' + \Delta y_m)ds\Big| \le K_0(3\int_0^t |\Delta y_m|^2 ds + \int_0^t |y_m''|^2 ds),$$

$$(3.25)\qquad \Big|2(k * \mathrm{div}\, G(\nabla y_m)(t),\Delta y_m(t))\Big| \le \frac{4k_0^2C^2T}{\epsilon}\int_0^t |\Delta y_m|^2 ds + \epsilon|\Delta y_m(t)|^2$$

for some $K_0 > 0$. If we take $\epsilon = \frac{\alpha}{4}$, by routine calculations, then we have from (3.22)-(3.25) that

$$
\begin{aligned}
&|y_m''(t)|^2 + |\nabla y_m'(t)|^2 + |\Delta y_m(t)|^2 \\
\le\ & K_1(\|y_0\|^2_{D(\Delta)} + \|y_1\|^2 + \|f\|^2_{H^1(0,T;L^2(\Omega))}) + K_2 \int_0^t (|y_m''|^2 + |\Delta y_m|^2) ds
\end{aligned} \tag{3.26}
$$

for some $K_1, K_2 > 0$. Therefore it is shown by using the Bellman-Gronwall's inequality that

$$
|y_m''(t)|^2 + |\nabla y_m'(t)|^2 + |\Delta y_m(t)|^2 \le C(\|y_0\|^2_{D(\Delta)} + \|y_1\|^2 + \|f\|^2_{H^1(0,T;L^2(\Omega))}), \quad \forall t \in [0,T] \tag{3.27}
$$

for some constant $C > 0$.

Step 2. Passage to the limits.
By (3.27), $\{y_m\}$ remains in a bounded sets of $\tilde{W}(0,T) \cap L^\infty(0,T;D(\Delta))$. Hence by the Rellich's extraction theorem, we can choose a subsequence $\{y_{m_k}\}$ of $\{y_m\}$ and find a $z \in \tilde{W}(0,T) \cap L^\infty(0,T;D(\Delta))$ such that

$$
\begin{aligned}
y_{m_k} \to z \ \text{ weakly-star in } \ & L^\infty(0,T;D(\Delta)) \\
\text{and weakly in } & L^2(0,T;D(\Delta)).
\end{aligned} \tag{3.28}
$$

And (3.28) implies that $\operatorname{div} G(\nabla y_{m_k})$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Therefore we can find an $F_\partial \in L^\infty(0,T;L^2(\Omega))$ such that for each $t \in [0,T]$

$$
\begin{aligned}
\operatorname{div} G(\nabla y_{m_k}) \to F_\partial \ \text{ weakly-star in } \ & L^\infty(0,t;L^2(\Omega)) \\
\text{and weakly in } & L^2(0,t;L^2(\Omega))
\end{aligned} \tag{3.29}
$$

as $k \to \infty$. Then by similar manipulations given in Dautray and Lions [3] and regularity arguments in Showalter [12], via (3.28), (3.29) and z'', $\Delta z \in L^\infty(0,T;L^2(\Omega))$, we can verify that z satisfies (2.7) in which $\operatorname{div} G(\nabla y) \in L^2(0,T;L^2(\Omega))$ is replaced by F_∂. In other words, z is a strong solution of the linear problem

$$
\begin{cases}
\dfrac{\partial^2 z}{\partial t^2} - \alpha \Delta z - \displaystyle\int_0^t k(t-s) F_\partial(s,x) ds = f \quad \text{in } \ Q, \\
z = 0 \quad \text{on } \ \Sigma, \\
z(0,x) = y_0(x), \quad \dfrac{\partial z}{\partial t}(0,x) = y_1(x) \quad \text{in } \ \Omega.
\end{cases} \tag{3.30}
$$

Step 3. The energy equality for z.
Since $\Delta z'(t)$ does not belong to $L^2(\Omega)$ in general by the above procedure, we can not take the inner product of $z''(t) - \alpha \Delta z(t) \in L^2(\Omega)$ and $\Delta z'(t)$ in $L^2(\Omega)$ as in the derivation of (3.19). This implies that the *strong* energy equality (3.31) given below can not be proved by direct computations as in the derivation of (3.20). However, we can prove the following lemme on the *strong* energy equality of strong solutions for (3.30) by applying the regularization arguments due to Lions and Magenes [8, pp. 276-279].

Lemma 3.1. *The strong solution z of (3.30) satisfies the following energy equality:*

$$
\begin{aligned}
&(\alpha+1)|\nabla z'(t)|^2 + |z''(t)|^2 + \alpha|\Delta z(t)|^2 \\
=\ & (\alpha+1)|\nabla y_1|^2 + |\alpha \Delta y_0 + f(0)|^2 + \alpha |\Delta y_0|^2 + 2(f(0), \Delta y_0) \\
& - 2(f(t), \Delta z(t)) - 2((k * F_\partial)(t), \Delta z(t)). \\
& + 2\int_0^t (k(0) F_\partial + k' * F_\partial + f', z'' + \Delta z) ds.
\end{aligned} \tag{3.31}
$$

A proof of Lemma 3.1 is given in Appendix.

Step 4. Strong convergence of approximate solutions.

In order to prove that z is a strong solution of (2.1) it is sufficient to prove $F_\partial = \operatorname{div} G(\nabla z)$. For this, we shall show $y_m(t) \to z(t)$ strongly in $D(\Delta)$ for all $t \in [0,T]$. To prove the strong convergence, we use the modified arguments in [6] for semilinear equations (cf. Dautray and Lions [3, pp.579-581]) and the classical compact imbedding theorem.

First we note the following trivial equalities:

$$|\phi_m|^2 + |\phi|^2 = |\phi_m - \phi|^2 + 2(\phi_m, \phi), \quad \forall \phi_m, \phi \in L^2(\Omega)$$
$$(\phi_m, \varphi_m) = (\phi_m - \phi, \varphi_m - \varphi) + (\phi_m, \varphi) + (\phi, \varphi_m - \varphi), \quad \forall \phi_m, \phi, \varphi_m, \varphi \in L^2(\Omega).$$

Adding (3.20) to (3.31) and using the above equalities, we have

$$\begin{aligned}(3.32)\qquad & (\alpha+1)|\nabla(y_m'(t) - z'(t))|^2 + |y_m''(t) - z''(t)|^2 + \alpha|\Delta(y_m(t) - z(t))|^2 \\ = \; & \Phi_m^0 + \sum_{i=1}^5 \Phi_m^i(t) - 2(k * \operatorname{div}(G(\nabla y_m) - G(\nabla z))(t), \Delta(y_m(t) - z(t))) \\ & + 2\int_0^t (k(0)\operatorname{div}(G(\nabla y_m) - G(\nabla z)), (y_m'' - z'') + \Delta(y_m - z))ds \\ & + 2\int_0^t (k' * \operatorname{div}(G(\nabla y_m) - G(\nabla z)), (y_m'' - z'') + \Delta(y_m - z))ds,\end{aligned}$$

where

$$\begin{aligned}(3.33)\qquad \Phi_m^0 = & (\alpha+1)(|\nabla y_m'(0)|^2 + |\nabla y_1|^2) + \alpha(|\Delta y_m(0)|^2 + |\Delta y_0|^2) \\ & + 2(f(0), \Delta(y_m(0) + y_0)) + |\alpha\Delta y_m(0) + f(0)|^2 + |\alpha \Delta y_0 + f(0)|^2,\end{aligned}$$

$$(3.34)\qquad \Phi_m^1(t) = -2(\alpha+1)(\nabla y_m'(t), \nabla z'(t)) - 2(y_m''(t), z''(t)) - 2\alpha(\Delta y_m(t), \Delta z(t)),$$

$$(3.35)\qquad \Phi_m^2(t) = 2\int_0^t (f', (y_m + z)'' + \Delta(y_m + z))ds - 2(f(t), \Delta(y_m(t) + z(t))),$$

$$\begin{aligned}(3.36)\qquad \Phi_m^3(t) = & -2(k * (\operatorname{div} G(\nabla y_m)(t) + F_\partial(t)), \Delta z(t)) \\ & - 2(k * \operatorname{div} G(\nabla z)(t), \Delta(y_m(t) - z(t))),\end{aligned}$$

$$\begin{aligned}(3.37)\qquad \Phi_m^4(t) = & 2\int_0^t (k(0)(\operatorname{div} G(\nabla y_m) + F_\partial), z'' + \Delta z)ds \\ & + 2\int_0^t (k(0)(\operatorname{div} G(\nabla z), (y_m - z)'' + \Delta(y_m - z))ds,\end{aligned}$$

$$\begin{aligned}(3.38)\qquad \Phi_m^5(t) = & 2\int_0^t (k' * (\operatorname{div} G(\nabla y_m) + F_\partial), z'' + \Delta z)ds \\ & + 2\int_0^t (k' * (\operatorname{div} G(\nabla z), (y_m - z)'' + \Delta(y_m - z))ds.\end{aligned}$$

It is verified by direct computations that

$$(3.39)\qquad \operatorname{div}(G(\nabla y_m) - G(\nabla z)) = \mathcal{K}_m + \mathcal{R}_m,$$

where

$$(3.40)\qquad \mathcal{K}_m = \frac{\Delta y_m - \Delta z}{\sqrt{1 + |\nabla y_m|^2}} - \sum_{i,j=1}^n \frac{y_{mx_i} y_{mx_j}(y_{mx_ix_j} - z_{x_ix_j})}{(1 + |\nabla y_m|^2)^{\frac{3}{2}}}$$

and

$$\mathcal{R}_m = \Delta z\Big(\frac{1}{\sqrt{1+|\nabla y_m|^2}} - \frac{1}{\sqrt{1+|\nabla z|^2}}\Big) + \sum_{i,j=1}^{n} z_{x_i x_j}\Big(\frac{z_{x_i} z_{x_j}}{(1+|\nabla z|^2)^{\frac{3}{2}}} - \frac{y_{m x_i} y_{m x_j}}{(1+|\nabla y_m|^2)^{\frac{3}{2}}}\Big). \tag{3.41}$$

By (3.39)-(3.41), the right hand side of (3.32) can be rewritten by

$$\Phi_m^0 + \sum_{i=1}^{5} \Phi_m^i(t) + \sum_{i=1}^{3} \Psi_m^i(t) - 2(k * \mathcal{K}_m(t), \Delta(y_m(t) - z(t))) + 2\int_0^t (k(0)\mathcal{K}_m + k' * \mathcal{K}_m, (y_m - z)'' + \Delta(y_m - z))ds, \tag{3.42}$$

where

$$\Psi_m^1(t) = 2\int_0^t (k(0)\mathcal{R}_m, (y_m - z)'' + \Delta(y_m - z))ds, \tag{3.43}$$

$$\Psi_m^2(t) = 2\int_0^t (k' * \mathcal{R}_m, (y_m - z)'' + \Delta(y_m - z))ds, \tag{3.44}$$

$$\Psi_m^3(t) = -2(k * \mathcal{R}_m(t), \Delta(y_m(t) - z(t))). \tag{3.45}$$

The term $\mathcal{K}_m$ can be estimated as

$$\|\mathcal{K}_m\|_{L^2(0,T;L^2(\Omega))} \le C_1\|y_m - z\|_{L^2(0,T;H^2(\Omega))} \le C_2\|\Delta y_m - \Delta z\|_{L^2(0,T;L^2(\Omega))}. \tag{3.46}$$

We set

$$\Phi_m(t) = \Big|\Phi_m^0 + \sum_{i=1}^{5} \Phi_m^i(t)\Big|, \qquad \Psi_m(t) = \sum_{i=1}^{3} |\Psi_m^i(t)|.$$

Then by routine calculations in (3.32) together with (3.42), mainly due to the estimation (3.46), we can derive the following inequality

$$|y_m''(t) - z''(t)|^2 + |\nabla(y_m'(t) - z'(t))|^2 + |\Delta(y_m(t) - z(t))|^2 \le C_4(\Phi_m(t) + \Psi_m(t)) + C_5\int_0^t (|\Delta(y_m - z)|^2 + |y_m'' - z''|^2)ds \tag{3.47}$$

for some $C_4, C_5 > 0$. By applying the extended Bellman-Gronwall's inequality to (3.47), we deduce

$$|y_m''(t) - z''(t)|^2 + |\nabla(y_m'(t) - z'(t))|^2 + |\Delta(y_m(t) - z(t))|^2 \le K_1(\Phi_m(t) + \Psi_m(t)) + K_2\int_0^t (\Phi_m(s) + \Psi_m(s))ds. \tag{3.48}$$

By virtue of the strong convergence of the initial values and (3.28), (3.29), we can extract a subsequence $\{y_{m_k}\}$ of $\{y_m\}$ such that

$$\Phi^0_{m_k} \to 2(\alpha+1)|\nabla y_1|^2 + 2\alpha|\Delta y_0|^2 + 4(f(0),\Delta y_0) + 2|\alpha\Delta y_0 + f(0)|^2, \tag{3.49}$$

$$\Phi^1_{m_k}(t) \to -2(\alpha+1)|\nabla z'(t)|^2 - 2|z''(t)|^2 - 2\alpha|\Delta z(t)|^2, \tag{3.50}$$

$$\Phi^2_{m_k}(t) \to 4\int_0^t (f', z''+\Delta z)ds - 4(f(t),\Delta z(t)), \tag{3.51}$$

$$\Phi^3_{m_k}(t) \to -4(k * F_\partial(t), \Delta z(t)), \tag{3.52}$$

$$\Phi^4_{m_k}(t) \to 4\int_0^t (k(0)F_\partial, z''+\Delta z)ds, \tag{3.53}$$

$$\Phi^5_{m_k}(t) \to 4\int_0^t (k' * F_\partial z''+\Delta z)ds, \tag{3.54}$$

as $k\to\infty$. Therefore by (3.31), the sum of limits in (3.49)-(3.54) is 0, so that

$$\Phi_{m_k}(t) \to 0 \quad \text{as} \quad k\to\infty. \tag{3.55}$$

Also from (3.27), we can easily verify that

$$\Phi_m(t) + \Psi_m(t) \le K_3, \quad \forall t\in[0,T] \tag{3.56}$$

for some $K_3>0$. Next we shall show the convergence of $\Psi_m(t)$ to 0. Since the imbedding $D(\Delta)\hookrightarrow H^1_0(\Omega)$ is compact, by the Aubin-Lions-Temam's compact imbedding theorem (cf. Temam [13, p.274]), the imbedding $W(D(\Delta),H^1_0(\Omega))\hookrightarrow L^2(0,T;H^1_0(\Omega))$ is also compact where $W(D(\Delta),H^1_0(\Omega)) = \{\phi|\ \phi\in L^2(0,T;D(\Delta)),\quad \phi'\in L^2(0,T;H^1_0(\Omega))\}$. This implies that the set $\{y_{mx_i}\}$ is precompact in $L^2(Q)=L^2(0,T;L^2(\Omega))$ for each $i=1,\cdots,n$. Hence we can deduce that there exists a subsequence $\{y_{m_k}\}$ of $\{y_m\}$ such that

$$y_{m_kx_i} \to z_{x_i} \quad \text{a.e.}\ \ x\in\Omega,\quad i=1,\cdots,n \tag{3.57}$$

for a.e. $t\in[0,T]$. Then it follows from (3.57) that

$$\frac{1}{\sqrt{1+|\nabla y_{m_k}(t,x)|^2}} \to \frac{1}{\sqrt{1+|\nabla z(t,x)|^2}} \quad \text{a.e.}\ \ x\in\Omega, \tag{3.58}$$

$$\frac{y_{m_kx_i}(t,x)y_{m_kx_j}(t,x)}{(1+|\nabla y_{m_k}(t,x)|^2)^{\frac32}} \to \frac{z_{x_i}(t,x)z_{x_j}(t,x)}{(1+|\nabla z(t,x)|^2)^{\frac32}} \quad \text{a.e.}\ \ x\in\Omega \tag{3.59}$$

for a.e. $t\in[0,T]$. We fix t such that (3.57)-(3.59) hold. Hence we have

$$|\mathcal{R}_m(t,x)| \le 2|\Delta z(t,x)| + 2\sum_{i,j=1}^n |z_{x_ix_j}(t,x)| \quad \text{a.e.}\ \ x\in\Omega, \tag{3.60}$$

$$\lim_{m_k\to\infty} \mathcal{R}_{m_k}(t,x) = 0 \quad \text{a.e.}\ \ x\in\Omega. \tag{3.61}$$

From (3.60) and (3.61) we see by the Lebesgue dominated convergence theorem that

$$\mathcal{R}_{m_k}(t)\to 0 \quad \text{strongly in} \quad L^2(\Omega), \tag{3.62}$$

so that

$$\Psi^3_{m_k}(t)\to 0 \quad \text{as} \quad m_k\to\infty. \tag{3.63}$$

At the same time we see $\mathcal{R}_{m_k}(s) \to 0$ strongly in $L^2(\Omega)$ for a.e. $s \in [0,t]$, so that by applying the Lebesgue dominated convergence theorem again we have

$$\Psi^1_{m_k}(t),\ \Psi^2_{m_k}(t) \to 0 \quad \text{as} \quad m_k \to \infty. \tag{3.64}$$

By (3.63) and (3.64), we deduce that

$$\Psi_{m_k}(t) \to 0 \quad \text{as} \quad m_k \to \infty \tag{3.65}$$

for a.e. $t \in [0,T]$. By the regularity result in Lemma 3.1, we can deduce $z \in C([0,T]; D(\Delta))$ and hence $\Psi_{m_k}(t)$ is continuous in $t \in [0,T]$. Therefore the convergence (3.62) holds for all $t \in [0,T]$. Finally, by applying (3.55) and (3.65) to (3.48) with $m = m_k$, we have

$$\begin{aligned} &(y_{m_k}(t), y'_{m_k}(t), y''_{m_k}(t)) \ \to \ (z(t), z'(t), z''(t)) \\ &\text{strongly in } \ D(\Delta) \times H^1_0(\Omega) \times L^2(\Omega), \quad \forall t \in [0,T]. \end{aligned} \tag{3.66}$$

Then from (3.40), (3.62) and (3.66) it follows readily that

$$F_\partial = \operatorname{div} G(\nabla z). \tag{3.67}$$

Therefore we have proved the existence of a strong solution of the problem (2.1) and regularity $z \in C([0,T]; D(\Delta)),\ z' \in C([0,T]; H^1_0(\Omega)),\ z'' \in C([0,T]; L^2(\Omega))$ by Lemma 3.1. The uniqueness of strong solutions is evident from the uniqueness of weak solutions.

Appendix

Proof of Lemma 3.1. By the same arguments as in Lions and Magenes [8, pp. 275-276] (see also [6, Lemma 4.1, Lemma 4.2]), we can verify that all functions in (3.31) has meaning for all $t \in [0,T]$, and that $z(t) \in D(\Delta)$, $z'(t) \in H^1_0(\Omega)$ and $z''(t) \in L^2(\Omega)$. For simplicity of notations we put $F_\partial = F$ in what follows. Let $\delta > 0$ and $t_0 \in (0,T)$ be fixed. We recall a continuous function $\mathcal{O}_\delta(t) = \mathcal{O}(t)$ and a step function $\mathcal{O}_0(t)$ given in the proof of [8, Lemma 8.3, p.276]. Let $\{\rho_n\}_{n=1}^\infty$ be the regularizing sequence of even functions introduced in the proof of [8, Lemma 8.3]. We shall extend $k(t)$ and $f(t)$ for all $t \in \mathbf{R}$, with the same properties on $[0,T]$. Especially we can suppose $k(t) = 0$ for $t \in \mathbf{R} \setminus [0,T]$. In the same way as in [8] we shall assume that z is extended on $\mathbf{R}$, which is possible by reflection. For the simplicity we shall denote by $[\ ,\ ]$ the scalar product in $L^2(\Omega)$ or the duality pairing between $H^1_0(\Omega)$ and $H^{-1}(\Omega)$, and we shall denote by $(\ ,\ )$ the duality pairing between $L^2(\mathbf{R_t}; H^1_0(\Omega))$ and $L^2(\mathbf{R_t}; H^{-1}(\Omega))$ or the scalar product on $L^2(\mathbf{R_t}; L^2(\Omega))$. We fix n and set $\rho_n = \rho$. Let $\rho * \psi$ be the mollifier of ψ. At first we have

$$\begin{aligned} &2(\rho * (\mathcal{O}_0 z''), \rho * (\mathcal{O}_0 z''')) + 2[(\rho * \rho * (\mathcal{O}_0 z''))(0), z''(0)] \\ &- 2[(\rho * \rho * (\mathcal{O}_0 z''))(t_0), z''(t_0)] = 0, \end{aligned} \tag{A.1}$$

by starting with

$$\int_{\mathbf{R_t}} \frac{d}{dt}[\rho * (\mathcal{O} z''), \rho * (\mathcal{O} z'')]dt = 0$$

and tending $\delta \to 0$ as in [8, p. 276]. Secondly we start with

$$-\alpha \int_{\mathbf{R_t}} \frac{d}{dt}[\Delta(\rho * (\mathcal{O} z')), \rho * (\mathcal{O} z')]dt = 0$$

and use the fact $\rho * (\mathcal{O}\Delta z') = \Delta(\rho * (\mathcal{O}z'))$ to obtain

$$-2\alpha(\rho * (\mathcal{O}\Delta z'), \rho * (\mathcal{O}z'')) - 2\alpha(\rho * (\mathcal{O}\Delta z'), \rho * (\mathcal{O}'z')) = 0. \tag{A.2}$$

We shall show that (A.2) tends to

$$\begin{aligned} &-2\alpha(\rho * (\mathcal{O}_0\Delta z'), \rho * (\mathcal{O}_0 z'')) - 2\alpha[(\rho * \rho * (\mathcal{O}_0\Delta z'))(0), z'(0)] \\ &+ 2\alpha[(\rho * \rho * (\mathcal{O}_0\Delta z'))(t_0), z'(t_0)] = 0 \end{aligned} \tag{A.3}$$

as $\delta \to 0$. To verify (A.3), it is sufficient to prove that

$$\begin{aligned} &(\rho * (\mathcal{O}\Delta z'), \rho * (\mathcal{O}'z')) \\ &\to [(\rho * \rho * (\mathcal{O}_0\Delta z'))(0), z'(0)] - [(\rho * \rho * (\mathcal{O}_0\Delta z'))(t_0), z'(t_0)] \end{aligned} \tag{A.4}$$

as $\delta \to 0$. The left hand side of (A.4) may be written as

$$(\rho * ((\mathcal{O} - \mathcal{O}_0)\Delta z'), \rho * (\mathcal{O}'z')) + (\rho * (\mathcal{O}_0\Delta z'), \rho * (\mathcal{O}'z')). \tag{A.5}$$

Since $(\mathcal{O} - \mathcal{O}_0)\Delta z' \to 0$ in $L^2(\mathbf{R}_t; H^{-1}(\Omega))$, $\rho * (\mathcal{O} - \mathcal{O}_0)\Delta z' \to 0$ in $L^\infty(\mathbf{R}_t; H^{-1}(\Omega))$. So that by $\int_{\mathbf{R_t}} |\mathcal{O}'| dt = 2$ and the boundedness of $\rho * (\mathcal{O}'z')$ in $L^1(\mathbf{R}_t; H_0^1(\Omega))$, we see that the first term of (A.5) tends to 0. The second term of (A.5) is written as

$$\frac{1}{\delta}\int_0^\delta [((\rho * \rho * (\mathcal{O}_0\Delta z'))(t), z'(t)]dt - \frac{1}{\delta}\int_{t_0-\delta}^{t_0} [((\rho * \rho * (\mathcal{O}_0\Delta z'))(t), z'(t)]dt. \tag{A.6}$$

Since the mapping $t \to [((\rho * \rho * (\mathcal{O}_0\Delta z'))(t), z'(t)]$ is continuous, (A.6) converges to

$$[(\rho * \rho * (\mathcal{O}_0\Delta z'))(0), z'(0)] - [(\rho * \rho * (\mathcal{O}_0\Delta z'))(t_0), z'(t_0)]$$

as $\delta \to 0$. We note here that the other similar cases to (A.3) can be proved by the similar fashion. We sum (A.1) and (A.3) and use the fact

$$z''' = \alpha\Delta z' + k(0)F + k' * F + f',$$

then we can obtain

$$\begin{aligned} &2[(\rho * \rho * (\mathcal{O}_0 z''))(0), z''(0)] - 2[(\rho * \rho * (\mathcal{O}_0 z''))(t_0), z''(t_0)] \\ &- 2\alpha[(\rho * \rho * (\mathcal{O}_0\Delta z'))(0), z'(0)] + 2\alpha[(\rho * \rho * (\mathcal{O}_0\Delta z'))(t_0), z'(t_0)] \\ &+ 2(\rho * (\mathcal{O}_0 W), \rho * (\mathcal{O}_0 z'')) = 0, \end{aligned} \tag{A.7}$$

where

$$W = k(0)F + k' * F + f'.$$

Next we shall prove the rest part of the equality. For the purpose we start again with

$$\int_{\mathbf{R_t}} \frac{d}{dt}[\Delta(\rho * (\mathcal{O}z')), \rho * (\mathcal{O}z')]dt = 0.$$

From this equality we have $2(\Delta(\rho * (\mathcal{O}z')), (\rho * (\mathcal{O}z'))') = 0$, and hence

$$2(\rho * (\Delta\mathcal{O}z'), \rho * (\mathcal{O}'z')) + 2(\Delta(\rho * (\mathcal{O}z')), \rho * (\mathcal{O}z'')) = 0. \tag{A.8}$$

By the same process as above, we can deduce that (A.8) tends to

$$\begin{aligned}(A.9)\qquad & 2(\rho * (\mathcal{O}_0 \Delta z'), \rho * (\mathcal{O}_0 z'')) + 2[(\rho * \rho * (\mathcal{O}_0 \Delta z'))(0), z'(0)] \\ & - 2[(\rho * \rho * (\mathcal{O}_0 \Delta z'))(t_0), z'(t_0)] = 0.\end{aligned}$$

Also, starting with

$$-\alpha \int_{\mathbf{R_t}} \frac{d}{dt}[\rho * (\mathcal{O}\Delta z), \rho * (\mathcal{O}\Delta z)]dt = 0,$$

we have

$$(A.10)\qquad -2\alpha(\rho * (\mathcal{O}'\Delta z), \rho * (\mathcal{O}\Delta z)) - 2\alpha(\rho * (\mathcal{O}\Delta z'), \rho * (\mathcal{O}\Delta z)) = 0.$$

Hence as shown as in (A.7), we deduce that (A.10) tends to

$$\begin{aligned}(A.11)\qquad & - 2\alpha(\rho * (\mathcal{O}_0 \Delta z'), \rho * (\mathcal{O}_0 \Delta z)) + 2\alpha[(\rho * \rho * (\mathcal{O}_0 \Delta z))(t_0), \Delta z(t_0)] \\ & - 2\alpha[(\rho * \rho * (\mathcal{O}_0 \Delta z))(0), \Delta z(0)] = 0.\end{aligned}$$

From the fact

$$- \int_{\mathbf{R_t}} \frac{d}{dt}[\rho * (\mathcal{O}k * F), \rho * (\mathcal{O}\Delta z)]dt = 0,$$

we immediately have

$$\begin{aligned}(A.12)\qquad & - (\rho * (\mathcal{O}k * F), \rho * (\mathcal{O}'\Delta z + \mathcal{O}\Delta z')) \\ & - (\rho * (\mathcal{O}'k * F + \mathcal{O}k(0)F + \mathcal{O}k' * F), \rho * (\mathcal{O}\Delta z)) = 0.\end{aligned}$$

Since

$$[(k * F)(0), (\rho * \rho * (\mathcal{O}_0 \Delta z))(0)] = 0,$$

(A.12) tends to

$$\begin{aligned}(A.13)\qquad & - (\rho * (\mathcal{O}_0 k * F), \rho * (\mathcal{O}_0 \Delta z')) - (\rho * (\mathcal{O}_0 k(0)F + \mathcal{O}_0 k' * F), \rho * (\mathcal{O}_0 \Delta z)) \\ & + [(\rho * \rho * (\mathcal{O}_0 k * F))(t_0), \Delta z(t_0)] + [k * F(t_0), (\rho * \rho * (\mathcal{O}_0 \Delta z))(t_0)] \\ & - [(\rho * \rho * (\mathcal{O}_0 (k * F))(0), \Delta z(0)] = 0.\end{aligned}$$

By starting from

$$- \int_{\mathbf{R_t}} \frac{d}{dt}[\rho * (\mathcal{O}f), \rho * (\mathcal{O}\Delta z)]dt = 0,$$

we see that

$$(A.14)\qquad -(\rho * (\mathcal{O}f), \rho * (\mathcal{O}'\Delta z + \mathcal{O}\Delta z')) - (\rho * (\mathcal{O}'f + \mathcal{O}f'), \rho * (\mathcal{O}\Delta z)) = 0.$$

Hence, in a similar way, we have that (A.14) tends to

$$\begin{aligned}(A.15)\qquad & - (\rho * (\mathcal{O}_0 f), \rho * (\mathcal{O}_0 \Delta z')) - (\rho * (\mathcal{O}_0 f'), \rho * (\mathcal{O}_0 \Delta z)) \\ & - [\rho * \rho * (\mathcal{O}_0 f))(0), \Delta z(0)] + [\rho * \rho * (\mathcal{O}_0 f))(t_0), \Delta z(t_0)] \\ & - [f(0), (\rho * \rho * (\mathcal{O}_0 \Delta z))(0)] + [f(t_0), (\rho * \rho * (\mathcal{O}_0 \Delta z))(t_0)] = 0.\end{aligned}$$

Now we take the sum of $(A.9)+(A.11)+2\times(A.13)+2\times(A.15)$. Using the equation $z''=\alpha\Delta z+k*F+f$, the sum is represented by

$$
\begin{aligned}
(\text{A.16})\qquad & -2(\rho*(\mathcal{O}_0 f'),\rho*(\mathcal{O}_0\Delta z))-2[\rho*\rho*(\mathcal{O}_0 f))(0),\Delta z(0)]\\
&+2[\rho*\rho*(\mathcal{O}_0 f))(t_0),\Delta z(t_0)]-2[f(0),(\rho*\rho*(\mathcal{O}_0\Delta z))(0)]\\
&+2[f(t_0),(\rho*\rho*(\mathcal{O}_0\Delta z))(t_0)]+2[z'(0),(\rho*\rho*(\mathcal{O}_0\Delta z'))(0)]\\
&-2[z'(t_0),(\rho*\rho*(\mathcal{O}_0\Delta z'))(t_0)]+2\alpha[\Delta z(t_0),(\rho*\rho*(\mathcal{O}_0\Delta z))(t_0)]\\
&-2\alpha[\Delta z(0),(\rho*\rho*(\mathcal{O}_0\Delta z))(0)]-2(\rho*(\mathcal{O}_0 k(0)F),\rho*(\mathcal{O}_0\Delta z))\\
&-2(\rho*(\mathcal{O}_0 k'*F),\rho*(\mathcal{O}_0\Delta z))+2[(\rho*\rho*(\mathcal{O}_0 k*F))(t_0),\Delta z(t_0)]\\
&-2[(\rho*\rho*(\mathcal{O}_0(k*F))(0),\Delta z(0)]\\
&+2[k*F(t_0),(\rho*\rho*(\mathcal{O}_0\Delta z))(t_0)]=0.
\end{aligned}
$$

Finally we subtract (A.16) from (A.7), then we have

$$
\begin{aligned}
(\text{A.17})\qquad & 2[(\rho*\rho*(\mathcal{O}_0 z''))(0),z''(0)]-2[(\rho*\rho*(\mathcal{O}_0 z''))(t_0),z''(t_0)]\\
&-2\alpha[(\rho*\rho*(\mathcal{O}_0\Delta z'))(0),z'(0)]+2\alpha[(\rho*\rho*(\mathcal{O}_0\Delta z'))(t_0),z'(t_0)]\\
&+2(\rho*(\mathcal{O}_0 W),\rho*(\mathcal{O}_0 z''))+2((\rho*(\mathcal{O}_0 W),\rho*(\mathcal{O}_0\Delta z))\\
&+2[\rho*\rho*(\mathcal{O}_0 f))(0),\Delta z(0)]-2[\rho*\rho*(\mathcal{O}_0 f))(t_0),\Delta z(t_0)]\\
&+2[f(0),(\rho*\rho*(\mathcal{O}_0\Delta z))(0)]-2[f(t_0),(\rho*\rho*(\mathcal{O}_0\Delta z))(t_0)]\\
&-2[(\rho*\rho*(\mathcal{O}_0\Delta z'))(0),z'(0)]+2[(\rho*\rho*(\mathcal{O}_0\Delta z'))(t_0),z'(t_0)]\\
&-2\alpha[(\rho*\rho*(\mathcal{O}_0\Delta z))(t_0),\Delta z(t_0)]+2\alpha[(\rho*\rho*(\mathcal{O}_0\Delta z))(0),\Delta z(0)]\\
&-2[(\rho*\rho*(\mathcal{O}_0 k*F))(t_0),\Delta z(t_0)]+2[(\rho*\rho*(\mathcal{O}_0(k*F))(0),\Delta z(0)]\\
&-2[k*F(t_0),(\rho*\rho*(\mathcal{O}_0\Delta z))(t_0)]=0.
\end{aligned}
$$

Setting $\rho*\rho=\sigma$, then σ is also an even function and $\int_0^{t_0}\sigma(t)dt=\frac{1}{2}$. By the quite similar manipulations as in [8, p.279], we can prove that the sum of first and second terms of (A.17) tend toward

$$
(\text{A.18})\qquad |z''(0)|^2-|z''(t_0)|^2=|\alpha\Delta y_0+f(0)|^2-|z''(t_0)|^2.
$$

Analogously, we can verify the following convergences

$$
\begin{aligned}
(\text{A.19})\qquad & -2\alpha[(\sigma*(\mathcal{O}_0\Delta z'))(0),z'(0)]+2\alpha[(\sigma*(\mathcal{O}_0\Delta z'))(t_0),z'(t_0)]\\
&\qquad\to\alpha|\nabla y_1|^2-\alpha|\nabla z'(t_0)|^2,\\
(\text{A.20})\qquad & 2[\sigma*(\mathcal{O}_0 f))(0),\Delta z(0)]-2[\sigma*(\mathcal{O}_0 f))(t_0),\Delta z(t_0)]\\
&\qquad\to(f(0),\Delta y_0)-(f(t_0),\Delta z(t_0)),\\
(\text{A.21})\qquad & 2[(f(0),\sigma*(\mathcal{O}_0\Delta z))(0)]-2[(f(t_0),\sigma*(\mathcal{O}_0\Delta z))(t_0)]\\
&\qquad\to(f(0),\Delta y_0)-(f(t_0),\Delta z(t_0)),\\
(\text{A.22})\qquad & -2[(\sigma*(\mathcal{O}_0\Delta z'))(0),z'(0)]+2[(\sigma*(\mathcal{O}_0\Delta z'))(t_0),z'(t_0)]\\
&\qquad\to|\nabla y_1|^2-|\nabla z'(t_0)|^2,\\
(\text{A.23})\qquad & -2\alpha[(\sigma*(\mathcal{O}_0\Delta z))(t_0),\Delta z(t_0)]+2\alpha[(\sigma*(\mathcal{O}_0\Delta z))(0),\Delta z(0)]\\
&\qquad\to-\alpha|\Delta z(t_0)|^2+\alpha|\Delta y_0|^2,\\
(\text{A.24})\qquad & -2[(\sigma*(\mathcal{O}_0 k*F))(t_0),\Delta z(t_0)]-2[k*F(t_0),(\sigma*(\mathcal{O}_0\Delta z))(t_0)]\\
&\qquad\to-2(k*F(t_0),\Delta z(t_0)),\\
(\text{A.25})\qquad & 2[(\sigma*(\mathcal{O}_0 k*F))(0),\Delta z(0)]\to((k*F)(0),\Delta z(0))=0.
\end{aligned}
$$

Clearly, the limits of the rest terms of (A.17) are obtained by

$$\text{(A.26)}\qquad 2(\rho * (\mathcal{O}_0 W), \rho * (\mathcal{O}_0 z'')) \to 2\int_0^{t_0} (W, z'')ds = 2\int_0^{t_0} (k(0)F + k' * F + f', z'')ds,$$

$$\text{(A.27)}\qquad 2(\rho * (\mathcal{O}_0 W), \rho * (\mathcal{O}_0 \Delta z)) \to 2\int_0^{t_0} (W, \Delta z)ds = 2\int_0^{t_0} (k(0)F + k' * F + f', \Delta z)ds.$$

By (A.17) and from (A.18) to (A.28), we can assert the equality (3.31). Therefore we proved Lemma 3.1.

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Department of Mathematics, Dong-A University, Pusan, 604-714 Korea
Department of Applied Mathematics, Kobe University, Kobe 657-8501, Japan