NEURONAL MODELING IN THE PRESENCE OF RANDOM REFRACTORINESS

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Abstract. Inclusion of refractoriness in a diffusion model for single neuron activity is the object of the present paper. An asymptotic analysis of the random process modeling the number of firings and the distribution of interspike intervals is performed under the approximation of exponentially distributed firings. In the cases of constant, uniform, exponential, Erlang, truncated normal and hyperexponential durations of refractory periods, asymptotic simple expressions are obtained and some numerical evaluations are performed.

1 Introduction and Background

Since the classical paper by Gerstein and Mandelbrot [7], numerous attempts have been made to formulate stochastic models for single neuron activity capable of reproducing features of the behavior exhibited by neural cells under spontaneous or stimulated conditions (see, for instance, [15], [18], [19] and the bibliography quoted therein). In particular, a quantitative description of the behavior of the neuron membrane potential as an instantaneous return process has been the object of various investigations (cf. [8], [10], [11], [17]) under the assumption that after each firing, instantly the membrane potential is either reset to a unique fixed value or to a value characterized by an assigned probability density function (pdf).

As is well-known, when mathematically modeling a complex system, in order to develop a feeling, or, better, to obtain quantitative information on its behavior without facing overwhelming difficulties, it is common practice to perform a careful selection of the available parameters and to retain only those that are believed to play an essential role, while disregarding those whose effects should be taken into account in successive refinements. This is probably the reason why the neuronal "dead time" or "refractoriness" has seldom been looked at as a relevant parameters in the large mass of theoretical works dealing with models of neuronal activity. However, such a practice can be justified, in first approximation, only when dealing with input conditions leading to low firing rates. Exceptions are a few works (see [1]$\sim$[5], [12], [13], [20]) inspired by the very early approaches of [16] and [22].

A first difficulty one faces when trying to embed refractoriness in a neuronal model is how to define quantitatively such a quantity. One may look exclusively at the "absolute" refractoriness, namely to the time interval (lasting about 1 ms) following a spike during which the neuron is unable to fire again whatever stimulus is acting on it; or one may focus attention on "relative" refractoriness (a time interval following the firing that may last as long as 100 ms, often including both positive and negative after-potential); or one may try to take into account that firing, especially in sensory neurons, is a consequence not only of the intensity of the acting stimulus, but also of its duration, so that in such cases refractoriness is not an intrinsic property of the neuron but also a function of its input.

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To overcome these preliminary difficulties, hereafter we shall refrain from making any specific assumption on the mechanisms that are ultimately responsible for generating refractoriness, and attack the problem in a totally different way. Indeed, we shall assume that each firing is accompanied by a refractory period whose duration is a random variable, and assume that the sequence of refractory periods associated to the sequence of firings is modeled by a sequence of independent and identically distributed (iid) random variables characterized by a preassigned pdf. For each such specified pdf the neuronal output will be determined, and some numerical computations will be performed to pinpoint similarities and differences implied by the types of considered random variables.

It should be mentioned that recently in [1], [2] and [3] the presence of refractoriness has been included in the mathematical characterization of the membrane potential by using an instantaneous return process based on diffusion models of single neuron activity, under the assumption that the firing threshold acts as a 'partially transparent' elastic barrier, whose behavior is intermediate between total absorption and total reflection. In [5] and [20] a diffusion process was, instead, invoked to describe the time course of the membrane potential and refractoriness was modeled as a sequence of iid random variables having a preassigned pdf, right along the lines indicated in the foregoing. A return process was again constructed, by means of which the distribution of the number of firings up to an assigned time could be determined.

Following this latter approach, let us now denote by $\{X(t), t \geq 0\}$ a regular, time-homogeneous diffusion process defined over an interval $I = (r_1, r_2)$ and characterized by drift and infinitesimal variance $A_1(x)$ and $A_2(x)$, respectively, that we assume to satisfy Feller condition [6]. Let $h(x)$ and $k(x)$ denote scale function and speed density of $X(t)$:

$$h(x) = \exp\left\{-2\int^x \frac{A_1(z)}{A_2(z)}\,dz\right\}, \quad k(x) = \frac{2}{A_2(x)\,h(x)}$$

and denote by

$$H(r_1, y] = \int_{r_1}^y h(z)\,dz, \qquad K(r_1, y] = \int_{r_1}^y k(z)\,dz$$

scale and the speed measures, respectively.

As is well-known, the first passage time (FPT) of $X(t)$ through S ($S \in I$), with $X(0) = x < S$, is defined as follows:

$$T_x = \inf_{t \geq 0}\{t : X(t) \geq S\}, \qquad X(0) = x < S. \tag{1.1}$$

Then,

$$g(S, t \mid x) = \frac{\partial}{\partial t} P(T_x < t), \qquad x < S \tag{1.2}$$

is the FPT pdf of $X(t)$ through S conditional upon $X(0) = x$.

In the neuronal modeling context the state S represents the neuron firing threshold, the FPT through S the firing time and $g(S, t \mid x)$ the firing time pdf. In the sequel we assume that one of the following cases holds:

(i) r_1 is a natural nonattracting boundary and $K(r_1, y] < +\infty$

(ii) r_1 is either a reflecting boundary or an entrance boundary.

Under such assumptions, for $x < S$ the first passage probability $P(S \mid x)$ from x to S is unity and the moments FPT are finite and given by (cf. [21]):

$$t_n(S \mid x) := \int_0^{+\infty} t^n\, g(S,t \mid x)\, dt = n \int_x^S h(z)\, dz \int_{r_1}^z k(u)\, t_{n-1}(S \mid u)\, du \qquad (n = 1, 2, \dots), \tag{1.3}$$

with $t_0(S \mid x) \equiv P(S \mid x) = 1$.

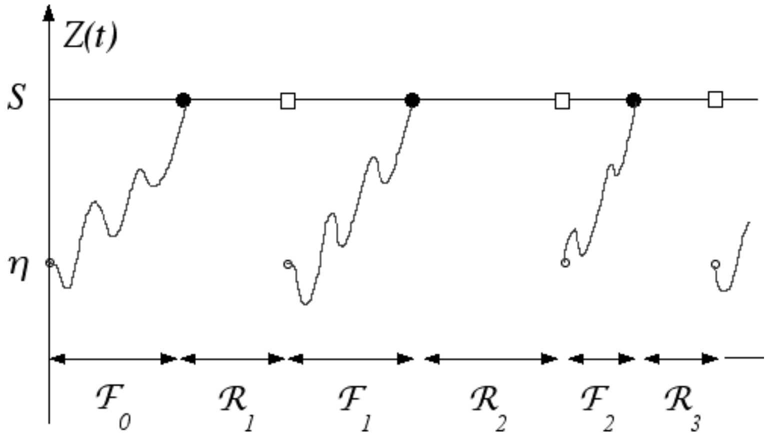

Figure 1: An hypothetical sample path of $Z(t)$. Circles indicate the firing times, and squares the ends of refractory periods. The instantaneous reset value has been denoted by η.

The return process $\{Z(t), t \geq 0\}$ in (r_1, S) is constructed as follows. Starting at $\eta \in (r_1, S)$ at time zero, a firing takes place when $X(t)$ attains the threshold S for the first time; then, a refractory period of random duration occurs, after which $Z(t)$ is instantaneously reset to η. The subsequent evolution of the process goes on as described by $X(t)$, until S is again reached. A new firing then occurs, followed by the refractory period, and so on (cf. Figure 1). The process $\{Z(t), t \geq 0\}$, describing the time course of the membrane potential, thus consists of recurrent cycles $\mathcal{F}_0, \mathcal{R}_1, \mathcal{F}_1, \mathcal{R}_2, \mathcal{F}_2, \dots$, each of random duration, where the durations F_i of $\mathcal{F}_i$ $(i = 0, 1, \dots)$ and the durations of refractory periods R_i of $\mathcal{R}_i$ $(i = 1, 2, \dots)$ are independently distributed random variables. Here, F_i $(i = 0, 1, \dots)$ describes the time interval elapsing between the i-th reset at η and the $(i+1)$-th FPT from η to S, whereas R_i $(i = 1, 2, \dots)$ describes the duration of i-th refractory period. Since $X(t)$ is time-homogeneous, $F_0, F_1, \dots$ can be assumed to be iid random variables, each with pdf $g(S, t \mid \eta)$ in which t denotes the time interval elapsing between a reset instant and the instant of release of next firing. Furthermore, $R_1, R_2, \dots$ are assumed to be iid random variables, each with pdf $\varphi(t)$ depending only on the duration of the refractory period.

We now aim at the description of the random process $\{M(t), t \geq 0\}$ representing the number of firings released by the neuron up to time t. To this purpose, for $\eta \in (r_1, S)$, let

$$q_k(t \mid \eta) = P\{M(t) = k \mid Z(0) = \eta\} \qquad (k = 0, 1, \dots) \tag{1.4}$$

be the probability of occurrence of k firings up to time t. Since $X(t)$ is time-homogeneous and the R_i's are iid,

$$
(1.5)\qquad
\begin{aligned}
q_0(t \mid \eta) &= 1 - \int_0^t g(S, \tau \mid \eta)\, d\tau \\
q_1(t \mid \eta) &= g(S, t \mid \eta) * \varphi(t) * \Big[1 - \int_0^t g(S, \tau \mid \eta)\, d\tau\Big] \\
&\quad + g(S, t \mid \eta) * \Big[1 - \int_0^t \varphi(\tau)\, d\tau\Big], \\
q_k(t \mid \eta) &= \big[g(S, t \mid \eta) * \varphi(t)\big]^{(k)} * \Big[1 - \int_0^t g(S, \tau \mid \eta)\, d\tau\Big] \\
&\quad + g(S, t \mid \eta) * \big[\varphi(t) * g(S, t \mid \eta)\big]^{(k-1)} * \Big[1 - \int_0^t \varphi(\tau)\, d\tau\Big] \qquad (k = 2, 3, \dots),
\end{aligned}
$$

where $(*)$ means convolution, exponent (r) indicates r-fold convolution, $g(S, t \mid \eta)$ is the FPT pdf of $X(t)$ through S starting from $X(0) = \eta < S$ and $\varphi(t)$ is the pdf of the refractory period.

It must be underlined that the present approach towards inclusion of refractoriness in models for neuron activity differs in an essential way from the approach attempted in [16] and from that successively pursued in [22]. Indeed, in both such contributions refractoriness was treated within the context of point processes and never within a continuous processes framework.

For $k = 0, 1, 2, \dots$, let now

$$
(1.6)\qquad \pi_k(\lambda \mid \eta) := \int_0^{+\infty} e^{-\lambda t}\, q_k(t \mid \eta)\, dt \qquad (\lambda > 0)
$$

be the Laplace transform of $q_k(t \mid \eta)$. Denoting by $g_\lambda(S \mid \eta)$ and by $\Phi(\lambda)$ the Laplace transforms of $g(S, t \mid \eta)$ and $\varphi(t)$, respectively, from (1.5) we have:

$$
(1.7)\qquad
\begin{aligned}
\pi_0(\lambda \mid \eta) &= \frac{1}{\lambda}\Big[1 - g_\lambda(S \mid \eta)\Big] \\
\pi_k(\lambda \mid \eta) &= \frac{1}{\lambda}\, g_\lambda(S \mid \eta)\Big[g_\lambda(S \mid \eta)\, \Phi(\lambda)\Big]^{k-1}\Big[1 - g_\lambda(S \mid \eta)\, \Phi(\lambda)\Big] \qquad (k = 1, 2, \dots).
\end{aligned}
$$

Eqs. (1.7) will be seen to play an important role to explore the statistical characteristics of the random variable that describes the number of firings. Indeed, let

$$
(1.8)\qquad E\big\{[M(t)]^r \mid \eta\big\} := \sum_{k \ge 1} k^r\, q_k(t \mid \eta) \qquad (r = 1, 2, \dots)
$$

denote the r-th order moment of the number of firings released by the neuron up to time t, and let

$$
(1.9)\qquad \psi_r(\lambda \mid \eta) := \int_0^{+\infty} e^{-\lambda t}\, E\big\{[M(t)]^r \mid \eta\big\}\, dt = \sum_{k \ge 1} k^r\, \pi_k(\lambda \mid \eta) \qquad (r = 1, 2, \dots)
$$

be its Laplace transform. Making use of (1.7), one has (cf. [20]):

$$
(1.10)\qquad \psi_1(\lambda \mid \eta) = \frac{g_\lambda(S \mid \eta)}{\lambda\big[1 - g_\lambda(S \mid \eta)\, \Phi(\lambda)\big]}\,, \qquad \psi_2(\lambda \mid \eta) = \frac{g_\lambda(S \mid \eta)\big[1 + g_\lambda(S \mid \eta)\, \Phi(\lambda)\big]}{\lambda\big[1 - g_\lambda(S \mid \eta)\, \Phi(\lambda)\big]^2}\,.
$$

Let now $I_0, I_1, I_2, \ldots$ be a sequence of random variables, with I_0 representing the time of occurrence of the first firing and I_k $(k = 1, 2, \ldots)$ the duration of the time interval elapsing between k-th and $(k+1)$-th firing. Furthermore, let $\gamma_k(t)$ denote the pdf's of I_k $(k = 0, 1, \ldots)$. Therefore, I_0 identifies with the FPT through threshold S starting at initial state $X(0) = \eta < S$, so that $\gamma_0(t) \equiv g(S, t \mid \eta)$. Due to time-homogeneity of $X(t)$ and since $R_1, R_2, \ldots$ are iid, the interspike intervals (ISI) $I_1, I_2, \ldots$ are also iid random variables having pdf (cf. [5])

$$\gamma(t) \equiv \gamma_k(t) = \int_0^t \varphi(\vartheta)\, g(S, t - \vartheta \mid \eta)\, d\vartheta \qquad (k = 1, 2, \ldots). \tag{1.11}$$

Hence, if the firing pdf is known, the ISI pdf can be determined for any assigned distribution of the refractory period. Note that, (1.11) is useful to evaluate ISI's first three moments and variance. Indeed,

$$\begin{aligned}
&E(I) = t_1(S \mid \eta) + E(R),\\
&E(I^2) = t_2(S \mid \eta) + 2\, E(R)\, t_1(S \mid \eta) + E(R^2),\\
&E(I^3) = t_3(S \mid \eta) + 3\, E(R)\, t_2(S \mid \eta) + 3\, E(R^2)\, t_1(S \mid \eta) + E(R^3),\\
&\mathrm{Var}(I) = \mathrm{Var}(S \mid \eta) + \mathrm{Var}(R),
\end{aligned} \tag{1.12}$$

where $E(R^r)$ is the r-th order moment of refractory periods, and where $t_r(S \mid \eta)$ denotes the r-th order moment of the FPT of $X(t)$ through S conditional upon $X(0) = \eta$.

For the Wiener neuronal model, in [13] the probabilities of occurrence of multiple firings up to time t are explicitly evaluated, and exact expressions for the first two moments of the number of firings released by the neuron up to time t are given. Furthermore, ISI pdf is determined for any preassigned pdf of the refractory period. Unfortunately, these theoretical results cannot be immediately extended to other types of diffusion processes. Indeed, for the evaluation of probabilities (1.5) and of ISI pdf (1.11), the explicit expression of the firing pdf $g(S, t \mid \eta)$ is required, which is known only in very particular cases.

Denoting now by $\Gamma(\lambda)$ the Laplace transform of $\gamma(t)$, from (1.11) one has

$$\Gamma(\lambda) := \int_0^{+\infty} e^{-\lambda t}\, \gamma(t)\, dt = g_\lambda(S \mid \eta)\, \Phi(\lambda), \tag{1.13}$$

so that (1.10) lead one to

$$\psi_1(\lambda \mid \eta) = \frac{g_\lambda(S \mid \eta)}{\lambda\big[1 - \Gamma(\lambda)\big]}\,, \qquad \psi_2(\lambda \mid \eta) = \frac{g_\lambda(S \mid \eta)\big[1 + \Gamma(\lambda)\big]}{\lambda\big[1 - \Gamma(\lambda)\big]^2}\,. \tag{1.14}$$

Use of (1.14) is made in [12] to disclose the asymptotic behavior of mean and variance of the number of firings released by the neuron up to time t. Indeed, under the assumption that the first three moments of the refractory period are finite, it is proved there that the following asymptotic expressions hold:

$$\begin{aligned}
&E\big\{M(t) \mid \eta\big\} \simeq \frac{1}{E(I)}\, t + \frac{1}{2}\,\frac{E(I^2)}{E^2(I)} - \frac{t_1(S \mid \eta)}{E(I)}\,,\\
&\mathrm{Var}\big\{M(t) \mid \eta\big\} \simeq \frac{\mathrm{Var}(I)}{E^3(I)}\, t + \frac{5}{4}\,\frac{[E(I^2)]^2}{E^4(I)} - \frac{2}{3}\,\frac{E(I^3)}{E^3(I)} - \frac{1}{2}\,\frac{E(I^2)}{E^2(I)}\\
&\qquad\qquad + \frac{t_1(S \mid \eta)}{E(I)} + \frac{t_2(S \mid \eta)}{E^2(I)} - \frac{E(I^2)}{E^3(I)}\, t_1(S \mid \eta) - \frac{t_1^2(S \mid \eta)}{E^2(I)}\,,
\end{aligned} \tag{1.15}$$

where $t_n(S \mid \eta)$ and $E(I^n)$ $(n = 1, 2, 3)$ are given in (1.3) and (1.12), respectively. Hence, for long times, mean and variance of the number of firings released by the neuron up to time t are approximately linear functions of t whatever diffusion neuronal model including arbitrary random refractory period is considered. Note that even though mean and variance are linear in t, they do not qualify to be representative of a Poisson process since they are not equal and, in addition, the dependence on t includes a constant, which does not occur for Poisson processes.

In the sequel, an asymptotic analysis of the random process that models the number of neuronal firings and of the ISI pdf will be performed under the assumption that the firing pdf admits an exponential behavior, like as in the cases of Ornstein-Uhlenbeck and Feller neuronal models (cf., for instance, [18]). In particular, in Section 2 the probabilities of occurrence of multiple firings up to time t will be proved to be related to the probabilities of occurrence of events for a Poisson process of parameter $[t_1(S \mid \eta)]^{-1}$. The first two moments of the number of firings released by the neuron up to time t will also be explicitly obtained. In Section 3, under the assumption of exponentially distributed firing times, we shall analyze the asymptotic behaviors of the probability of occurrence of a single firing up to time t, of the mean and variance of the number of firings up to time t and of ISI pdf. The cases of constant, uniform, exponential, Erlang, truncated normal and hyperexponential distributions of the refractory periods will be specifically considered in Sections 4∼9. Finally, in Section 10 some numerical evaluations will be presented for the probability of single firing occurrence up to time t and for ISI pdf.

2 Exponential behavior of the firing pdf

If $\{X(t), t \geq 0\}$ possesses a steady state distribution, we expect that for firing thresholds sufficiently far from the starting point, an exponential behavior of the firing pdf takes place (cf. [9]), namely:

$$g(S, t \mid \eta) \simeq \frac{1}{t_1(S \mid \eta)} \exp\Big\{-\frac{t}{t_1(S \mid \eta)}\Big\} \qquad (S \gg \eta). \tag{2.1}$$

Under assumption (2.1), let us denote by $\{\widehat{M}(t), t \geq 0\}$ the random process representing the number of firings released up to time t, by $\widehat{q}_k(t \mid \eta)$ $(k = 0, 1, \dots)$ the probability of release of k firings up to time t and by $\widehat{\pi}_k(\lambda \mid \eta)$ its Laplace transform. Recalling (1.7), one obtains:

$$\begin{aligned} &\widehat{\pi}_0(\lambda \mid \eta) = \frac{t_1(S \mid \eta)}{1 + \lambda\, t_1(S \mid \eta)} \\ &\widehat{\pi}_k(\lambda \mid \eta) = \frac{1}{\lambda}\, \frac{1}{1 + \lambda\, t_1(S \mid \eta)} \Big[\frac{\Phi(\lambda)}{1 + \lambda\, t_1(S \mid \eta)}\Big]^{k-1} \Big[1 - \frac{\Phi(\lambda)}{1 + \lambda\, t_1(S \mid \eta)}\Big] \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (k = 1, 2, \dots), \end{aligned} \tag{2.2}$$

that implies:

$$\sum_{j \geq k} \widehat{\pi}_j(\lambda \mid \eta) = \frac{1}{\lambda}\, \frac{1}{1 + \lambda\, t_1(S \mid \eta)} \Big[\frac{\Phi(\lambda)}{1 + \lambda\, t_1(S \mid \eta)}\Big]^{k-1} \qquad (k = 1, 2, \dots). \tag{2.3}$$

Eq. (2.3) will be seen to play an important role in certain forthcoming calculations.

Under assumption (2.1), the following proposition shows that the probabilities of the number of firings released up to time t can be expressed as function of the probabilities of occurrence of multiple events in $(0, t)$ for a particular Poisson process.

Proposition 2.1 *The probability function of* $\{\widehat{M}(t), t \geq 0\}$ *is given by*

$$\widehat{q}_0(t \mid \eta) = p_0(t), \tag{2.4}$$

$$\begin{aligned}\widehat{q}_1(t \mid \eta) &= 1 - p_0(t) - \varphi(t) * \big[1 - p_0(t) - p_1(t)\big] \\ &= 1 - p_0(t) - \frac{1}{t_1(S \mid \eta)}\, \varphi(t) * \int_0^t p_1(\tau)\, d\tau,\end{aligned} \tag{2.5}$$

$$\begin{aligned}\widehat{q}_k(t \mid \eta) &= \big[\varphi(t)\big]^{(k-1)} * \sum_{j \geq k} p_j(t) - \big[\varphi(t)\big]^{(k)} * \sum_{j \geq k+1} p_j(t) \\ &= \frac{1}{t_1(S \mid \eta)} \left[\big[\varphi(t)\big]^{(k-1)} * \int_0^t p_{k-1}(\tau)\, d\tau - \big[\varphi(t)\big]^{(k)} * \int_0^t p_k(\tau)\, d\tau\right] \\ &\qquad (k = 2, 3, \dots),\end{aligned} \tag{2.6}$$

where

$$p_j(t) = \frac{1}{j!} \left[\frac{t}{t_1(S \mid \eta)}\right]^j \exp\left\{-\frac{t}{t_1(S \mid \eta)}\right\} \qquad (j = 0, 1, \dots) \tag{2.7}$$

is the probability of occurrence of j *events in* $(0, t)$ *for the Poisson process* $\{N(t), t \geq 0\}$ *of parameter* $[t_1(S \mid \eta)]^{-1}$.

Proof. Taking the inverse Laplace transform of the first of (2.2), identity (2.4) immediately follows. Furthermore, denoting by $\mathcal{P}_j(\lambda)$ the Laplace transform of $p_j(t)$ $(j = 0, 1, \dots)$, from (2.7) one has:

$$\sum_{j \geq k} \mathcal{P}_j(\lambda) = \int_0^{+\infty} e^{-\lambda t} \left[\sum_{j \geq k} p_j(t)\right] dt = \frac{1}{\lambda \big[1 + \lambda\, t_1(S \mid \eta)\big]^k} \qquad (k = 1, 2, \dots). \tag{2.8}$$

Hence, (2.3) can be written as:

$$\sum_{j \geq k} \widehat{\pi}_j(\lambda \mid \eta) = [\Phi(\lambda)]^{k-1} \sum_{j \geq k} \mathcal{P}_j(\lambda) \qquad (k = 1, 2, \dots). \tag{2.9}$$

Taking the inverse Laplace transform of (2.9) one then obtains:

$$\sum_{j \geq k} \widehat{q}_j(t \mid \eta) = \begin{cases} 1 - p_0(t), & k = 1 \\ [\varphi(t)]^{(k-1)} * \sum_{j \geq k} p_j(t), & k = 2, 3, \dots . \end{cases} \tag{2.10}$$

Making use of

$$\widehat{q}_k(t \mid \eta) = \sum_{j \geq k} \widehat{q}_j(t \mid \eta) - \sum_{j \geq k+1} \widehat{q}_j(t \mid \eta) \qquad (k = 1, 2, \dots), \tag{2.11}$$

the first equalities in (2.5) and (2.6) then immediately follow. Furthermore, from the definition of incomplete gamma function (cf. [14] , p. 940, n. 8.352.1)

$$\gamma\big(j + 1, \mu\, t\big) = \mu^{j+1} \int_0^t \tau^j\, e^{-\mu\, \tau}\, d\tau = j! \left[1 - e^{-\mu\, t} \sum_{i=0}^{j} \frac{(\mu\, t)^i}{i!}\right] \qquad (j = 0, 1, \dots), \tag{2.12}$$

and making use of (2.7) we have:

$$\sum_{j\geq k} p_j(t) = 1 - \sum_{j=0}^{k-1} p_j(t) = 1 - \exp\Big\{-\frac{t}{t_1(S\mid\eta)}\Big\}\sum_{j=0}^{k-1}\frac{1}{j!}\Big[\frac{t}{t_1(S\mid\eta)}\Big]^j$$

$$= \frac{1}{t_1(S\mid\eta)}\int_0^t p_{k-1}(\tau)\,d\tau \qquad (k=1,2,\dots). \tag{2.13}$$

Hence, by virtue of (2.13), the last equalities in (2.5) and (2.6) finally follow. □

The following remark shows that, under assumption (2.1), the probability of a single firing up to time t is always greater than, or equal to, the probability of occurrence of a single event in $(0,t)$ for the Poisson process $\{N(t), t\geq 0\}$.

Remark 2.1 *For all* $t\geq 0$ *one has:*

$$\widehat{q}_1(t\mid\eta)\geq p_1(t). \tag{2.14}$$

Proof. ¿From (2.5) and (2.7) one obtains:

$$\widehat{q}_1(t\mid\eta) - p_1(t) = u(t) - \int_0^t \varphi(t-\tau)\,u(\tau)\,d\tau, \tag{2.15}$$

with

$$u(t) := 1 - p_0(t) - p_1(t) = 1 - \exp\Big\{-\frac{t}{t_1(S\mid\eta)}\Big\} - \frac{t}{t_1(S\mid\eta)}\exp\Big\{-\frac{t}{t_1(S\mid\eta)}\Big\}. \tag{2.16}$$

Since $u(t)$ is bounded, monotonically increasing and nonnegative, and since $\varphi(t)$ is nonnegative for $t>0$, one has:

$$\int_0^t \varphi(t-\tau)\,u(\tau)\,d\tau \leq u(t)\int_0^t \varphi(t-\tau)\,d\tau = u(t)\int_0^t \varphi(\vartheta)\,d\vartheta \leq u(t).$$

Hence, (2.15) leads immediately to (2.14). □

Let now

$$E\big\{[\widehat{M}(t)]^r\mid\eta\big\} := \sum_{k\geq 1} k^r\,\widehat{q}_k(t\mid\eta) \qquad (r=1,2,\dots) \tag{2.17}$$

be the r-th order moment of the number of firings released up to time t for the process $\{\widehat{M}(t), t\geq 0\}$. Under assumption (2.1), the following proposition shows that the first two moments of the number of firings released up to time t can be expressed as functions of the probabilities (2.7) of the Poisson process $\{N(t), t\geq 0\}$.

Proposition 2.2 *The first two moments of* $\{\widehat{M}(t), t \geq 0\}$ *are given by:*

$$
\begin{aligned}
E\big\{\widehat{M}(t) \mid \eta\big\} &= 1 - p_0(t) + \sum_{k\geq 2}\Big\{\big[\varphi(t)\big]^{(k-1)} * \sum_{j\geq k} p_j(t)\Big\} \\
&= 1 - p_0(t) + \frac{1}{t_1(S \mid \eta)} \sum_{j\geq 1}\Big\{\big[\varphi(t)\big]^{(j)} * \int_0^t p_j(\tau)\, d\tau\Big\},
\end{aligned}
\tag{2.18}
$$

$$
\begin{aligned}
E\big\{[\widehat{M}(t)]^2 \mid \eta\big\} &= 1 - p_0(t) + \sum_{k\geq 2}(2\,k-1)\Big\{\big[\varphi(t)\big]^{(k-1)} * \sum_{j\geq k} p_j(t)\Big\} \\
&= 1 - p_0(t) + \frac{1}{t_1(S \mid \eta)} \sum_{j\geq 1}(2\,j+1)\Big\{\big[\varphi(t)\big]^{(j)} * \int_0^t p_j(\tau)\, d\tau\Big\},
\end{aligned}
$$

with $p_j(t)$ $(j = 0, 1, \dots)$ *given by (2.7).*

Proof. By virtue of (2.17) one has

$$
E\big\{\widehat{M}(t) \mid \eta\big\} = \sum_{k\geq 1} P\big\{\widehat{M}(t) \geq k \mid \eta\big\}, \quad E\big\{[\widehat{M}(t)]^2 \mid \eta\big\} = \sum_{k\geq 1}(2\,k-1)\, P\big\{\widehat{M}(t) \geq k \mid \eta\big\},
$$

so that, making use of (2.10) and (2.13), relations (2.18) immediately follow.

Under the assumption of exponential behavior of the firing pdf, we shall now focus the attention on the sequence of random variables $\widehat{I}_0, \widehat{I}_1, \dots$, where $\widehat{I}_0$ identifies with the FPT through the threshold S starting at initial state $X(0) = \eta < S$ and $\widehat{I}_k$ $(k = 1, 2, \dots)$ describes the duration of the interval elapsing between the k-th firing and the $(k+1)$-th firing. Hence, assuming that (2.1) holds, $\widehat{I}_0$ is exponentially distributed with mean $t_1(S \mid \eta)$, whereas the pdf of $\widehat{I}_k$ is:

$$
\widehat{\gamma}(t) \equiv \widehat{\gamma}_k(t) = \int_0^t \frac{\varphi(\vartheta)}{t_1(S \mid \eta)} \exp\Big\{-\frac{t-\vartheta}{t_1(S \mid \eta)}\Big\}\, d\vartheta \qquad (k = 1, 2, \dots). \tag{2.19}
$$

The first three moments and the variance of ISI's $\widehat{I}_1, \widehat{I}_2, \dots$ are then:

$$
\begin{aligned}
&E(\widehat{I}) = t_1(S \mid \eta) + E(R), \\
&E(\widehat{I}^{\,2}) = 2\, t_1^2(S \mid \eta) + 2\, E(R)\, t_1(S \mid \eta) + E(R^2), \\
&E(\widehat{I}^{\,3}) = 6\, t_1^3(S \mid \eta) + 6\, E(R)\, t_1^2(S \mid \eta) + 3\, E(R^2)\, t_1(S \mid \eta) + E(R^3), \\
&\mathrm{Var}(\widehat{I}) = t_1^2(S \mid \eta) + \mathrm{Var}(R),
\end{aligned}
\tag{2.20}
$$

where $E(R^r)$ denotes the r-th order moment of refractory periods and $t_1(S \mid \eta)$ is the mean of the firing times. Since

$$
\widehat{\Gamma}(\lambda) = \int_0^{+\infty} e^{-\lambda\, t}\, \widehat{\gamma}(t)\, dt = \frac{\Phi(\lambda)}{1 + \lambda\, t_1(S \mid \eta)}\,, \tag{2.21}
$$

is the Laplace transform of $\widehat{\gamma}(t)$, denoting by

$$
\widehat{\psi}_r(\lambda \mid \eta) := \int_0^{+\infty} e^{-\lambda\, t}\, E\big\{[\widehat{M}(t)]^r \mid \eta\big\}\, dt \qquad (r = 1, 2, \dots),
$$

the Laplace transform of (2.17), from (2.18) one obtains:

$$\widehat{\psi}_1(\lambda \mid \eta) = \frac{1}{\lambda\left[1+\lambda\, t_1(S \mid \eta)\right]\left[1-\widehat{\Gamma}(\lambda)\right]}\,, \quad \widehat{\psi}_2(\lambda \mid \eta) = \frac{1+\widehat{\Gamma}(\lambda)}{\lambda\left[1+\lambda\, t_1(S \mid \eta)\right]\left[1-\widehat{\Gamma}(\lambda)\right]^2}\,. \tag{2.22}$$

In the sequel, for some types of refractoriness, use of (2.21) and (2.22) will be made to evaluate ISI pdf and the first two moments of the number of firings released up to time t.

Note that if (2.1) holds, probabilities $\widehat{q}_k(t|\eta)$ $(k=0,1,\dots)$, moments $E\big\{[\widehat{M}(t)]^r \mid \eta\big\}$ $r=1,2,\dots$ and ISI density $\widehat{\gamma}(t)$ can be viewed as asymptotic approximations of the corresponding quantities determined in Section 1.

3 Asymptotic behaviors

Assuming that (2.1) holds, in this section we analyze the asymptotic behavior of $\widehat{q}_1(t \mid \eta)$, $E\big\{\widehat{M}(t) \mid \eta\big\}$, $\mathrm{Var}\big\{\widehat{M}(t) \mid \eta\big\}$ and $\widehat{\gamma}(t)$ for long times.

Proposition 3.1 *If*

$$\begin{gathered}\lim_{t\to+\infty} \exp\Big\{\frac{t}{t_1(S \mid \eta)}\Big\} \int_t^{+\infty} \varphi(\tau)\, d\tau < +\infty,\\ \zeta_r := \lim_{t\to+\infty} \int_0^t \tau^r\, \varphi(\tau)\, \exp\Big\{\frac{\tau}{t_1(S \mid \eta)}\Big\}\, d\tau < +\infty \qquad (r=0,1)\end{gathered} \tag{3.1}$$

then,

$$\lim_{t\to+\infty} \frac{\widehat{q}_1(t \mid \eta)}{p_1(t)} = \zeta_0. \tag{3.2}$$

Proof. ¿From (2.5) and (2.7) one has:

$$\begin{aligned}\frac{\widehat{q}_1(t \mid \eta)}{p_1(t)} &= \frac{t_1(S \mid \eta)}{t}\, \exp\Big\{\frac{t}{t_1(S \mid \eta)}\Big\} \Big[1 - p_0(t) - \int_0^t \varphi(\tau)\Big\{1-p_0(t-\tau)-p_1(t-\tau)\Big\}\, d\tau\Big]\\ &= \int_0^t \varphi(\tau)\, \exp\Big\{\frac{\tau}{t_1(S \mid \eta)}\Big\}\, d\tau + \frac{t_1(S \mid \eta)}{t}\Big[\exp\Big\{\frac{t}{t_1(S \mid \eta)}\Big\} \int_t^{+\infty} \varphi(\tau)\, d\tau - 1\\ &\quad + \int_0^t \Big(1 - \frac{\tau}{t_1(S \mid \eta)}\Big)\, \varphi(\tau)\, \exp\Big\{\frac{\tau}{t_1(S \mid \eta)}\Big\}\, d\tau\Big],\end{aligned}$$

so that, recalling (3.1), one obtains (3.2). □

For $\{\widehat{M}(t), t \geq 0\}$, the following proposition discloses the behavior for long times of mean and variance of the number of firings released up to time t.

Proposition 3.2 *For large t one has:*

$$\begin{aligned}E\big\{\widehat{M}(t) \mid \eta\big\} &\simeq \frac{1}{E(\widehat{I})}\, t + \frac{E(R^2)}{2\,[E(\widehat{I})]^2},\\ \mathrm{Var}\big\{\widehat{M}(t) \mid \eta\big\} &\simeq \frac{\mathrm{Var}(\widehat{I})}{[E(\widehat{I})]^3}\, t + \frac{1}{[E(\widehat{I})]^4}\Big\{\frac{5}{4}\,[E(R^2)]^2 + \frac{3}{2}\, t_1^2(S \mid \eta)\, E(R^2)\\ &\qquad + E(R)\, E(R^2)\, t_1(S \mid \eta) - \frac{1}{2}\,[E(R)]^2\, E(R^2) - \frac{2}{3}\, E(R^3)\, E(\widehat{I})\Big\}.\end{aligned} \tag{3.3}$$

Proof. It follows from (1.15) by noting that, under assumption (2.1), $t_2(S \mid \eta) \simeq 2\, t_1^2(S \mid \eta)$ and $t_3(S \mid \eta) \simeq 6\, t_1^3(S \mid \eta)$. □

Hence, for long times, mean and variance of $\{\widehat{M}(t), t \geq 0\}$ are linear functions of t whose coefficients depend only on the mean firing time and on the first three moments of the refractory period.

Proposition 3.3 *If $\zeta_0 < +\infty$, with ζ_0 defined in (3.1), then*

$$\lim_{t\to+\infty}\Big[\exp\Big\{\frac{t}{t_1(S \mid \eta)}\Big\}\,\widehat{\gamma}(t)\Big] = \frac{\zeta_0}{t_1(S \mid \eta)}\,. \tag{3.4}$$

Proof. It follows immediately from (2.19). □

Table 1 shows the pdf and its first three moments for the cases of constant, uniform, exponential, Erlang, truncated normal and hyperexponential distributions of the refractory period, all having mean $1/\xi$. Setting $\alpha = \xi\, t_1(S \mid \eta)$, we note that conditions (3.1) hold if $\alpha > 0$ in the constant, uniform and Gaussian cases, if $\alpha > 1$ in the exponential case, if $\alpha > 1/h$ in Erlang case and if $\alpha > \max\big\{[hp_1]^{-1}, [hp_2]^{-1}, \ldots, [hp_h]^{-1}\big\}$ in hyperexponential case. Hence, if these conditions on α are satisfied, from Propositions 3.1 it follows:

$$\widehat{q}_1(t \mid \eta) \simeq \frac{\zeta_0\, t}{t_1(S \mid \eta)}\,\exp\Big\{-\frac{t}{t_1(S \mid \eta)}\Big\} \qquad (t \to +\infty) \tag{3.5}$$

while from Proposition 3.3 one obtains:

$$\widehat{\gamma}(t) \simeq \frac{\zeta_0}{t_1(S \mid \eta)}\,\exp\Big\{-\frac{t}{t_1(S \mid \eta)}\Big\} \qquad (t \to +\infty). \tag{3.6}$$

For the same choices of the refractoriness pdf of Table 1, the explicit expression of ζ_0 is indicated in Table 2. Note that ζ_0 depends only on α and always tends to 1 as α increases.

Making use of (3.3), for the same choices of the refractoriness pdf of Table 1, the asymptotic behaviors for long times of $E\big\{\widehat{M}(t) \mid \eta\big\}$ and of $\mathrm{Var}\big\{\widehat{M}(t) \mid \eta\big\}$ are indicated in Table 3. In particular, Table 3*(a)* refers to the mean. It shows that the coefficient of $t/t_1(S \mid \eta)$ is always equal to $\alpha/(\alpha+1)$ and goes to 1 as α increases. Furthermore, the constant term is always expressed in terms of α; it depends on the choice of the refractoriness pdf and tends to vanish as α increases. Table 3*(b)* refers to the variance. It shows that both the coefficient of $t/t_1(S \mid \eta)$ and the constant term are always expressed in terms of α and depend on the refractoriness pdf. Furthermore, as α increases the coefficient of $t/t_1(S \mid \eta)$ goes to 1 and the constant term tends to vanish. These considerations lead one to conjecture that only for large α, i.e. if $t_1(S \mid \eta) \gg E(R)$, the behavior of $\{\widehat{M}(t), t \geq 0\}$ can be approximated by that of Poisson process $\{N(t), t \geq 0\}$. Under the assumption of an exponentially distributed firing density, hereafter we shall consider the cases of constant, uniform, exponential, Erlang, truncated normal and hyperexponential distributions of the refractory periods (cf. Table 1). Closed form expressions for the probabilities of occurrence of multiple firings up to time t and for ISI pdf will be obtained.

	$\varphi(t)$	$E(R)$	$E(R^2)$	$E(R^3)$
Constant	$\delta\Big(t-\frac{1}{\xi}\Big)$	$\dfrac{1}{\xi}$	$\dfrac{1}{\xi^2}$	$\dfrac{1}{\xi^3}$
Uniform	$\begin{cases} \dfrac{\xi}{2}, & 0<t<\dfrac{2}{\xi} \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1}{\xi}$	$\dfrac{4}{3\,\xi^2}$	$\dfrac{2}{\xi^3}$
Exponential	$\begin{cases} \xi\, e^{-\xi\, t}, & t>0 \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1}{\xi}$	$\dfrac{2}{\xi^2}$	$\dfrac{6}{\xi^3}$
Erlang	$\begin{cases} \dfrac{(\xi\, h)^h}{(h-1)!}\, t^{h-1}\, e^{-\xi\, h\, t}, & t>0 \\ 0, & \text{otherwise} \end{cases}$ $(h=1,2,\dots)$	$\dfrac{1}{\xi}$	$\dfrac{h+1}{h\,\xi^2}$	$\dfrac{(h+1)(h+2)}{h^2\,\xi^3}$
Gaussian	$\begin{cases} \dfrac{2\,\xi}{\pi}\,\exp\Big\{-\dfrac{\xi^2\,t^2}{\pi}\Big\}, & t>0 \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1}{\xi}$	$\dfrac{\pi}{2\,\xi^2}$	$\dfrac{\pi}{\xi^3}$
Hyperexponential	$\begin{cases} h\,\xi\,\sum_{i=1}^{h} p_i^2\, e^{-h\, p_i\, \xi\, t}, & t>0 \\ 0, & \text{otherwise.} \end{cases}$ $(h=1,2,\dots,\ 0<p_i<1$ $p_1+p_2+\ldots+p_h=1)$	$\dfrac{1}{\xi}$	$\dfrac{2}{(h\,\xi)^2}\sum_{i=1}^{h}\dfrac{1}{p_i}$	$\dfrac{6}{(h\,\xi)^3}\sum_{i=1}^{h}\dfrac{1}{p_i^2}$

Table 1: Density $\varphi(t)$ of the duration of the refractory periods and first three moments.

Constant	$e^{1/\alpha}$	$(\alpha > 0)$
Uniform	$\frac{\alpha}{2}\Big(e^{2/\alpha} - 1\Big)$	$(\alpha > 0)$
Exponential	$\frac{\alpha}{\alpha - 1}$	$(\alpha > 1)$
Erlang	$\Big(\frac{\alpha\, h}{\alpha\, h - 1}\Big)^h$	$(\alpha > 1/h)$
Gaussian	$\exp\Big\{\frac{\pi}{4\,\alpha^2}\Big\}\Big[1 + \mathrm{Erf}\Big(\frac{\sqrt{\pi}}{2\,\alpha}\Big)\Big]$	$(\alpha > 0)$
Hyperexponential	$\sum_{i=1}^{h} \frac{\alpha\, h\, p_i^2}{\alpha\, h\, p_i - 1}$	$\alpha > \max\big\{[hp_1]^{-1}, [hp_2]^{-1}, \dots, [hp_h]^{-1}\big\}$

Table 2: The limit ζ_0 in (3.2) is listed for the same choices of the refractoriness pdf of Table 1.

4 Constant Refractory Period

We assume that the refractoriness pdf is given by:

$$\varphi(t) = \delta\Big(t - \frac{1}{\xi}\Big) \qquad (\xi > 0), \tag{4.1}$$

where $\delta(t)$ denotes the Dirac delta-function.

In the presence of constant refractoriness, the following proposition shows that (2.5) and (2.6) can be expressed as a linear combinations of probabilities of the Poisson process $\{N(t), t \geq 0\}$.

Proposition 4.1 *Under assumption (4.1), relation (2.4) holds and*

$$\begin{aligned} \widehat{q}_1(t \mid \eta) &= 1 - p_0(t) - H\Big(t - \frac{1}{\xi}\Big) \sum_{j \geq 2} p_j\Big(t - \frac{1}{\xi}\Big), \\ \widehat{q}_k(t \mid \eta) &= H\Big(t - \frac{k-1}{\xi}\Big) \sum_{j \geq k} p_j\Big(t - \frac{k-1}{\xi}\Big) - H\Big(t - \frac{k}{\xi}\Big) \sum_{j \geq k+1} p_j\Big(t - \frac{k}{\xi}\Big) \\ &\qquad (k = 2, 3, \dots), \end{aligned} \tag{4.2}$$

where $p_j(t)$ $(j = 0, 1, \dots)$ is given in (2.7) and where

$$H(t) = \begin{cases} 0, & t \leq 0 \\ 1, & t > 0 \end{cases} \tag{4.3}$$

denotes the Heaviside unit step function.

(a) Asymptotic behavior of $E\{\widehat{M}(t) \mid \eta\}$	
Constant	$\dfrac{\alpha}{\alpha+1}\,\dfrac{t}{t_1(S \mid \eta)} + \dfrac{1}{2\,(\alpha+1)^2}$
Uniform	$\dfrac{\alpha}{\alpha+1}\,\dfrac{t}{t_1(S \mid \eta)} + \dfrac{2}{3\,(\alpha+1)^2}$
Exponential	$\dfrac{\alpha}{\alpha+1}\,\dfrac{t}{t_1(S \mid \eta)} + \dfrac{1}{(\alpha+1)^2}$
Erlang	$\dfrac{\alpha}{\alpha+1}\,\dfrac{t}{t_1(S \mid \eta)} + \dfrac{h+1}{2\,h\,(\alpha+1)^2}$
Gaussian	$\dfrac{\alpha}{\alpha+1}\,\dfrac{t}{t_1(S \mid \eta)} + \dfrac{\pi}{4\,(\alpha+1)^2}$
Hyperexponential	$\dfrac{\alpha}{\alpha+1}\,\dfrac{t}{t_1(S \mid \eta)} + \dfrac{1}{h^2\,(\alpha+1)^2}\displaystyle\sum_{i=1}^{h}\dfrac{1}{p_i}$

(b) Asymptotic behavior of $\mathrm{Var}\{\widehat{M}(t) \mid \eta\}$	
Constant	$\dfrac{\alpha^3}{(\alpha+1)^3}\,\dfrac{t}{t_1(S \mid \eta)} + \dfrac{1}{(\alpha+1)^4}\left(\dfrac{3\,\alpha^2}{2} + \dfrac{\alpha}{3} + \dfrac{1}{12}\right)$
Uniform	$\dfrac{\alpha}{(\alpha+1)^3}\left(\alpha^2 + \dfrac{1}{3}\right)\dfrac{t}{t_1(S \mid \eta)} + \dfrac{2}{(\alpha+1)^4}\left(\alpha^2 + \dfrac{1}{9}\right)$
Exponential	$\dfrac{\alpha}{(\alpha+1)^3}\,(\alpha^2+1)\,\dfrac{t}{t_1(S \mid \eta)} + \dfrac{\alpha}{(\alpha+1)^4}\,(3\,\alpha-2)$
Erlang	$\dfrac{\alpha}{(\alpha+1)^3}\left(\alpha^2 + \dfrac{1}{h}\right)\dfrac{t}{t_1(S \mid \eta)} + \dfrac{1}{(\alpha+1)^4}\left[\dfrac{3\,(h+1)\,\alpha^2}{2\,h} + \dfrac{(h+1)\,(h-4)\,\alpha}{3\,h^2} + \dfrac{h^2-1}{12\,h^2}\right]$
Gaussian	$\dfrac{\alpha}{(\alpha+1)^3}\left(\alpha^2 + \dfrac{\pi}{2} - 1\right)\dfrac{t}{t_1(S \mid \eta)} + \dfrac{\pi}{(\alpha+1)^4}\left(\dfrac{3\,\alpha^2}{4} - \dfrac{\alpha}{6} + \dfrac{5\,\pi}{16} - \dfrac{11}{12}\right)$
Hyperexponential	$\dfrac{\alpha}{(\alpha+1)^3}\left[\alpha^2 + \dfrac{1}{h^2}\left(2\displaystyle\sum_{i=1}^{h}\dfrac{1}{p_i} - h^2\right)\right]\dfrac{t}{t_1(S \mid \eta)} + \dfrac{1}{(\alpha+1)^4}\left\{\dfrac{3\,\alpha^2}{h^2}\displaystyle\sum_{i=1}^{h}\dfrac{1}{p_i} + \dfrac{\alpha}{h^3}\left[h\displaystyle\sum_{i=1}^{h}\dfrac{1}{p_i} - 2\displaystyle\sum_{i=1}^{h}\dfrac{1}{p_i^2}\right] + \dfrac{1}{h^4}\left[5\left(\displaystyle\sum_{i=1}^{h}\dfrac{1}{p_i}\right)^2 - h^2\displaystyle\sum_{i=1}^{h}\dfrac{1}{p_i} - 4\,h\displaystyle\sum_{i=1}^{h}\dfrac{1}{p_i^2}\right]\right\}$

Table 3: Asymptotic behavior of $E\{\widehat{M}(t) \mid \eta\}$ in *(a)* and of $\mathrm{Var}\{\widehat{M}(t) \mid \eta\}$ in *(b)* for the same choices of the refractoriness pdf's of Table 1.

Proof. The Laplace transform of (4.1) is

$$\Phi(\lambda) = e^{-\lambda/\xi}, \tag{4.4}$$

so that $[\Phi(\lambda)]^k = e^{-\lambda\, k/\xi}$ $(k = 1, 2, \ldots)$ is (cf. [14], p. 1144, n. 27) the Laplace transform of

$$[\varphi(t)]^{(k)} = \delta\Big(t - \frac{k}{\xi}\Big) \qquad (k = 1, 2, \ldots). \tag{4.5}$$

Making use of (4.5), one has:

$$\begin{aligned}
\big[\varphi(t)\big]^{(k)} * \int_0^t p_k(\tau)\, d\tau &= \int_0^t d\tau\, \delta\Big(\tau - \frac{k}{\xi}\Big) \int_0^{t-\tau} p_k(\vartheta)\, d\vartheta \\
&= H\Big(t - \frac{k}{\xi}\Big) \int_0^{t-k/\xi} p_k(\vartheta)\, d\vartheta \\
&= t_1(S \mid \eta)\, H\Big(t - \frac{k}{\xi}\Big) \sum_{j \geq k+1} p_j\Big(t - \frac{k}{\xi}\Big) \qquad (k = 1, 2, \ldots),
\end{aligned} \tag{4.6}$$

where the last equality follows from (2.13). Hence, by virtue of (4.6), Equations (2.5) and (2.6) lead to (4.2). □

Note that Proposition 4.1 in particular yields the results obtained in [16] and [22] concerning the output probability function for a non-linear switching element with finite dead time $\tau = 1/\xi$ subject to a Poisson distributed sequence of impulses. Indeed, assuming that its net input in $(0, t)$ consists of a sequence of over-threshold pulses whose occurrence times are Poisson distributed with parameter $[t_1(S \mid \eta)]^{-1}$, Proposition 4.1 gives the output probability function. In particular, from the first of (4.2) one has:

$$\widehat{q}_1(t \mid \eta) = \begin{cases} 1 - p_0(t)\,, & 0 \leq t \leq 1/\xi \\[2ex] p_0\Big(t - \dfrac{1}{\xi}\Big) + p_1\Big(t - \dfrac{1}{\xi}\Big) - p_0(t), & t > 1/\xi. \end{cases} \tag{4.7}$$

Setting

$$t_1^*(S \mid \eta) := \frac{1}{\xi} + t_1(S \mid \eta)\, \exp\Big\{-\frac{1}{\xi\, t_1(S \mid \eta)}\Big\}, \tag{4.8}$$

we note that $\widehat{q}_1(t \mid \eta)$ is monotonic increasing for $t < t_1^*(S \mid \eta)$ and monotonic decreasing for $t > t_1^*(S \mid \eta)$. Furthermore, since $t_1^*(S \mid \eta) \geq t_1(S \mid \eta)$, the value $t = t_1^*(S \mid \eta)$ that maximize $\widehat{q}_1(t \mid \eta)$ is greater than, or equal to, the value $t = t_1(S \mid \eta)$ that maximize $p_1(t)$.

Under assumption (4.1), a study of the first two moments of process $\{\widehat{M}(t), t \geq 0\}$ can be performed.

Proposition 4.2 *Under assumption (4.1), one has:*

$$\begin{aligned}
E\big\{\widehat{M}(t) \mid \eta\big\} &= \sum_{k=0}^{\lfloor t\,\xi \rfloor} \Big[1 - \sum_{j=0}^{k} p_j\Big(t - \frac{k}{\xi}\Big)\Big], \\
E\big\{[\widehat{M}(t)]^2 \mid \eta\big\} &= \sum_{k=0}^{\lfloor t\,\xi \rfloor} (2\,k + 1)\, \Big[1 - \sum_{j=0}^{k} p_j\Big(t - \frac{k}{\xi}\Big)\Big],
\end{aligned} \tag{4.9}$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

Proof. Making use of (2.10), (2.13) and (4.6) one has:

$$P\{\widehat{M}(t) \geq k \mid \eta\} = \sum_{j \geq k} \widehat{q}_j(t \mid \eta) = \begin{cases} 1 - p_0(t), & k = 1 \\ H\Big(t - \dfrac{k-1}{\xi}\Big) \displaystyle\sum_{j \geq k} p_j\Big(t - \dfrac{k-1}{\xi}\Big) & k = 2, 3, \dots \end{cases} \tag{4.10}$$

so that, by virtue of (2.18), one obtains (4.9). □

We point out that, due to (4.1), Eq. (2.19) leads to:

$$\widehat{\gamma}(t) = \begin{cases} 0, & t < \dfrac{1}{\xi} \\ \dfrac{1}{t_1(S \mid \eta)} \exp\Big\{-\dfrac{1}{t_1(S \mid \eta)}\Big(t - \dfrac{1}{\xi}\Big)\Big\}, & t > \dfrac{1}{\xi}, \end{cases} \tag{4.11}$$

that is seen to coincide with the result (8) in [22] related to the pulse-interval pdf for a Poisson process of parameter $[t_1(S \mid \eta)]^{-1}$ with a fixed dead time $1/\xi$. Hence, in the presence of constant refractoriness, if $t > 1/\xi$, ISI pdf is simply obtained by replacing t with $t - 1/\xi$ in the exponential firing density (2.1). Furthermore, as $\xi \to +\infty$, (4.11) identifies with the exponential firing density (2.1).

5 Uniform Refractory Period

Assume that the refractoriness pdf is given by:

$$\varphi(t) = \begin{cases} \dfrac{\xi}{2}, & 0 < t < \dfrac{2}{\xi} \\ 0, & \text{otherwise} \end{cases} \qquad (\xi > 0). \tag{5.1}$$

Proposition 5.1 *Under assumption (5.1), relation (2.4) holds and there hold:*

$$\begin{aligned} \widehat{q}_1(t \mid \eta) &= \frac{1}{t_1(S \mid \eta)}\, f_1(t) - \frac{\xi}{2\, t_1^2(S \mid \eta)} \sum_{n=0}^{\min\left(1,\, \lfloor \xi t/2 \rfloor\right)} \binom{1}{n} (-1)^n f_2\Big(t - \frac{2n}{\xi}\Big), \\ \widehat{q}_k(t \mid \eta) &= \Big(\frac{\xi}{2\, t_1(S \mid \eta)}\Big)^{k-1} \Big\{\frac{1}{t_1(S \mid \eta)} \sum_{n=0}^{\min\left(k-1,\, \lfloor \xi t/2 \rfloor\right)} \binom{k-1}{n} (-1)^n f_k\Big(t - \frac{2n}{\xi}\Big) \\ &\quad - \frac{\xi}{2\, t_1^2(S \mid \eta)} \sum_{n=0}^{\min\left(k,\, \lfloor \xi t/2 \rfloor\right)} \binom{k}{n} (-1)^n f_{k+1}\Big(t - \frac{2n}{\xi}\Big)\Big\} \qquad (k = 2, 3, \dots) \end{aligned} \tag{5.2}$$

where

$$f_k(t) = (-1)^k [t_1(S \mid \eta)]^{2k-1} \sum_{j=0}^{k-1} \binom{2k-j-2}{k-1} \frac{1}{j!} \Big[\frac{t}{t_1(S \mid \eta)}\Big]^j \Big[\exp\Big\{-\frac{t}{t_1(S \mid \eta)}\Big\} - (-1)^j\Big] \qquad (k = 1, 2, \dots). \tag{5.3}$$

Proof. The Laplace transform of (5.1) is:

$$\Phi(\lambda) = \frac{\xi}{2\,\lambda}\left(1 - \exp\Big\{-\frac{2\,\lambda}{\xi}\Big\}\right), \tag{5.4}$$

so that, from (2.3) one has:

$$\sum_{j\geq k} \widehat{\pi}_j(\lambda \mid \eta) = \Big(\frac{\xi}{2}\Big)^{k-1} \frac{1}{\lambda^k\,\big[1+\lambda\; t_1(S\mid\eta)\big]^k}\left(1 - \exp\Big\{-\frac{2\,\lambda}{\xi}\Big\}\right)^{k-1} \qquad (k=1,2\ldots). \tag{5.5}$$

Note that (cf. [4], p. 238, n. 5)

$$\begin{aligned}\mathcal{L}^{-1}\left[\frac{1}{\lambda^k\Big(\lambda+\dfrac{1}{t_1(S\mid\eta)}\Big)^k}\right] &= \frac{\sqrt{\pi}}{(k-1)!}\,\big[t_1(S\mid\eta)\,t\big]^{k-1/2}\,\exp\Big\{-\frac{t}{2\,t_1(S\mid\eta)}\Big\}\\ &\quad\times t^{k-1/2}\; I_{k-1/2}\left[\frac{t}{2\,t_1(S\mid\eta)}\right] \qquad (k=1,2,\dots),\end{aligned} \tag{5.6}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform and $I_\nu(z)$ is the Bessel function of first kind. Since (cf. [14], p. 967, n. 8467)

$$I_{k-1/2}(z) = \frac{1}{\sqrt{2\,\pi\,z}}\left[e^z\sum_{n=0}^{k-1}\frac{(-1)^n\,(k+n-1)!}{n!\,(k-n-1)!\,\big(2\,z\big)^n} + (-1)^k\,e^{-z}\sum_{n=0}^{k-1}\frac{(k+n-1)!}{n!\,(k-n-1)!\,\big(2\,z\big)^n}\right] \qquad (k=1,2,\dots),$$

the right-hand side of (5.6) is seen to coincide with $f_k(t)$ given in (5.3). Then, making use of (5.6), one obtains (cf. [4] , p. 244, n. 23):

$$\mathcal{L}^{-1}\left[\frac{1}{\lambda^k\Big(\lambda+\dfrac{1}{t_1(S\mid\eta)}\Big)^k}\left(1-\exp\Big\{-\frac{2\lambda}{\xi}\Big\}\right)^{k-1}\right] = \sum_{n=0}^{\min\left(k-1,\,\lfloor\xi\,t/2\rfloor\right)}\binom{k-1}{n}(-1)^n f_k\Big(t-\frac{2\,n}{\xi}\Big) \qquad (k=1,2,\dots). \tag{5.7}$$

Taking now the inverse Laplace transform of (5.5), by virtue of (5.7), one obtains:

$$\sum_{j\geq k}\widehat{q}_j(t\mid\eta) = \frac{1}{t_1(S\mid\eta)}\left(\frac{\xi}{2\,t_1(S\mid\eta)}\right)^{k-1}\sum_{n=0}^{\min\left(k-1,\,\lfloor\xi\,t/2\rfloor\right)}\binom{k-1}{n}(-1)^n\,f_k\Big(t-\frac{2\,n}{\xi}\Big) \qquad (k=1,2,\dots). \tag{5.8}$$

Hence, making use of (5.8) in (2.11), relations (5.2) are easily proved. □

Note that, as easily seen, the first of (5.2) can also be written as:

$$\widehat{q}_1(t \mid \eta) = \begin{cases} 1 + \xi\, t_1(S \mid \eta) - \dfrac{\xi\, t}{2} - \Big[1 + \xi\, t_1(S \mid \eta) + \dfrac{\xi\, t}{2}\Big] \exp\Big\{-\dfrac{t}{t_1(S \mid \eta)}\Big\}, & 0 < t \leq 2/\xi \\ \Big[\dfrac{\xi\, t}{2} - 1 + \xi\, t_1(S \mid \eta)\Big] \exp\Big\{-\dfrac{1}{t_1(S \mid \eta)}\Big(t - \dfrac{2}{\xi}\Big)\Big\} & \\ \qquad -\Big[1 + \xi\, t_1(S \mid \eta) + \dfrac{\xi\, t}{2}\Big] \exp\Big\{-\dfrac{t}{t_1(S \mid \eta)}\Big\}, & t > 2/\xi \end{cases} \tag{5.9}$$

Proposition 5.2 *Under assumption (5.1), one has:*

$$E\big\{M(t) \mid \eta\big\} = \frac{1}{t_1(S \mid \eta)} \sum_{k \geq 0} \Big[\frac{\xi}{2\, t_1(S \mid \eta)}\Big]^k \sum_{n=0}^{\min\left(k,\, \lfloor \xi\, t/2 \rfloor\right)} \binom{k}{n} (-1)^n f_{k+1}\Big(t - \frac{2n}{\xi}\Big), \tag{5.10}$$

$$E\big\{[M(t)]^2 \mid \eta\big\} = \frac{1}{t_1(S \mid \eta)} \sum_{k \geq 0} (2\,k+1) \Big[\frac{\xi}{2\, t_1(S \mid \eta)}\Big]^k \sum_{n=0}^{\min\left(k,\, \lfloor \xi\, t/2 \rfloor\right)} \binom{k}{n} (-1)^n f_{k+1}\Big(t - \frac{2n}{\xi}\Big),$$

with $f_k(t)$ defined in (5.3).

Proof. The proof follows from (2.18) making use of (5.8). □

It is interesting to note that, by virtue of (5.1), from (2.19) one obtains:

$$\widehat{\gamma}(t) = \begin{cases} \dfrac{\xi}{2}\Big[1 - \exp\Big\{-\dfrac{t}{t_1(S \mid \eta)}\Big\}\Big], & 0 \leq t < \dfrac{2}{\xi} \\ \dfrac{\xi}{2} \exp\Big\{-\dfrac{t}{t_1(S \mid \eta)}\Big\}\Big[\exp\Big\{\dfrac{2}{\xi\, t_1(S \mid \eta)}\Big\} - 1\Big], & t \geq \dfrac{2}{\xi}. \end{cases} \tag{5.11}$$

We remark that (5.11) becomes the exponential firing density (2.1) as $\xi \to +\infty$.

6 Exponential Refractory Period

Let now consider the case of exponential refractoriness pdf:

$$\varphi(t) = \begin{cases} \xi\, e^{-\xi\, t}, & t > 0 \\ 0, & \text{otherwise} \end{cases} \qquad (\xi > 0). \tag{6.1}$$

Proposition 6.1 *Under assumption (6.1), relation (2.4) holds and one has:*

(i) if $\xi = [t_1(S \mid \eta)]^{-1}$, then

$$\widehat{q}_k(t \mid \eta) = e^{-\xi\, t} \Big[\frac{(\xi\, t)^{2k-1}}{(2k-1)!} + \frac{(\xi\, t)^{2k}}{(2k)!}\Big] \qquad (k = 1, 2, \dots); \tag{6.2}$$

(ii) if $\xi \neq [t_1(S \mid \eta)]^{-1}$*, then*

$$\widehat{q}_k(t \mid \eta) = A_k(t)\, e^{-t/t_1(S|\eta)} + B_k(t)\, e^{-\xi\, t}\,, \tag{6.3}$$

where for $k = 1, 2, \dots$ *one has*

$$\begin{aligned}A_k(t) = {}& \frac{(-1)^k}{k!}\left[\frac{\xi\, t}{1-\xi\, t_1(S \mid \eta)}\right]^k + (-1)^k\left[\frac{\xi\, t_1(S \mid \eta)}{1-\xi\, t_1(S \mid \eta)}\right]^{k-1}\\ &\times\Bigg\{\frac{\xi\, t_1(S \mid \eta)}{1-\xi\, t_1(S \mid \eta)}\sum_{r=0}^{k-1}\binom{2k-r-1}{k-1}\frac{1}{[t_1(S \mid \eta)]^r\,[1-\xi\, t_1(S \mid \eta)]^{k-r}}\,\frac{t^r}{r!}\\ &\qquad+\sum_{r=0}^{k-1}\binom{2k-r-2}{k-1}\frac{1}{[t_1(S \mid \eta)]^r\,[1-\xi\, t_1(S \mid \eta)]^{k-r}}\,\frac{t^r}{r!}\Bigg\},\end{aligned}$$

$$\begin{aligned}B_k(t) = {}& -\frac{1}{[1-\xi\, t_1(S \mid \eta)]^k}\sum_{r=0}^{k-1}(-1)^{k-r}\binom{2k-r-1}{k}\left[\frac{\xi\, t_1(S \mid \eta)}{1-\xi\, t_1(S \mid \eta)}\right]^{k-r}\frac{(\xi\, t)^r}{r!}\\ &-\frac{1}{[1-\xi\, t_1(S \mid \eta)]^k}\sum_{r=0}^{k-1}(-1)^{k-r}\binom{2k-r-2}{k-1}\left[\frac{\xi\, t_1(S \mid \eta)}{1-\xi\, t_1(S \mid \eta)}\right]^{k-r-1}\frac{(\xi\, t)^r}{r!}.\end{aligned} \tag{6.4}$$

Proof. The Laplace transform of (6.1) is:

$$\Phi(\lambda) = \frac{\xi}{\lambda+\xi}, \tag{6.5}$$

so that, from (2.3) it follows:

$$\sum_{j\geq k}\widehat{\pi}_j(\lambda \mid \eta) = \frac{\xi^{k-1}}{[t_1(S \mid \eta)]^k}\,\frac{1}{\lambda\,(\lambda+\xi)^{k-1}\Big(\lambda+\dfrac{1}{t_1(S \mid \eta)}\Big)^k} \qquad (k = 1, 2, \dots). \tag{6.6}$$

We now consider separately the following two cases: *(i)* $\xi = [t_1(S \mid \eta)]^{-1}$ and *(ii)* $\xi \neq [t_1(S \mid \eta)]^{-1}$.
(i) Let $\xi = [t_1(S \mid \eta)]^{-1}$. Then, from (6.6) one has:

$$\sum_{j\geq k}\widehat{\pi}_j(\lambda \mid \eta) = \xi^{2k-1}\,\frac{1}{\lambda\,(\lambda+\xi)^{2k-1}} \qquad (k = 1, 2, \dots). \tag{6.7}$$

Since (cf. [4] , p. 232, n. 18)

$$\mathcal{L}^{-1}\left[\frac{1}{\lambda\big(\lambda+a\big)^n}\right] = \frac{1}{a^n}\left[1-e^{-a\,t}\sum_{r=0}^{n-1}\frac{(a\,t)^r}{r!}\right] \qquad (n = 1, 2, \dots), \tag{6.8}$$

taking the inverse Laplace transform of (6.7), one obtains:

$$\sum_{j\geq k}\widehat{q}_j(t \mid \eta) = 1-e^{-\xi\, t}\sum_{r=0}^{2\,k-2}\frac{(\xi\, t)^r}{r!} \qquad (k = 1, 2, \dots). \tag{6.9}$$

Hence, by virtue of (2.11) and (6.9), one is immediately lead to (6.2).

(ii) Let $\xi \neq [t_1(S \mid \eta)]^{-1}$. We note that

$$\mathcal{L}^{-1}\left[\frac{1}{\lambda\,(\lambda+\xi)^{k-1}\Big(\lambda+\dfrac{1}{t_1(S\mid\eta)}\Big)^k}\right]=\int_0^t d_k(\tau)\,d\tau \qquad (k=1,2,\dots), \tag{6.10}$$

where (cf. [4] , p. 232, n. 21)

$$\begin{aligned} &d_1(t)=e^{-t/t_1(S|\eta)}\,,\\ &d_k(t)=e^{-t/t_1(S|\eta)}\,(-1)^{k-1}\sum_{j=0}^{k-1}\binom{2k-j-3}{k-2}\left[\frac{t_1(S\mid\eta)}{1-\xi\,t_1(S\mid\eta)}\right]^{2k-j-2}\frac{t^j}{j!}\\ &\qquad +e^{-\xi\,t}\sum_{j=0}^{k-2}(-1)^{k-j}\binom{2k-j-3}{k-1}\left[\frac{t_1(S\mid\eta)}{1-\xi\,t_1(S\mid\eta)}\right]^{2k-j-2}\frac{t^j}{j!} \qquad (k=2,3,\dots). \end{aligned} \tag{6.11}$$

Taking the inverse Laplace transform of (6.6) and making use of (6.10) and (6.11) one has:

$$\begin{aligned} &\sum_{j\geq 1}\widehat{q}_j(t\mid\eta)=1-e^{-t/t_1(S|\eta)}\,,\\ &\sum_{j\geq k}\widehat{q}_j(t\mid\eta)=\frac{\xi^{k-1}}{[t_1(S\mid\eta)]^k}\int_0^t d_k(\tau)\,d\tau=U_k+V_k(t)\,e^{-t/t_1(S|\eta)}+Z_k(t)\,e^{-\xi\,t}\\ &\qquad\qquad (k=2,3,\dots), \end{aligned} \tag{6.12}$$

where for $k=2,3,\dots$ we have set:

$$\begin{aligned} U_k&=\big[-\xi\,t_1(S\mid\eta)\big]^{k-1}\sum_{j=0}^{k-1}\binom{2\,k-j-3}{k-2}\frac{1}{[1-\xi\,t_1(S\mid\eta)]^{2\,k-j-2}}\\ &\quad+\sum_{j=0}^{k-2}(-1)^{k-j}\binom{2\,k-j-3}{k-1}\frac{[\xi\,t_1(S\mid\eta)]^{k-j-2}}{[1-\xi\,t_1(S\mid\eta)]^{2\,k-j-2}}\,,\\ V_k(t)&=-\big[-\xi\,t_1(S\mid\eta)\big]^{k-1}\sum_{j=0}^{k-1}\binom{2\,k-j-3}{k-2}\frac{1}{[1-\xi\,t_1(S\mid\eta)]^{2\,k-j-2}}\sum_{i=0}^{j}\frac{1}{i!}\Big(\frac{t}{t_1(S\mid\eta)}\Big)^i,\\ Z_k(t)&=-\sum_{j=0}^{k-2}(-1)^{k-j}\binom{2\,k-j-3}{k-1}\frac{[\xi\,t_1(S\mid\eta)]^{k-j-2}}{[1-\xi\,t_1(S\mid\eta)]^{2\,k-j-2}}\sum_{i=0}^{j}\frac{(\xi\,t)^i}{i!}. \end{aligned}$$

Hence, by virtue of (6.12), from (2.11) one obtains:

$$\begin{aligned} &\widehat{q}_1(t\mid\eta)=1-e^{-t/t_1(S|\eta)}-U_2-V_2(t)\,e^{-t/t_1(S|\eta)}-Z_2(t)\,e^{-\xi\,t},\\ &\widehat{q}_k(t\mid\eta)=U_k-U_{k+1}+[V_k(t)-V_{k+1}(t)]\,e^{-t/t_1(S|\eta)}\\ &\qquad +[Z_k(t)-Z_{k+1}(t)]\,e^{-\xi\,t} \qquad (k=2,3,\dots). \end{aligned} \tag{6.13}$$

Some rather cumbersome calculations show that

$$U_2 = 1, \qquad -1 - V_2(t) = A_1(t), \qquad Z_2(t) = B_1(t),$$
$$U_k - U_{k+1} = 0, \qquad V_k(t) - V_{k+1}(t) = A_k(t), \qquad Z_k(t) - Z_{k+1}(t) = B_k(t)$$
$$(k = 2, 3, \dots),$$

where $A_k(t)$ and $B_k(t)$ $(k = 1, 2, \dots)$ are given in (6.4). Hence, (6.13) ultimately lead to (6.3). This completes the proof of Propositions 6.1 □

Note that in particular for $k = 1$ from (6.2) and (6.3) one obtains:

$$(6.14)\quad \widehat{q}_1(t \mid \eta) = \begin{cases} e^{-\xi\, t}\, \xi\, t \Big(1 + \dfrac{\xi\, t}{2}\Big), & \xi = [t_1(S \mid \eta)]^{-1} \\ \dfrac{1}{[1 - \xi\, t_1(S \mid \eta)]^2} \Big\{ e^{-\xi\, t} + e^{-t/t_1(S|\eta)} \Big[-1 - \xi\, t + \xi^2\, t_1(S \mid \eta)\, t\Big]\Big\}, & \\ & \xi \neq [t_1(S \mid \eta)]^{-1}. \end{cases}$$

Remark 6.1 *Under assumption (6.1), if* $\xi = [t_1(S \mid \eta)]^{-1}$*,* $\widehat{q}_k(t \mid \eta)$ *identifies with* $P\big\{\lfloor N(t)/2 \rfloor = k\big\}$ $(k = 0, 1, \dots)$*, where* $\{N(t), t \geq 0\}$ *is the Poisson process of parameter* $[t_1(S \mid \eta)]^{-1}$*.*

Proof. Making use of (2.7), one has:

$$(6.15)\quad \begin{aligned} &P\Big(\Big\lfloor \frac{N(t)}{2} \Big\rfloor = 0\Big) = p_0(t) \equiv e^{-\xi\, t} \\ &P\Big(\Big\lfloor \frac{N(t)}{2} \Big\rfloor = k\Big) = P\Big\{2\,(k-1) < N(t) \leq 2\,k\Big\} = p_{2k-1}(t) + p_{2k}(t) \\ &\qquad \equiv e^{-\xi\, t} \left[\frac{(\xi\, t)^{2k-1}}{(2k-1)!} + \frac{(\xi\, t)^{2k}}{(2k)!}\right] \qquad (k = 1, 2, \dots). \end{aligned}$$

By comparing (6.2) and (6.15), the thesis follows. □

With $\xi = [t_1(S \mid \eta)]^{-1}$, let us now set:

$$t_k^*(S \mid \eta) := \frac{\sqrt{2\,k\,(2\,k-1)}}{\xi} \equiv \sqrt{2\,k\,(2\,k-1)}\; t_1(S \mid \eta) \qquad (k = 1, 2, \dots).$$

Then, $\widehat{q}_k(t \mid \eta)$ $(k = 1, 2, \dots)$ is monotonic increasing for $t < t_k^*(S \mid \eta)$ and monotonic decreasing for $t > t_k^*(S \mid \eta)$. Hence, if $\xi = [t_1(S \mid \eta)]^{-1}$, the value $t = t_k^*(S \mid \eta)$ that maximize $\widehat{q}_k(t \mid \eta)$ is greater then the value $t = k\, t_1(S \mid \eta)$ that maximize $p_k(t)$ for all $k = 1, 2, \dots$

Proposition 6.2 *Under assumption (6.1), one has:*

$$(6.16)\quad \begin{aligned} E\big\{\widehat{M}(t) \mid \eta\big\} &= \frac{\xi\, t}{1 + \xi\, t_1(S \mid \eta)} + \frac{1}{[1 + \xi\, t_1(S \mid \eta)]^2} \left[1 - \exp\Big\{-\frac{1 + \xi\, t_1(S \mid \eta)}{t_1(S \mid \eta)}\, t\Big\}\right], \\ E\big\{[\widehat{M}(t)]^2 \mid \eta\big\} &= \frac{\xi^2\, t^2}{[1 + \xi\, t_1(S \mid \eta)]^2} + \frac{\big[3 + \xi^2\, t_1^2(S \mid \eta)\big]\, \xi\, t}{[1 + \xi\, t_1(S \mid \eta)]^3} + \frac{1 + 3\,\xi^2\, t_1^2(S \mid \eta) - 2\,\xi\, t_1(S \mid \eta)}{[1 + \xi\, t_1(S \mid \eta)]^4} \\ &\quad + \Big\{\frac{2\,\xi\, t}{[1 + \xi\, t_1(S \mid \eta)]^3} - \frac{3\,\xi^2\, t_1^2(S \mid \eta) - 2\,\xi\, t_1(S \mid \eta) + 1}{[1 + \xi\, t_1(S \mid \eta)]^4}\Big\} \exp\Big\{-\frac{1 + \xi\, t_1(S \mid \eta)}{t_1(S \mid \eta)}\, t\Big\}. \end{aligned}$$

Proof. Making use of (6.5) we re-write (2.22) in the following form:

$$\widehat{\psi}_1(\lambda \mid \eta) = \frac{\lambda+\xi}{\lambda^2\big[1+\xi\, t_1(S\mid\eta)+\lambda\, t_1(S\mid\eta)\big]} = \frac{1}{t_1(S\mid\eta)}\left\{\frac{1}{\lambda\,(\lambda+a)}+\frac{\xi}{\lambda^2\,(\lambda+a)}\right\},$$

(6.17)

$$\begin{aligned}\widehat{\psi}_2(\lambda \mid \eta) &= \widehat{\psi}_1(\lambda \mid \eta) + \frac{2\,\xi\,(\lambda+\xi)}{\lambda^3\big[1+\xi\, t_1(S\mid\eta)+\lambda\, t_1(S\mid\eta)\big]^2}\\ &= \widehat{\psi}_1(\lambda \mid \eta) + \frac{2\,\xi}{t_1^2(S\mid\eta)}\left\{\frac{1}{\lambda^2\,\big(\lambda+a\big)^2}+\frac{\xi}{\lambda^3\,\big(\lambda+a\big)^2}\right\},\end{aligned}$$

where, for simplicity, we have set $\big[1+\xi\, t_1(S\mid\eta)\big]/t_1(S\mid\eta)=a$. Since

$$\begin{aligned}&\mathcal{L}^{-1}\Big[\frac{1}{\lambda\,(\lambda+a)}\Big] = \int_0^t e^{-a\,\tau}\,d\tau = \frac{1}{a}\Big(1-e^{-a\,t}\Big),\\ &\mathcal{L}^{-1}\Big[\frac{1}{\lambda^2\,(\lambda+a)}\Big] = \int_0^t \tau\, e^{-a\,(t-\tau)}\,d\tau = \frac{t}{a}-\frac{1}{a^2}\Big(1-e^{-a\,t}\Big),\\ &\mathcal{L}^{-1}\Big[\frac{1}{\lambda^2\,(\lambda+a)^2}\Big] = \int_0^t \tau\,(t-\tau)\,e^{-a\,\tau}\,d\tau = \frac{a\,t+2}{a^3}\,e^{-a\,t}+\frac{a\,t-2}{a^3},\\ &\mathcal{L}^{-1}\Big[\frac{1}{\lambda^3\,(\lambda+a)^2}\Big] = \frac{1}{2}\int_0^t \tau\,(t-\tau)^2\,e^{-a\,\tau}\,d\tau = -\frac{(a\,t+3)}{a^4}\,e^{-a\,t}+\frac{a^2\,t^2-4\,a\,t+6}{2\,a^4},\end{aligned}$$

taking the inverse Laplace transforms of (6.17), relations (6.16) follow. □

Note that, by virtue of (6.1), Eq. (2.19) leads to:

$$\widehat{\gamma}(t) = \begin{cases} \xi^2\, t\, e^{-\xi\, t}, & \xi = [t_1(S\mid\eta)]^{-1}\\[2ex] \dfrac{\xi\Big[e^{-\xi\,t}-e^{-t/t_1(S\mid\eta)}\Big]}{1-\xi\, t_1(S\mid\eta)}, & \xi \neq [t_1(S\mid\eta)]^{-1}. \end{cases} \tag{6.18}$$

Hence, if $\xi \neq [t_1(S\mid\eta)]^{-1}$, when $\xi \to +\infty$ ISI pdf (6.18) becomes the exponential firing density (2.1).

7 Erlang Refractory Period

We now consider an Erlang distributed refractory period, with pdf:

$$\varphi(t) = \begin{cases} \dfrac{(\xi\, h)^h}{(h-1)!}\, t^{h-1}\, e^{-\xi\, h\, t}, & t>0\\[2ex] 0, & \text{otherwise} \end{cases} \qquad (\xi>0,\ h=1,2,\dots). \tag{7.1}$$

Hence, refractory period may be thought of as consisting of h independent exponential stages each with mean $(\xi\, h)^{-1}$, so that (7.1) identifies with the pdf of the sum of h independent and exponentially distributed random variables having mean $(\xi\, h)^{-1}$.

Proposition 7.1 *Under assumption (7.1), for $h = 1, 2, \ldots$ relation (2.4) holds and*

(i) if $\xi = [h\, t_1(S \mid \eta)]^{-1}$, then

$$\widehat{q}_k(t \mid \eta) = e^{-\xi\, h\, t} \sum_{r=(h+1)k-h}^{(h+1)k} \frac{(\xi\, h\, t)^r}{r!} \qquad (k = 1, 2, \ldots); \tag{7.2}$$

(ii) if $\xi \neq [h\, t_1(S \mid \eta)]^{-1}$, then

$$\begin{aligned} \widehat{q}_1(t \mid \eta) &= 1 - e^{-t/t_1(S|\eta)} - U_{2,h} - V_{2,h}(t)\, e^{-t/t_1(S|\eta)} - Z_{2,h}(t)\, e^{-\xi\, h\, t}, \\ \widehat{q}_k(t \mid \eta) &= U_{k,h} - U_{k+1,h} + \big[V_{k,h}(t) - V_{k+1,h}(t)\big]\, e^{-t/t_1(S|\eta)} \\ &\quad + \big[Z_{k,h}(t) - Z_{k+1,h}(t)\big]\, e^{-\xi\, h\, t} \qquad (k = 2, 3, \ldots), \end{aligned} \tag{7.3}$$

where for $k = 2, 3, \ldots$ one has:

$$\begin{aligned} U_{k,h} &= \big[-\xi\, h\, t_1(S \mid \eta)\big]^{h(k-1)} \sum_{j=0}^{k-1} \binom{h(k-1)+k-j-2}{h(k-1)-1} \frac{1}{[1 - \xi\, h\, t_1(S \mid \eta)]^{h(k-1)+k-j-1}} \\ &\quad + \sum_{j=0}^{h(k-1)-1} \binom{h(k-1)+k-j-2}{k-1} \frac{[-\xi\, h\, t_1(S \mid \eta)]^{h(k-1)-j-1}}{[1 - \xi\, h\, t_1(S \mid \eta)]^{h(k-1)+k-j-1}}\,, \\ V_{k,h}(t) &= -\big[-\xi\, h\, t_1(S \mid \eta)\big]^{h(k-1)} \sum_{j=0}^{k-1} \binom{h(k-1)+k-j-2}{h(k-1)-1} \\ &\quad \times \frac{1}{[1 - \xi\, h\, t_1(S \mid \eta)]^{h(k-1)+k-j-1}} \sum_{i=0}^{j} \frac{1}{i!} \Big[\frac{t}{t_1(S \mid \eta)}\Big]^i, \\ Z_{k,h}(t) &= - \sum_{j=0}^{h(k-1)-1} \binom{h(k-1)+k-j-2}{k-1} \frac{[-\xi\, h\, t_1(S \mid \eta)]^{h(k-1)-j-1}}{[1 - \xi\, h\, t_1(S \mid \eta)]^{h(k-1)+k-j-1}} \sum_{i=0}^{j} \frac{(\xi\, h\, t)^i}{i!}. \end{aligned} \tag{7.4}$$

Proof. The proof is an extension of the proof provided for the exponential case. The Laplace transform of (7.1) is:

$$\Phi(\lambda) = \Big(\frac{\xi\, h}{\lambda + \xi\, h}\Big)^h \qquad (h = 1, 2, \ldots), \tag{7.5}$$

so that from (2.3) for $h = 1, 2, \ldots$ it follows:

$$\sum_{j \geq k} \widehat{\pi}_j(\lambda \mid \eta) = \frac{(\xi\, h)^{h(k-1)}}{[t_1(S \mid \eta)]^k} \frac{1}{\lambda\, (\lambda + \xi\, h)^{h(k-1)} \Big(\lambda + \dfrac{1}{t_1(S \mid \eta)}\Big)^k} \qquad (k = 1, 2, \ldots). \tag{7.6}$$

We now consider separately the following two cases: *(i)* $\xi = [h\, t_1(S \mid \eta)]^{-1}$ and *(ii)* $\xi \neq [h\, t_1(S \mid \eta)]^{-1}$.
(i) Let $\xi = [h\, t_1(S \mid \eta)]^{-1}$. From (7.6) one has:

$$\sum_{j \geq k} \widehat{\pi}_j(\lambda \mid \eta) = (\xi\, h)^{h(k-1)+k} \frac{1}{\lambda\, (\lambda + \xi\, h)^{h(k-1)+k}} \qquad (k = 1, 2, \ldots), \tag{7.7}$$

so that, recalling (6.8), one finds:

$$\sum_{j\geq k}\widehat{q}_j(t\mid\eta)=1-e^{-\xi\,h\,t}\sum_{r=0}^{(h+1)\,(k-1)}\frac{(\xi\,h\,t)^r}{r!}\qquad(k=1,2,\ldots).\tag{7.8}$$

Hence, by virtue of (2.11) and (7.8), one is immediately led to (7.2).

(ii) Let $\xi\neq[h\,t_1(S\mid\eta)]^{-1}$. We note that for $h=1,2,\ldots$ there holds:

$$\mathcal{L}^{-1}\left[\frac{1}{\lambda\,(\lambda+\xi\,h)^{h(k-1)}\Big(\lambda+\dfrac{1}{t_1(S\mid\eta)}\Big)^k}\right]=\int_0^t d_{k,h}(\tau)\;d\tau\qquad(k=1,2,\ldots),\tag{7.9}$$

where (cf. [4] , p. 232, n. 21):

$$\begin{aligned}d_{1,h}(t)&=e^{-t/t_1(S|\eta)}\,,\\ d_{k,h}(t)&=e^{-t/t_1(S|\eta)}\,(-1)^s\sum_{j=0}^{k-1}\binom{s+k-j-2}{s-1}\left[\frac{t_1(S\mid\eta)}{1-\xi\,h\,t_1(S\mid\eta)}\right]^{s+k-j-1}\frac{t^j}{j!}\\ &\quad+e^{-\xi\,h\,t}\sum_{j=0}^{s-1}(-1)^{s-j-1}\binom{s+k-j-2}{k-1}\left[\frac{t_1(S\mid\eta)}{1-\xi\,h\,t_1(S\mid\eta)}\right]^{s+k-j-1}\frac{t^j}{j!}\\ &\qquad\qquad(k=2,3,\ldots),\end{aligned}\tag{7.10}$$

with $s=h(k-1)$. Taking now the inverse Laplace transform of (7.6) and making use of (7.9) and (7.10), one obtains:

$$\begin{aligned}\sum_{j\geq 1}\widehat{q}_j(t\mid\eta)&=1-e^{-t/t_1(S|\eta)}\,,\\ \sum_{j\geq k}\widehat{q}_j(t\mid\eta)&=\frac{(\xi\,h)^{h(k-1)}}{[t_1(S\mid\eta)]^k}\int_0^t d_{k,h}(\tau)\;d\tau\\ &=U_{k,h}+V_{k,h}(t)\,e^{-t/t_1(S|\eta)}+Z_{k,h}(t)\,e^{-\xi\,h\,t}\qquad(k=2,3,\ldots),\end{aligned}\tag{7.11}$$

where $U_{k,h}$, $V_{k,h}(t)$ and $Z_{k,h}(t)$ are given in (7.4). Hence, by virtue of (7.11), from (2.11) one finally obtains (7.3). □

Note that for $k=1$, from (7.2) and (7.3) in particular one obtains:

$$\widehat{q}_1(t\mid\eta)=\begin{cases}e^{-\xi\,h\,t}\displaystyle\sum_{r=1}^{h+1}\frac{(\xi\,h\,t)^r}{r!}\,, & \xi=[h\,t_1(S\mid\eta)]^{-1}\\[2ex] \left\{\left[\dfrac{\xi\,h\,t_1(S\mid\eta)-h-1}{\xi\,h\,t_1(S\mid\eta)-1}+\dfrac{t}{t_1(S\mid\eta)}\right]\left[\dfrac{\xi\,h\,t_1(S\mid\eta)}{\xi\,h\,t_1(S\mid\eta)-1}\right]^h-1\right\}e^{-t/t_1(S|\eta)} & \\ \quad+\displaystyle\sum_{r=0}^{h-1}\frac{(\xi\,h\,t)^r}{r!}\left\{1-\left[\frac{\xi\,h\,t_1(S\mid\eta)}{\xi\,h\,t_1(S\mid\eta)-1}\right]^{h-r}\left[1-\frac{h-r}{\xi\,h\,t_1(S\mid\eta)-1}\right]\right\}e^{-\xi\,h\,t} & \\ & \xi\neq[h\,t_1(S\mid\eta)]^{-1}.\end{cases}\tag{7.12}$$

Remark 7.1 *Under assumption (7.1), if* $\xi = [h\,t_1(S \mid \eta)]^{-1}$*,* $\widehat{q}_k(t \mid \eta)$ *identifies with* $P\big\{\lfloor N(t)/(h{+}1)\rfloor = k\big\}$ $(k = 0, 1, \dots)$*, where* $\{N(t), t \geq 0\}$ *is a Poisson process of parameter* $[t_1(S \mid \eta)]^{-1}$*.*

Proof. Making use of (2.7), one has:

$$
\begin{aligned}
&P\Big(\Big\lfloor\frac{N(t)}{h+1}\Big\rfloor = 0\Big) = p_0(t) \equiv e^{-\xi\,h\,t},\\
&P\Big(\Big\lfloor\frac{N(t)}{h+1}\Big\rfloor = k\Big) = P\Big\{(h+1)\,(k-1) < N(t) \leq (h+1)\,k\Big\}\\
&\qquad = \sum_{r=0}^{(h+1)\,k} p_r(t) - \sum_{r=0}^{(h+1)\,(k-1)} p_r(t) \equiv e^{-\xi\,h\,t} \sum_{r=(h+1)\,k-h}^{(h+1)\,k} \frac{(\xi\,h\,t)^r}{r!} \qquad (k = 1, 2, \dots).
\end{aligned}
\tag{7.13}
$$

By comparison of (7.2) and (7.13), the thesis follows. □

By virtue of (7.1), from (2.19) one has:

$$
\widehat{\gamma}(t) = \begin{cases} \xi\,\dfrac{(\xi\,h\,t)^h}{(h-1)!}\,e^{-\xi\,h\,t}, & \xi = [h\,t_1(S \mid \eta)]^{-1}\\[2ex] \dfrac{1}{t_1(S \mid \eta)}\Big[\dfrac{\xi\,h\,t_1(S \mid \eta)}{\xi\,h\,t_1(S \mid \eta) - 1}\Big]^h \Big\{e^{-t/t_1(S|\eta)} - e^{-\xi\,h\,t}\displaystyle\sum_{r=0}^{h-1}\Big[\frac{\xi\,h\,t_1(S \mid \eta) - 1}{t_1(S \mid \eta)}\Big]^r \frac{t^r}{r!}\Big\}, & \\[2ex] & \xi \neq [h\,t_1(S \mid \eta)]^{-1}\,. \end{cases}
\tag{7.14}
$$

If $\xi \neq [h\,t_1(S \mid \eta)]^{-1}$, ISI pdf (7.14) becomes the exponential firing density (2.1) when $\xi \to +\infty$.

8 Truncated Gaussian Refractory Period

Let us now consider the case of Gaussian refractoriness pdf truncated in $(0, +\infty)$:

$$
\varphi(t) = \begin{cases} \dfrac{2\,\xi}{\pi}\,\exp\Big\{-\dfrac{\xi^2\,t^2}{\pi}\Big\}, & t > 0\\[2ex] 0, & \text{otherwise} \end{cases} \qquad (\xi > 0).
\tag{8.1}
$$

The Laplace transform of $\varphi(t)$ is:

$$
\Phi(\lambda) = \exp\Big\{\frac{\lambda^2\pi}{4\,\xi^2}\Big\}\,\mathrm{Erfc}\Big(\frac{\lambda\sqrt{\pi}}{2\,\xi}\Big) = \exp\Big\{\frac{\lambda^2\pi}{4\,\xi^2}\Big\}\Big[1 - \mathrm{Erf}\Big(\frac{\lambda\sqrt{\pi}}{2\,\xi}\Big)\Big],
\tag{8.2}
$$

where

$$
\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du, \qquad \mathrm{Erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{+\infty} e^{-u^2}\,du \qquad (x \in \mathbb{R})
$$

denote the error function and the complementary error function, respectively. Making use of (8.2), from (2.3) there follows:

$$\sum_{j\geq k}\widehat{\pi}_j(\lambda\mid\eta)=\frac{1}{[t_1(S\mid\eta)]^k}\,\frac{1}{\lambda\Big(\lambda+\dfrac{1}{t_1(S\mid\eta)}\Big)^k}\left[\exp\Big\{\frac{\lambda^2\pi}{4\,\xi^2}\Big\}\,\mathrm{Erfc}\Big(\frac{\lambda\sqrt{\pi}}{2\,\xi}\Big)\right]^{k-1} \tag{8.3}$$
$$(k=1,2,\ldots).$$

Although (8.3) is much too complicated to lead to any useful expression for its inverse Laplace transforms, use of it for $k=2$ can be made to calculate $\widehat{q}_1(t\mid\eta)$.

Proposition 8.1 *Under assumption (8.1), relation (2.4) holds and one has:*

$$\begin{aligned}\widehat{q}_1(t\mid\eta)&=\mathrm{Erfc}\Big(\frac{\xi\,t}{\sqrt{\pi}}\Big)-\left[1+\frac{1}{\xi\,t_1(S\mid\eta)}\right]\exp\Big\{-\frac{t}{t_1(S\mid\eta)}\Big\}+\frac{1}{\xi\,t_1(S\mid\eta)}\exp\Big\{-\frac{\xi^2\,t^2}{\pi}\Big\}\\&\quad+\exp\Big\{-\frac{t}{t_1(S\mid\eta)}+\frac{\pi}{4\,\xi^2\,t_1^2(S\mid\eta)}\Big\}\left[1+\frac{t}{t_1(S\mid\eta)}-\frac{\pi}{2\,\xi^2\,t_1^2(S\mid\eta)}\right]\\&\quad\times\left[\mathrm{Erf}\Big(\frac{\xi\,t}{\sqrt{\pi}}-\frac{\sqrt{\pi}}{2\,\xi\,t_1(S\mid\eta)}\Big)+\mathrm{Erf}\Big(\frac{\sqrt{\pi}}{2\,\xi\,t_1(S\mid\eta)}\Big)\right].\end{aligned} \tag{8.4}$$

Proof. Setting $k=2$ in (8.3),

$$\sum_{j\geq 2}\widehat{\pi}_j(\lambda\mid\eta)=\frac{1}{[t_1(S\mid\eta)]^2}\,\frac{1}{\Big(\lambda+\dfrac{1}{t_1(S\mid\eta)}\Big)^2}\left[\frac{1}{\lambda}\exp\Big\{\frac{\lambda^2\pi}{4\,\xi^2}\Big\}\,\mathrm{Erfc}\Big(\frac{\lambda\sqrt{\pi}}{2\,\xi}\Big)\right]. \tag{8.5}$$

Since

$$\mathcal{L}^{-1}\left[\frac{1}{(\lambda+a)^2}\right]=t\,e^{-a\,t},$$

and (cf. [4] , p. 266, n. 3)

$$\mathcal{L}^{-1}\left[\frac{1}{\lambda}\,e^{b\,\lambda^2}\,\mathrm{Erfc}\big(\lambda\sqrt{b}\big)\right]=\mathrm{Erf}\Big(\frac{t}{2\sqrt{b}}\Big)\qquad(\mathrm{Re}\,b>0),$$

after taking the inverse Laplace transform of (8.5), one obtains:

$$\begin{aligned}\sum_{j\geq 2}\widehat{q}_j(t\mid\eta)&=\frac{1}{[t_1(S\mid\eta)]^2}\int_0^t(t-\tau)\exp\Big\{-\frac{t-\tau}{t_1(S\mid\eta)}\Big\}\,\mathrm{Erf}\Big(\frac{\xi\,\tau}{\sqrt{\pi}}\Big)\,d\tau\\&=\frac{2\,\xi}{\pi}\,\frac{1}{[t_1(S\mid\eta)]^2}\int_0^t d\vartheta\,\exp\Big\{-\frac{\xi^2\,\vartheta^2}{\pi}\Big\}\int_0^{t-\vartheta}x\,\exp\Big\{-\frac{x}{t_1(S\mid\eta)}\Big\}\,dx\\&=\frac{2\,\xi}{\pi}\int_0^t\exp\Big\{-\frac{\xi^2\,\vartheta^2}{\pi}\Big\}\left[1-\exp\Big\{-\frac{t-\vartheta}{t_1(S\mid\eta)}\Big\}\Big(1+\frac{t-\vartheta}{t_1(S\mid\eta)}\Big)\right]d\vartheta.\end{aligned} \tag{8.6}$$

Since

$$\int_0^t \exp\Big\{-\frac{\xi^2\,\vartheta^2}{\pi}\Big\}\,d\vartheta = \frac{\pi}{2\,\xi}\,\mathrm{Erf}\Big(\frac{\xi\,t}{\sqrt{\pi}}\Big),$$

$$\int_0^t \exp\Big\{-\Big[\frac{\xi^2\,\vartheta^2}{\pi}-\frac{\vartheta}{t_1(S\mid\eta)}\Big]\Big\}\,d\vartheta = \frac{\pi}{2\,\xi}\,\exp\Big\{\frac{\pi}{4\,\xi^2\,t_1^2(S\mid\eta)}\Big\}$$
$$\times\Big[\mathrm{Erf}\Big(\frac{\xi\,t}{\sqrt{\pi}}-\frac{\sqrt{\pi}}{2\,\xi\,t_1(S\mid\eta)}\Big)+\mathrm{Erf}\Big(\frac{\sqrt{\pi}}{2\,\xi\,t_1(S\mid\eta)}\Big)\Big],$$

$$\int_0^t (t-\vartheta)\,\exp\Big\{-\Big[\frac{\xi^2\,\vartheta^2}{\pi}-\frac{\vartheta}{t_1(S\mid\eta)}\Big]\Big\}\,d\vartheta = \frac{\pi}{2\,\xi^2}\Big[\exp\Big\{-\frac{\xi^2\,t^2}{\pi}+\frac{t}{t_1(S\mid\eta)}\Big\}-1\Big]$$
$$+\frac{\pi}{2\,\xi}\,\exp\Big\{\frac{\pi}{4\,\xi^2\,t_1^2(S\mid\eta)}\Big\}\Big[t-\frac{\pi}{2\,\xi^2\,t_1(S\mid\eta)}\Big]$$
$$\times\Big[\mathrm{Erf}\Big(\frac{\xi\,t}{\sqrt{\pi}}-\frac{\sqrt{\pi}}{2\,\xi\,t_1(S\mid\eta)}\Big)+\mathrm{Erf}\Big(\frac{\sqrt{\pi}}{2\,\xi\,t_1(S\mid\eta)}\Big)\Big],$$

relation (8.6) can also be written as:

$$\sum_{j\geq 2}\widehat{q}_j(t\mid\eta) = \mathrm{Erf}\Big(\frac{\xi\,t}{\sqrt{\pi}}\Big)+\frac{1}{\xi\,t_1(S\mid\eta)}\Big[\exp\Big\{-\frac{t}{t_1(S\mid\eta)}\Big\}-\exp\Big\{-\frac{\xi^2\,t^2}{\pi}\Big\}\Big]$$
$$-\exp\Big\{-\frac{t}{t_1(S\mid\eta)}+\frac{\pi}{4\,\xi^2\,t_1^2(S\mid\eta)}\Big\}\Big[1+\frac{t}{t_1(S\mid\eta)}-\frac{\pi}{2\,\xi^2\,t_1^2(S\mid\eta)}\Big]$$
$$\times\Big[\mathrm{Erf}\Big(\frac{\xi\,t}{\sqrt{\pi}}-\frac{\sqrt{\pi}}{2\,\xi\,t_1(S\mid\eta)}\Big)+\mathrm{Erf}\Big(\frac{\sqrt{\pi}}{2\,\xi\,t_1(S\mid\eta)}\Big)\Big]. \tag{8.7}$$

Hence, making use of (2.11) for $k=1$, by virtue of (2.4) and (8.7), Eq. (8.4) immediately follows. □

Due to (8.1), Eq. (2.19) leads to the ISI pdf:

$$\widehat{\gamma}(t) = \frac{1}{t_1(S\mid\eta)}\,\exp\Big\{-\frac{t}{t_1(S\mid\eta)}+\frac{\pi}{4\,\xi^2\,t_1^2(S\mid\eta)}\Big\}$$
$$\times\Big[\mathrm{Erf}\Big(\frac{\xi\,t}{\sqrt{\pi}}-\frac{\sqrt{\pi}}{2\,\xi\,t_1(S\mid\eta)}\Big)+\mathrm{Erf}\Big(\frac{\sqrt{\pi}}{2\,\xi\,t_1(S\mid\eta)}\Big)\Big], \tag{8.8}$$

that is seen to coincide with the result (14) in [22] related to the pulse-interval pdf for a Poisson process of parameter $[t_1(S\mid\eta)]^{-1}$ with the truncated in $(0,+\infty)$ Gaussian dead time pdf (8.1). Note that as $\xi\to+\infty$, (8.8) becomes the exponential firing density (2.1).

9 Hyperexponential Refractory Period

We now consider the case of hyperexponential refractoriness pdf:

$$\varphi(t) = \begin{cases} h\,\xi\,\displaystyle\sum_{i=1}^{h} p_i^2\,e^{-h\,p_i\,\xi\,t}, & t>0 \\ 0, & \text{otherwise.}\end{cases} \qquad (\xi>0) \tag{9.1}$$

where $0<p_i<1$ and $p_1+p_2+\ldots+p_h=1$. We remark that if $h=1$ or $p_i=1/h$ for $i=1,2,\ldots,h$, Eq. (9.1) identifies with the exponential refractoriness pdf given in (6.1).

The Laplace transform of $\varphi(t)$ is:

$$\Phi(\lambda) = h\,\xi \sum_{i=1}^{h} \frac{p_i^2}{\lambda + h\,p_i\,\xi}\,. \tag{9.2}$$

Making use of (9.2), from (2.3) it follows:

$$\sum_{j\geq k} \widehat{\pi}_j(\lambda \mid \eta) = \frac{1}{[t_1(S\mid\eta)]^k}\,\frac{1}{\lambda\Big(\lambda + \dfrac{1}{t_1(S\mid\eta)}\Big)^k}\left[h\,\xi \sum_{i=0}^{h} \frac{p_i^2}{\lambda + h\,p_i\,\xi}\right]^{k-1} \qquad (k = 1, 2, \dots). \tag{9.3}$$

Use of (9.3) for $k = 2$ can be made to calculate $\widehat{q}_1(t \mid \eta)$.

Proposition 9.1 *Under assumption (9.1), relation (2.4) holds and one has:*

$$\widehat{q}_1(t \mid \eta) = \sum_{i=1}^{h} p_i\,H_i(t) - \exp\Big\{-\frac{t}{t_1(S\mid\eta)}\Big\}, \tag{9.4}$$

where for $i = 1, 2, \dots, h$ we have set:

$$H_i(t) = \begin{cases} \Big[1 + h\,p_i\,\xi\,t + \dfrac{(h\,p_i\,\xi\,t)^2}{2}\Big]\,e^{-h\,p_i\,\xi\,t}, & \xi = [h\,p_i\,t_1(S\mid\eta)]^{-1} \\[2ex] \dfrac{h\,p_i\,\xi\,t_1(S\mid\eta)}{1 - h\,p_i\,\xi\,t_1(S\mid\eta)}\left[\dfrac{h\,p_i\,\xi\,t_1(S\mid\eta) - 2}{1 - h\,p_i\,\xi\,t_1(S\mid\eta)} - \dfrac{t}{t_1(S\mid\eta)}\right] e^{-t/t_1(S\mid\eta)} & \\[2ex] \qquad + \dfrac{1}{[1 - h\,p_i\,\xi\,t_1(S\mid\eta)]^2}\,e^{-h\,p_i\,\xi\,t}, & \xi \neq [h\,p_i\,t_1(S\mid\eta)]^{-1} \end{cases}$$

Proof. Setting $k = 2$ in (9.3) one has:

$$\sum_{j\geq 2} \widehat{\pi}_j(\lambda \mid \eta) = \frac{h\,\xi}{[t_1(S\mid\eta)]^2}\,\frac{1}{\lambda\Big(\lambda + \dfrac{1}{t_1(S\mid\eta)}\Big)^2} \sum_{i=0}^{h} \frac{p_i^2}{\lambda + h\,p_i\,\xi}\,. \tag{9.5}$$

We note that

$$\mathcal{L}^{-1}\left[\frac{1}{\lambda\Big(\lambda + \dfrac{1}{t_1(S\mid\eta)}\Big)^2 (\lambda + h\,p_i\,\xi)}\right] = \frac{1}{(h\,p_i\,\xi)^3}\left\{1 - e^{-h\,p_i\,\xi\,t}\left[1 + h\,p_i\,\xi\,t + \frac{(h\,p_i\,\xi\,t)^2}{2}\right]\right\}$$

if $\xi = [h\,p_i\,t_1(S\mid\eta)]^{-1}$, whereas

$$\mathcal{L}^{-1}\left[\frac{1}{\lambda\Big(\lambda + \dfrac{1}{t_1(S\mid\eta)}\Big)^2 (\lambda + h\,p_i\,\xi)}\right] = \frac{t_1^2(S\mid\eta)}{h\,p_i\,\xi} - \frac{t_1^2(S\mid\eta)}{h\,p_i\,\xi}\,\frac{e^{-h\,p_i\,\xi\,t}}{[1 - h\,p_i\,\xi\,t_1(S\mid\eta)]^2}$$
$$+ \frac{t_1^3(S\mid\eta)}{1 - h\,p_i\,\xi\,t_1(S\mid\eta)}\left\{\frac{2 - h\,p_i\,\xi\,t_1(S\mid\eta)}{1 - h\,p_i\,\xi\,t_1(S\mid\eta)} + \frac{t}{t_1(S\mid\eta)}\right\} e^{-t/t_1(S\mid\eta)}$$

if $\xi \neq [h\,p_i\,t_1(S\mid\eta)]^{-1}$. Taking the inverse Laplace transforms of (9.5) one obtains:

$$\sum_{j\geq 2} \widehat{q}_j(t \mid \eta) = 1 - \sum_{i=1}^{h} p_i\,H_i(t) \tag{9.6}$$

with $H_i(t)$ $(i = 1, 2, \ldots, h)$ given in (9.1). Hence, making use of (2.11) for $k = 1$, by virtue of (2.4) and (9.6), Eq. (9.4) finally follows. $\square$

By virtue of (9.1), from (2.19) one obtains:

$$\widehat{\gamma}(t) = \sum_{i=1}^{h} p_i \, L_i(t), \tag{9.7}$$

where

$$L_i(t) = \begin{cases} \dfrac{t}{[t_1(S \mid \eta)]^2} \, e^{-t/t_1(S|\eta)}, & \xi = [h \, p_i \, t_1(S \mid \eta)]^{-1} \\[2ex] \dfrac{h \, p_i \, \xi}{1 - h \, p_i \, \xi \, t_1(S \mid \eta)} \left(e^{-h \, p_i \, \xi \, t} - e^{-t/t_1(S|\eta)} \right), & \xi \neq [h \, p_i \, t_1(S \mid \eta)]^{-1}. \end{cases}$$

If $\xi \neq [h \, p_i \, t_1(S \mid \eta)]^{-1}$ for $i = 1, 2, \ldots, h$, then (9.7) becomes the exponential firing density (2.1) as $\xi \to +\infty$.

10 Some Numerical Results

In this Section, some numerical results are provided for the probability $\widehat{q}_1(t \mid \eta)$ and for ISI pdfs $\widehat{\gamma}(t)$ in the cases considered in Table 1 of constant (C), uniform (U), exponential (E_1), Erlang (E_h), truncated normal (G) and hyperexponential (H_h) distributions of the refractory period.

To this purpose, we perform a time scaling by changing t into $t \, t_1(S \mid \eta)$; furthermore, we set $\alpha = \xi \, t_1(S \mid \eta)$.

First of all, we note that for the refractoriness pdf considered in Table 1 it is possible to prove that $\widehat{q}_k\{t \, t_1(S \mid \eta) \mid \eta\}$ $(k = 0, 1, \ldots)$ does not depend on ξ and on $t_1(S \mid \eta)$ singularly, but only depends on α.

Table 4 shows the probability $\widehat{q}_1\{t \, t_1(S \mid \eta) \mid \eta\}$ obtained from (4.7), (5.9), (6.14), (7.12), (8.4) and (9.4) after changing t to $t \, t_1(S \mid \eta)$. Note that as α increases $\widehat{q}_1\{t \, t_1(S \mid \eta) \mid \eta\}$ tends to $t \, e^{-t}$, i.e. to the probability of occurrence of one event in $(0, t)$ for a Poisson process of unit parameter. In Figure 2 the probability $\widehat{q}_1\{t \, t_1(S \mid \eta) \mid \eta\}$ is plotted as function of t for deterministic *(a)*, uniform *(b)*, exponential *(c)*, Erlang with $h = 2$ *(d)*, truncated Gaussian *(e)* and hyperexponential with $h = 2$, $p_1 = 0.25$ and $p_2 = 0.75$ *(f)* refractoriness pdf, with $\alpha = 0.5, 1, 5$. The dotted curve is the graph of $t \, e^{-t}$, i.e. the asymptotic behavior of $\widehat{q}_1\{t \, t_1(S \mid \eta) \mid \eta\}$ as $\alpha \to +\infty$. Figure 2 shows that already for $\alpha = 5$ probabilities $\widehat{q}_1\{t \, t_1(S \mid \eta) \mid \eta\}$ become indistinguishable. This is also emphasized in Table 6 in which the values of these probabilities are listed for the same choices of refractoriness pdf of Figure 2 with $t = 1, 2, \ldots, 10$ and $\alpha = 5$ and $\alpha = 7$.

Table 5 shows the behaviors of functions $t_1(S \mid \eta) \, \widehat{\gamma}\{t \, t_1(S \mid \eta)\}$ obtained from (4.11), (5.11), (6.18), (7.14), (8.8) and (9.7) after changing t to $t \, t_1(S \mid \eta)$. Note that as α increases, $t_1(S \mid \eta) \, \widehat{\gamma}\{t \, t_1(S \mid \eta)\}$ approaches e^{-t}, i.e. the interarrival times pdf of a Poisson process of unit parameter. In Figure 3 $t_1(S \mid \eta) \, \widehat{\gamma}\{t \, t_1(S \mid \eta)\}$ is plotted as function of t for the same choices of Figure 2. The dotted curve is e^{-t}, that depicts the asymptotic behavior of $t_1(S \mid \eta) \, \widehat{\gamma}\{t \, t_1(S \mid \eta)\}$ as α increases. Figure 3 shows that if $t > 1$, already for $\alpha = 5$, the functions $t_1(S \mid \eta) \, \widehat{\gamma}\{t \, t_1(S \mid \eta)\}$ become indistinguishable. This is also emphasized in Tables 7 in which the values of these functions are listed for the same choices of refractoriness pdf of Figure 3 for $t = 1, 2, \ldots, 10$ and $\alpha = 5$ and $\alpha = 7$.

	$\widehat{q}_1\{t\, t_1(S \mid \eta) \mid \eta\}$
Constant	$\begin{cases} 1-e^{-t}\,, & 0 \le t \le 1/\alpha \\ e^{1/\alpha}\Big(t+1-\dfrac{1}{\alpha}-e^{-1/\alpha}\Big)\,e^{-t}, & t>1/\alpha, \end{cases}$
Uniform	$\begin{cases} 1+\alpha-\dfrac{\alpha\, t}{2}-\Big[1+\alpha+\dfrac{\alpha\, t}{2}\Big]\,e^{-t}\,, & 0<t\le 2/\alpha \\ \Big[\dfrac{\alpha\, t}{2}-1+\alpha\Big]\,e^{-t+2/\alpha}-\Big[1+\alpha+\dfrac{\alpha\, t}{2}\Big]\,e^{-t}, & t>2/\alpha \end{cases}$
Exponential	$\begin{cases} t\Big(1+\dfrac{t}{2}\Big)\,e^{-t}\,, & \alpha=1 \\ \dfrac{1}{(1-\alpha)^2}\Big[e^{-\alpha\, t}+e^{-t}\Big(-1-\alpha\, t+\alpha^2\, t\Big)\Big], & \alpha\neq 1 \end{cases}$
Erlang	$\begin{cases} e^{-t}\displaystyle\sum_{r=1}^{h+1}\frac{t^r}{r!}\,, & \alpha=1/h \\ \Big[\Big(\dfrac{\alpha\, h-h-1}{\alpha\, h-1}+t\Big)\Big(\dfrac{\alpha\, h}{\alpha\, h-1}\Big)^{h}-1\Big]\,e^{-t} & \\ +\displaystyle\sum_{r=0}^{h-1}\frac{(\alpha\, h\, t)^r}{r!}\Big[1-\Big(\frac{\alpha\, h}{\alpha\, h-1}\Big)^{h-r}\Big(1-\frac{h-r}{\alpha\, h-1}\Big)\Big]\,e^{-\alpha\, h\, t}, & \alpha\neq 1/h \end{cases}$
Gaussian	$\mathrm{Erfc}\Big(\dfrac{\alpha\, t}{\sqrt{\pi}}\Big)-\Big(1+\dfrac{1}{\alpha}\Big)\,e^{-t}+\dfrac{1}{\alpha}\,\exp\Big\{-\dfrac{\alpha^2\, t^2}{\pi}\Big\}$ $+\exp\Big\{-t+\dfrac{\pi}{4\,\alpha^2}\Big\}\Big(1+t-\dfrac{\pi}{2\,\alpha^2}\Big)\Big[\mathrm{Erf}\Big(\dfrac{\alpha\, t}{\sqrt{\pi}}-\dfrac{\sqrt{\pi}}{2\,\alpha}\Big)+\mathrm{Erf}\Big(\dfrac{\sqrt{\pi}}{2\,\alpha}\Big)\Big]$
Hyperexponential	$\displaystyle\sum_{i=1}^{h} p_i\, C_i(t)-e^{-t}$ $C_i(t)=\begin{cases} \Big(1+t+\dfrac{t^2}{2}\Big)\,e^{-t}, & p_i=[h\,\alpha]^{-1} \\ \dfrac{h\, p_i\, \alpha}{1-h\, p_i\, \alpha}\Big(\dfrac{h\, p_i\, \alpha-2}{1-h\, p_i\, \alpha}-t\Big)\,e^{-t}+\dfrac{e^{-h\, p_i\, \alpha\, t}}{(1-h\, p_i\, \alpha)^2}, & p_i\neq[h\,\alpha]^{-1} \end{cases}$

Table 4: The probabilities $\widehat{q}_1\{t\, t_1(S \mid \eta) \mid \eta\}$ are listed for the same choices of the refractory periods pdf as in Table 1.

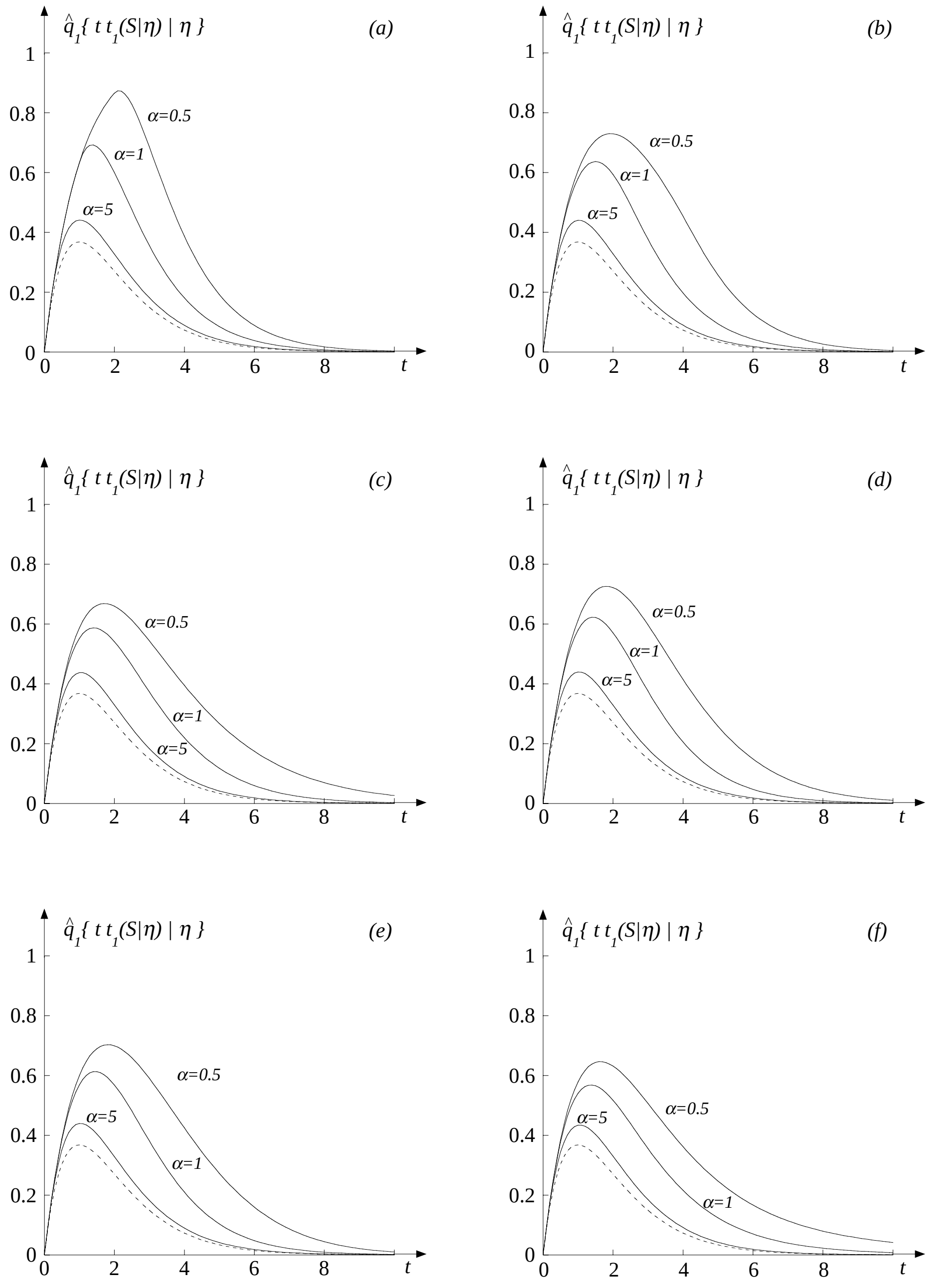

Figure 2: For the deterministic *(a)*, uniform *(b)*, exponential *(c)*, Erlang with $h = 2$ *(d)*, truncated Gaussian *(e)* and hyperexponential with $h = 2$, $p_1 = 0.25$ and $p_2 = 0.75$ *(f)* refractoriness pdf, the probabilities $\widehat{q}_1\{t\,t_1(S \mid \eta) \mid \eta\}$ are plotted with $\alpha = 0.5, 1, 5$. The dotted curve is a plot of $t\,e^{-t}$.

	$t_1(S \mid \eta)\,\widehat{\gamma}\{t\, t_1(S \mid \eta)\}$
Constant	$\begin{cases} 0, & t < 1/\alpha \\ e^{-t+1/\alpha}, & t > 1/\alpha \end{cases}$
Uniform	$\begin{cases} \dfrac{\alpha}{2}\Big(1-e^{-t}\Big), & 0 \le t < 2/\alpha \\ \dfrac{\alpha}{2}\Big(e^{2/\alpha}-1\Big)\, e^{-t}, & t \ge 2/\alpha \end{cases}$
Exponential	$\begin{cases} t\, e^{-t}, & \alpha = 1 \\ \dfrac{\alpha\left(e^{-\alpha\, t} - e^{-t}\right)}{1-\alpha}, & \alpha \neq 1 \end{cases}$
Erlang	$\begin{cases} \dfrac{t^h}{h!}\, e^{-t}, & \alpha = 1/h \\ \Big(\dfrac{\alpha\, h}{\alpha\, h - 1}\Big)^h \Big\{e^{-t} - e^{-\alpha\, h\, t} \displaystyle\sum_{r=0}^{h-1} \Big(\alpha\, h - 1\Big)^r \dfrac{t^r}{r!}\Big\}, & \alpha \neq 1/h \end{cases}$
Gaussian	$\exp\Big\{\dfrac{\pi}{4\,\alpha^2}\Big\}\Big[\mathrm{Erf}\Big(\dfrac{\alpha\, t}{\sqrt{\pi}} - \dfrac{\sqrt{\pi}}{2\,\alpha}\Big) + \mathrm{Erf}\Big(\dfrac{\sqrt{\pi}}{2\,\alpha}\Big)\Big]\, e^{-t}$
Hyperexponential	$\displaystyle\sum_{i=1}^{h} p_i\, D_i(t),$ $D_i(t) = \begin{cases} t\, e^{-t}, & \alpha = (h\, p_i)^{-1} \\ \dfrac{h\, p_i\, \alpha}{1 - h\, p_i\, \alpha}\Big(e^{-h\, p_i\, \alpha\, t} - e^{-t}\Big), & \alpha \neq (h\, p_i)^{-1} \end{cases}$

Table 5: The function $t_1(S \mid \eta)\,\widehat{\gamma}\{t\, t_1(S \mid \eta)\}$ is listed for the same choices of refractory period pdf's of Table 1.

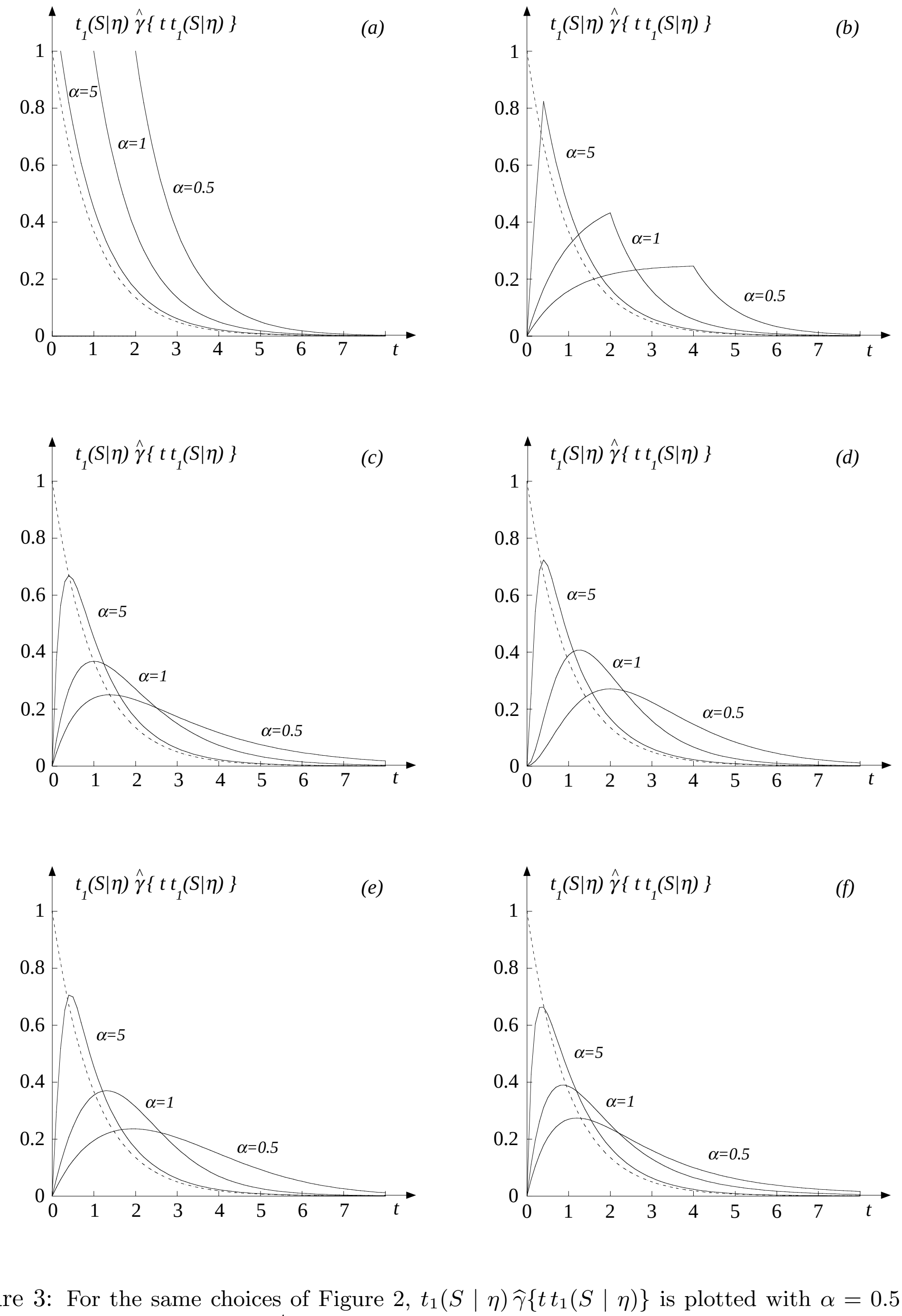

Figure 3: For the same choices of Figure 2, $t_1(S \mid \eta)\,\widehat{\gamma}\{t\,t_1(S \mid \eta)\}$ is plotted with $\alpha = 0.5, 1, 5$. The dotted curve is a plot of e^{-t}.

	t	D	U	E_1	E_2	G	H_2
$\alpha = 5$							
	1	0.44091 E0	0.44030 E0	0.43728 E0	0.43955 E0	0.43948 E0	0.43336 E0
	2	0.32750 E0	0.32838 E0	0.32988 E0	0.32878 E0	0.32897 E0	0.33032 E0
	3	0.18129 E0	0.18202 E0	0.18359 E0	0.18242 E0	0.18257 E0	0.18513 E0
	4	0.89064 E-1	0.89482 E-1	0.90433 E-1	0.89719 E-1	0.89804 E-1	0.91571 E-1
	5	0.40995 E-1	0.41203 E-1	0.41691 E-1	0.41324 E-1	0.41366 E-1	0.42324 E-1
	6	0.18109 E-1	0.18206 E-1	0.18436 E-1	0.18263 E-1	0.18282 E-1	0.18748 E-1
	7	0.77756 E-2	0.78187 E-2	0.79220 E-2	0.78442 E-2	0.78527 E-2	0.80661 E-2
	8	0.32702 E-2	0.32888 E-2	0.33337 E-2	0.32999 E-2	0.33035 E-2	0.33974 E-2
	9	0.13538 E-2	0.13616 E-2	0.13806 E-2	0.13663 E-2	0.13679 E-2	0.14081 E-2
	10	0.55348 E-3	0.55674 E-3	0.56466 E-3	0.55869 E-3	0.55933 E-3	0.57620 E-3
$\alpha = 7$							
	1	0.42024 E0	0.42003 E0	0.41900 E0	0.41979 E0	0.41977 E0	0.41715 E0
	2	0.31072 E0	0.31117 E0	0.31202 E0	0.31139 E0	0.31149 E0	0.31260 E0
	3	0.17174 E0	0.17210 E0	0.17287 E0	0.17229 E0	0.17237 E0	0.17368 E0
	4	0.84307 E-1	0.84513 E-1	0.84964 E-1	0.84625 E-1	0.84668 E-1	0.85488 E-1
	5	0.38788 E-1	0.38890 E-1	0.39118 E-1	0.38946 E-1	0.38967 E-1	0.39393 E-1
	6	0.17129 E-1	0.17176 E-1	0.17282 E-1	0.17202 E-1	0.17212 E-1	0.17414 E-1
	7	0.73532 E-2	0.73741 E-2	0.74217 E-2	0.73860 E-2	0.73902 E-2	0.74814 E-2
	8	0.30921 E-2	0.31011 E-2	0.31217 E-2	0.31062 E-2	0.31080 E-2	0.31477 E-2
	9	0.12799 E-2	0.12837 E-2	0.12924 E-2	0.12858 E-2	0.12866 E-2	0.13035 E-2
	10	0.52321 E-3	0.52479 E-3	0.52840 E-3	0.52568 E-3	0.52600 E-3	0.53305 E-3

Table 6: The values of $\widehat{q}_1\{t\,t_1(S \mid \eta) \mid \eta\}$ are listed for $\alpha = 5, 7$ and $t = 1, 2, \ldots, 10$ and for the same choices of refractoriness pdf of Figure 2.

	t	D	U	E_1	E_2	G	H_2
$\alpha = 5$							
	1	0.44933 E0	0.45233 E0	0.45143 E0	0.45361 E0	0.45469 E0	0.43696 E0
	2	0.16530 E0	0.16640 E0	0.16911 E0	0.16708 E0	0.16730 E0	0.17070 E0
	3	0.60810 E-1	0.61216 E-1	0.62233 E-1	0.61466 E-1	0.61545 E-1	0.63599 E-1
	4	0.22371 E-1	0.22520 E-1	0.22895 E-1	0.22612 E-1	0.22641 E-1	0.23463 E-1
	5	0.82297 E-2	0.82847 E-2	0.84224 E-2	0.83185 E-2	0.83292 E-2	0.86368 E-2
	6	0.30276 E-2	0.30478 E-2	0.30984 E-2	0.30602 E-2	0.30641 E-2	0.31778 E-2
	7	0.11138 E-2	0.11212 E-2	0.11399 E-2	0.11258 E-2	0.11272 E-2	0.11691 E-2
	8	0.40973 E-3	0.41247 E-3	0.41933 E-3	0.41415 E-3	0.41468 E-3	0.43008 E-3
	9	0.15073 E-3	0.15174 E-3	0.15426 E-3	0.15236 E-3	0.15255 E-3	0.15822 E-3
	10	0.55452 E-4	0.55822 E-4	0.56750 E-4	0.56049 E-4	0.56121 E-4	0.58205 E-4
$\alpha = 7$							
	1	0.42437 E0	0.42582 E0	0.42813 E0	0.42664 E0	0.42694 E0	0.42312 E0
	2	0.15612 E0	0.15665 E0	0.15789 E0	0.15696 E0	0.15706 E0	0.15923 E0
	3	0.57433 E-1	0.57628 E-1	0.58085 E-1	0.57741 E-1	0.57780 E-1	0.58687 E-1
	4	0.21128 E-1	0.21200 E-1	0.21368 E-1	0.21242 E-1	0.21256 E-1	0.21593 E-1
	5	0.77727 E-2	0.77991 E-2	0.78609 E-2	0.78144 E-2	0.78197 E-2	0.79437 E-2
	6	0.28594 E-2	0.28691 E-2	0.28919 E-2	0.28748 E-2	0.28767 E-2	0.29223 E-2
	7	0.10519 E-2	0.10555 E-2	0.10639 E-2	0.10576 E-2	0.10583 E-2	0.10751 E-2
	8	0.38698 E-3	0.38830 E-3	0.39137 E-3	0.38906 E-3	0.38932 E-3	0.39549 E-3
	9	0.14236 E-3	0.14285 E-3	0.14398 E-3	0.14313 E-3	0.14322 E-3	0.14549 E-3
	10	0.52372 E-4	0.52550 E-4	0.52967 E-4	0.52653 E-4	0.52689 E-4	0.53524 E-4

Table 7: The values of $t_1(S \mid \eta)\,\widehat{\gamma}\{t\,t_1(S \mid \eta)\}$ are listed for $\alpha = 5, 7$ and $t = 1, 2, \ldots, 10$ and for the same choices of refractoriness pdf of Figure 3.

11 Concluding Remarks

The aim of this paper has been to provide some quantitative information on the role of refractoriness in affecting the output distribution of single neurons whose activity is modeled by means of stationary diffusion processes. This task has been achieved under the reasonable assumption that the neuron firing pdf can be considered to be exponential by virtue of certain asymptotic results holding for diffusion processes that admit a steady-state distribution. In particular, denoting by η the reset state, exact formulas have been obtained for the probabilities $\widehat{q}_k(t \mid \eta)$ of the number of spikes released up to any specified instant t and for the first two moments, as well as for the ISI distribution $\widehat{\gamma}(t)$. Manageable asymptotic expressions have finally been determined for the case of refractory periods modeled as random variables of various types, and some numerical evaluation have been performed to shed light on the effects on different types of random refractoriness. In all cases, besides $\widehat{q}_0(t \mid \eta)$ that is unaffected by refractoriness, $\widehat{q}_1(t \mid \eta)$ and $\widehat{\gamma}(t)$ have been obtained in a closed form, whereas $\widehat{q}_k(t \mid \eta)$, $k > 1$, have been determined for the C, U, E_1 and E_h cases. Finally, for C, U and E_1 refractoriness distributions, also mean $E\big\{\widehat{M}(t) \mid \eta\big\}$ and second order moment $E\big\{[\widehat{M}(t)]^2 \mid \eta\big\}$ have been explicitly determined.

Our study indicates that future endeavors should focus on a systematic computational analysis to be associated to the theoretical results presented in this paper for the probabilities and moments of the number of firings released by the neuron, as well as for ISI pdf. In particular, the computational analysis carried out for $\widehat{q}_1\{t\, t_1(S \mid \eta) \mid \eta\}$ should be extended also to the probabilities of occurrence of multiple firings. Several further developments can also be envisaged, such as analysis of the role of various types of random refractoriness, within specific neuronal models (based, for instance, on Ornstein-Uhlenbeck and Feller processes) to obtain ISI pdf under various choices of refractoriness distribution.

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