GRÖBNER BASES ON PATH ALGEBRAS AND THE HOCHSCHILD COHOMOLOGY ALGEBRAS

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Received October 4, 2005

Abstract. We generalize the techniques developed in the previous paper [10] on free algebras and free bimodules to path algebras and projective bimodules. We develop the theory of Gröber bases on path algebras and their projective bimodules, and use it to construct projective resolutions of bimodules over a quotient algebra of a path algebra. It gives an effective way to calculate the Hochschild cohomology of algebras expressed as quotients of path algebras. We also give a formula for the cup product in the cohomology in terms of our resolution. It gives a way to determine the ring structure of the cohomology.

1 Introduction In our previous paper [10] we developed the theory of Gröbner bases on free algebras and free bimodules. We utilized it to construct free bimodule resolutions of algebras admitting Gröbner bases. In this paper we develop the theory of Gröber bases on path algebras and their projective bimodules, and use it to construct projective resolutions of bimodules over a quotient algebra of a path algebra. It gives a way to calculate the Hochschild cohomology of the algebras expressed as quotients of path algebras.

We consider a possibly infinite Gröbner basis G on a path algebra F of a quiver over a commutative ring. We give an algorithmic way to construct a projective bimodule resolution of a bimodule over the quotient algebra $A = F/I(G)$ where $I(G)$ is the ideal of F generated by G. In the construction Gröbner bases on projective bimodules play a crucial role as Gröbner bases on free bimodules do in the previous paper. We treat Gröbner bases from a viewpoint of rewriting systems.

We give basic results on Gröbner bases on a path algebra F and on projective F-bimodules in Sections 2 and 3. We omit most of the proofs of these basic results in this paper, because it is not difficult for the reader to modify the proofs given in [10] for our generalized situation. We need thereby suitable compatible well-founded partial orders on F and on projective F-bimodules.

Let Σ be a quiver and let $F = K\Sigma^*$ be the path algebra of Σ over a commutative ring K. Let A be the quotient algebra of F modulo the ideal $I(G)$ generated by a Gröbner basis G on F. Moreover, we consider a Gröbner basis H for an A-subbimodule L of a projective A-bimodule $A\cdot X\cdot A$ generated by a set X, that is, H is a set of monic elements of the projective F-bimodule $F\cdot X\cdot F$ generated by X such that the system $T_H = \{\mathrm{lt}(h) \to -\mathrm{rt}(h) \mid h \in H\}$ is a complete rewriting system on $F\cdot X\cdot F$ modulo G, where $\mathrm{lt}(h)$ is the leading term of h and $\mathrm{rt}(h) = h - \mathrm{lt}(h)$. Defining a morphism ∂ of A-bimodules from the projective A-bimodule $A\cdot H\cdot A$ generated by H to $A\cdot X\cdot A$ by $\partial([h]) = h$, where $[h]$ is the formal generator corresponding to $h \in H$, we have an exact sequence $A\cdot H\cdot A \xrightarrow{\partial} A\cdot X\cdot A \xrightarrow{\eta} (A\cdot X\cdot A)/L$ of A-bimodules, where η is the natural surjection.

2000 *Mathematics Subject Classification.* Primary 16E05, 16E40, 16S15.

Key words and phrases. path algebra, projective bimodule, Gröbner basis, complete rewriting system, Hochschild cohomology, cup product.

The key techniques used in the previous paper work in our new situation, and we can construct a Gröbner basis C on $F \cdot H \cdot F$ for the A-subbimodule $\mathrm{Ker}(\partial)$ of $A \cdot H \cdot A$. Again, the K-linear map $\boldsymbol{\beta}$ defined on $F \cdot X \cdot F$ plays a crucial role for our construction. This C is made from the critical pairs of reductions on $F \cdot X \cdot F$ with respect to H and G. With this C we have the projective A-bimodule $A \cdot C \cdot A$ generated by C and an exact sequence $A \cdot C \cdot A \to A \cdot H \cdot A \to A \cdot X \cdot A \to (A \cdot X \cdot A)/L$. Applying this construction inductively to the A-bimodule A itself, we have a projective A-bimodule resolution of A

$$\mathbf{H} : \cdots \to A \cdot C_n \cdot A \to A \cdot C_{n-1} \cdot A \to \cdots \to A \cdot C_0 \cdot A \to A.$$

Taking the functor $\mathrm{Hom}_{A,A}(.,A)$ on $\mathbf{H}$, we have the Hochschild cohomology $H(A)$ of A as the cohomology group of the complex $\mathrm{Hom}_{A,A}(\mathbf{H}, A)$. It has a ring structure with the cup product (Yoneda product). We give a formula for the cup product in terms of our resolution. We construct a diagonal mapping $\Delta : \mathbf{H} \to \mathbf{H} \otimes \mathbf{H}$ on the resolution $\mathbf{H}$ above, and define the cup product by $f \cup g = (f \otimes g) \circ \Delta$ for cocycles f and g. This gives an effective method to determine the algebra structure of $H(A)$. In the last section we apply our construction to special example algebras and determine their Hochschild cohomology algebras.

2 Gröbner bases on path algebras Let Σ be a quiver (finite directed graph). For $n \geq 0, \Sigma^n$ denotes the set of directed paths in Σ of length n. Accordingly, Σ^0 is the set of vertices and Σ^1 is the set of arrows of Σ. The set of all paths and the set of paths of length $\leq n$ are denoted by Σ^* and $\Sigma^{\leq n}$, respectively. We farther set $\Sigma^+ = \Sigma^* \setminus \Sigma^0$.

If x is a path from $v \in \Sigma^0$ to $v' \in \Sigma^0$, v (resp. v') is the *source* (resp. *terminal*) of x denoted by $\sigma(x)$ (resp. $\tau(x)$). For two paths x and y, if $\tau(x) = \sigma(y)$ we have another path xy concatenating x and y at $\tau(x) = \sigma(y)$. The set $\Sigma^* \cup \{0\}$ forms a semigroup with zero;

$$x \cdot y = \begin{cases} xy & \text{if} \quad \tau(x) = \sigma(y) \\ 0 & \text{otherwise} \end{cases}$$

for $x, y \in \Sigma^*$. In particular,

$$v \cdot v' = \begin{cases} v & \text{if} \quad v = v' \\ 0 & \text{if} \quad v \neq v' \end{cases}$$

for $v, v' \in \Sigma^0$. The set of paths with source v (resp. terminal v') is denoted by ${}_v\Sigma^*$ (resp. $\Sigma^*_{v'}$). In fact, ${}_v\Sigma^*$ is equal to the set $v \cdot \Sigma^* \setminus \{0\} = \{vx \,|\, x \in \Sigma^*, vx \neq 0\}$ and $\Sigma^*_{v'}$ is equal to the set $\Sigma^* \cdot v' \setminus \{0\} = \{xv' \,|\, x \in \Sigma^*, xv' \neq 0\}$. We also consider the set ${}_v\Sigma^*_{v'} = {}_v\Sigma^* \cap \Sigma^*_{v'} = \{vxv' \,|\, x \in \Sigma^*, vxv' \neq 0\}$ of paths with source v and target v'. Two paths x and y are called *parallel* if $\sigma(x) = \sigma(y)$ and $\tau(x) = \tau(y)$, that is, x and y are contained in some ${}_v\Sigma^*_{v'}$. For parallel x and y we write $x \,\|\, y$.

In accordance with the terminology in free monoids, we call a path in Σ^* a *word* over Σ and a path in Σ^+ a *nonempty word*. A nonempty word x is written as $x = a_1 a_2 \cdots a_n$ with $a_i \in \Sigma^1$ such that $\tau(a_i) = \sigma(a_{i+1})$ for $i = 1, \ldots, n-1$. The length n of the word x is denoted by $|x|$. For an empty word $v \in \Sigma^0$, we set $|v| = 0$. If $x = y \cdot z$ for $x, y, z \in \Sigma^*$, y is called a *prefix* of x and z is called a *suffix* of x. Moreover, if $x = y \cdot w \cdot z$, w is called a *subword* of x. A prefix y (suffix, subword) of x is *proper*, if $|x| > |y| > 0$.

Let K be a commutative ring with 1 and let $F = K \cdot \Sigma^*$ be the path algebra of Σ over K. F is the free K-module spanned by Σ^* with the multiplication induced by the semigroup operation of $\Sigma^* \cup \{0\}$ above. For $v, v' \in \Sigma^0$, ${}_vF$, $F_{v'}$ and ${}_vF_{v'}$ denote the K-submodules of

F spanned by ${}_v\Sigma^*$, $\Sigma^*_{v'}$ and ${}_v\Sigma^*_{v'}$, respectively. An element f of F is uniquely written as a finite sum

$$f = \sum_{i=1}^{n} k_i x_i \tag{2.1}$$

with $k_i \in K \setminus \{0\}$ and x_i are different words in Σ^*. The element f is *uniform*, if $x_i \| x_j$ for all i, j. So, f is uniform if $f \in {}_vF_{v'}$ for some $v, v' \in \Sigma^0$, and for this f we define the source $\sigma(f) = v$ and the terminal $\tau(f) = v'$. Two uniform elements f and g are *parallel* if $\sigma(f) = \sigma(g)$ and $\tau(f) = \tau(g)$.

We fix a compatible well-order $\succ$ on Σ^*, that is, $\succ$ is a strict total order on Σ^* such that there is no infinite decreasing sequence $x_1 \succ x_2 \succ \cdots$, and for any $x, y, z, w \in \Sigma^*$, $x \succ y$ implies $zxw \succ zyw$ as long as both zxw and zyw are nonzero. A typical such order is the *length-lexicographic order* $\succ_{\text{llex}}$ based on a linear order $>$ on $\Sigma^0 \cup \Sigma^1$ defined as follows. For $x = a_1 \cdots a_m$ and $y = b_1 \cdots b_n$ with $a_1, \dots, a_m, b_1, \dots, b_n \in \Sigma^1$, $x \succ_{\text{llex}} y$ if and only if (i) $m > n$, or (ii) $m = n = 0$ (that is, x an y are vetices) and $x > y$ in Σ^0, or (iii) $m = n > 0$ and x is lexicographically greater than y as words over Σ^1 with respect to the order $>$ on $\Sigma^0 \cup \Sigma^1$ restricted to Σ^1.

Let f be an element of F written as (2.1). If x_1 is the maximal among x_i $(i = 1, \dots, n)$ with respect to $\succ$, k_1x_1 is called the *leading term* of f and denoted by $\text{lt}(f)$. Let $\text{rt}(f) = f - \text{lt}(f)$.

We extend the order $\succ$ on Σ^* to a (partial) order on F, which is also denoted by $\succ$, as follows. First, $f \succ 0$ for any $f \neq 0$. For nonzero elements f and g of F with $\text{lt}(f) = k \cdot x$ and $\text{lt}(g) = \ell \cdot y$ $(k, \ell \in K \setminus \{0\}$, $x, y \in \Sigma^*)$, define $f \succ g$ if and only if either $x \succ y$, or $x = y$ and $\text{rt}(f) \succ \text{rt}(g)$. Then, $\succ$ is also well-founded, that is, there is no infinite sequence $f_1 \succ f_2 \succ \cdots$ in F. Moreover, if $f \succ g$ and f is uniform, then $x \cdot f \cdot y \succ x \cdot g \cdot y$ for any $x \in \Sigma^*_{\sigma(f)}$ and $y \in {}_{\tau(f)}\Sigma^*$.

A *rewriting rule* is a pair (u, v) such that $u \in \Sigma^+$, $v \in F$, $u \succ v$ and $u - v$ is uniform. A rule (u, v) is written as $u \to v$. A *rewriting system* R is a (not necessarily finite) set of rewriting rules. If $f \in F$ has a nonzero term $k \cdot x$ and $x = x_1ux_2$ with $x_1, x_2 \in \Sigma^*$ and $u \to v \in R$, the rule $u \to v$ can be applied to f and f is transformed to $g = f - k \cdot x_1(u - v)x_2$. In this situation we write $f \to_R g$, and we call $\to_R$ the *one-step reduction* by R.

Let $\to_R^*$ denotes the reflexive transitive closure of $\to_R$, and let $\leftrightarrow_R^*$ be the reflexive symmetric and transitive closure of $\to_R$. Set

$$G_R = \{u - v \mid u \to v \in R\},$$

and let $I(R)$ be the (two-sided) ideal of F generated by G_R. Then, the relation $\leftrightarrow_R^*$ is equal to the congruence on F modulo $I(R)$. The quotient algebra $A = F/I(R) = F/{\leftrightarrow_R^*}$ is said to be *defined by* the rewriting system R.

The relation $\to_R$ is noetherian (terminating), that is, there is no infinite sequence

$$f_1 \to_R f_2 \to_R \cdots \to_R f_n \to_R \cdots$$

in F, because $f \to_R g$ implies $f \succ g$ and $\succ$ is well-founded. If two elements f and g of F have a common R-descendant, that is, there is $h \in F$ such that $f \to_R^* h$ and $f \to_R^* h$, we say that $f \downarrow_R g$ holds. R is called *confluent* if for any $f, g, h \in F$ such that $h \to_R^* f$ and $h \to_R^* g$, $f \downarrow_R g$ holds. A noetherian and confluent system is called *complete*, but here a confluent system is complete because a rewriting system we consider is always noetherian.

An element f in F is *irreducible* (R-*irreducible* to specify R) if there is no $g \in F$ such that $f \to_R g$. In particular, an irreducible monic monomial $x \in \Sigma^*$ is called an *irreducible* *word*, and $\mathrm{Irr}(R)$ denotes the set of irreducible words. Clearly, $\mathrm{Irr}(R) = \Sigma^* \setminus \Sigma^* \cdot \mathrm{Left}(R) \cdot \Sigma^*$, where $\mathrm{Left}(R) = \{u \mid u \to v \in R\}$, and $f \in F$ is irreducible if and only if f is a K-linear combination of irreducible words. An element $f \in F$ is *R-reducible* if it is not R-irreducible. A word x is a *minimal R-reducible word* if it is R-reducible but any proper prefix y of x ($x = yx'$, $x' \in \Sigma^+$) is R-irreducible. Since R is noetherian, for any $f \in F$ there is an irreducible $\hat{f} \in F$ such that $f \to_R^* \hat{f}$. If R is confluent (so complete), such an $\hat{f}$ is unique, and is called the *normal form* of f.

Let I be an ideal of F and let $A = F/I$ be the quotient algebra. For $v, v' \in \Sigma^0$, ${}_vA$, $A_{v'}$ and ${}_vA_{v'}$ are the set of elements of A coming from elements of ${}_vF$, $F_{v'}$ and ${}_vF_v$, and are isomorphic to ${}_vF/(I \cap {}_vF)$, $F_{v'}/(I \cap F_{v'})$ and ${}_vF_{v'}/(I \cap {}_vF_{v'})$ as K-modules, respectively. We have

$$A = \bigoplus_{v,v' \in \Sigma^0} {}_vA_{v'}.$$

A subset G of F is *monic* (resp. *uniform*) if every $g \in G$ is monic (resp. uniform). A set G of generators of an ideal I is called a *Gröbner basis* of I, if it is monic, uniform and the system

$$R_G = \{\mathrm{lt}(g) \to -\mathrm{rt}(g) \mid g \in G\}$$

associated with G is a complete rewriting system on F. We confuse a Gröbner basis G with the associated rewriting system R_G. We write $g = u - v \in G$, implicitly assuming that $u = \mathrm{lt}(g)$ and $v = -\mathrm{rt}(g)$, and we simply write $\to_G$ for the relation $\to_{R_G}$. We say $f \in F$ is *G-irreducible* if it is R_G-irreducible, and $\mathrm{Left}(G)$ and $\mathrm{Irr}(G)$ denote $\mathrm{Left}(R_G)$ and $\mathrm{Irr}(R_G)$ respectively.

Now we state the fundamental results on complete rewriting systems and Gröbner bases as follows.

Proposition 2.1. *Let G be a Gröbner basis of an ideal I of a path algebra F, and let $A = F/I$ be the quotient algebra of F by I and let $\rho : F \to A$ be the canonical surjection. Then, ρ is injective on* $\mathrm{Irr}(G)$ *and* $\rho(\mathrm{Irr}(G))$ *forms a free K-basis of $A = F/I$. Any f has the unique normal form $\hat{f}$, and we have*

$$\hat{f} = \hat{g} \Leftrightarrow f \leftrightarrow_G^* g \Leftrightarrow f - g \to_G^* 0 \Leftrightarrow \rho(f) = \rho(g)$$

for any $f, g \in F$. In particular, we have

$$I = \{f \in F \mid \hat{f} = 0\} = \{f \in F \mid f \to_G^* 0\}.$$

Corollary 2.2. *An algebra over K isomorphic to the quotient F/I of a path algebra F over K modulo an ideal I with a Gröbner basis is free as a K-module.*

Let R be a rewriting system on $F = K\Sigma^*$. Let $u_1 \to v_1, u_2 \to v_2 \in R$. Suppose u_1 overlaps properly with u_2, that is, $u_1 = u_1'z$, $u_2 = zu_2'$ with $u_1', u_2' \in \Sigma^*$ and $z \in \Sigma^+$. We have two reductions $p_1 : u_1u_2' \to v_1u_2'$ and $p_2 : u_1'u_2 \to u_1'v_2$ applying the rules to $u_1u_2' = u_1'u_2$ in two different ways. We call $(v_1u_2', u_1'v_2)$ a *critical pair of elements of overlapping type* and (p_1, p_2) a *critical pair of reductions*. Next suppose u_1 contains u_2 as subword, that is, $u_1 = u'u_2u''$ with $u', u'' \in \Sigma^*$. Applying the rules to u_1 in two ways, we have a *critical pair* $(v_1, u'v_2u'')$ of elements *of inclusion type* and a critical pair $(u_1 \to v_1, u'u_2u'' \to u'v_2u'')$ of reductions. A critical pair (v_1', v_2') of elements is *resolvable* if $v_1' \downarrow_R v_2'$ holds.

The following is also basic in the rewriting theory.

Proposition 2.3. *A system R is complete if all the critical pairs are resolvable.*

A rewriting system R is *normalized* if the right-hand side v of any rule $u \to v$ from R is R-irreducible and the left-hand side u is $(R \setminus \{u \to v\})$-irreducible. A set G of monic uniform elements is *normalized* if so is R_G. If G is normalized, there is no critical pair of inclusion type. Subsets G and G' are said to be *equivalent* if they generate the same ideal.

Proposition 2.4. *For any Gröbner basis G on F, there exists a normalized Gröbner basis G' equivalent to G. If G is finite, we can choose G' to be finite.*

The Gröbner basis theory on path algebras over a field was developed in [5], [7] (see also [6]). Here we discussed Gröbner bases on path algebras over a commutative ring from a viewpoint of rewriting systems. We refer to [4] and [9] for the general theory of rewriting systems and [12] for its relationship to the Gröbner bases theory.

In the rest of this paper, G is a normalized Gröbner basis of an ideal I of the path algebra $F = K\Sigma^*$ of a quiver Σ with respect to a fixed compatible well-order $\succ$ on Σ^*, $A = F/I$ is the quotient algebra and $\rho: F \to A$ is the natural surjection.

3 Gröbner bases on projective bimodules In this section we consider projective bimodules over F and A. An *edged set* is a set X of elements ξ such that the source $\sigma(\xi) \in \Sigma^0$ and the terminal $\tau(\xi) \in \Sigma^0$ of ξ are assigned. For a nonempty edged set X we consider the projective F-bimodule $F \cdot X \cdot F$ generated by X, that is, $F \cdot X \cdot F$ is the free K-module generated by $\Sigma^* X \Sigma^* = \bigcup_{\xi \in X} \Sigma^*_{\sigma(\xi)} \times {}_{\tau(\xi)}\Sigma^*$ with two-sided F-action. The set $\Sigma^*_{\sigma(\xi)} \times {}_{\tau(\xi)}\Sigma^*$ for $\xi \in X$ is written as $\Sigma^*[\xi]\Sigma^*$ and an element (x, y) in it with $x \in \Sigma^*_{\sigma(\xi)}$ and $y \in {}_{\tau(\xi)}\Sigma^*$ is written as $x[\xi]y$. Then, an element f of $F \cdot X \cdot F$ is uniquely written as a finite sum

$$f = \sum k_i x_i [\xi_i] y_i, \tag{3.1}$$

with $k_i \in K \setminus \{0\}$, $x_i \in \Sigma^*_{\sigma(\xi_i)}$, $y_i \in {}_{\tau(\xi_i)}\Sigma^*$ and $\xi_i \in X$, where (x_i, ξ_i, y_i) are different for i. For $f = \sum k_i x_i$, $g = \sum \ell_j y_j \in F$ with $k_i, \ell_j \in K$ and $x_i, y_j \in \Sigma^*$, $f[\xi]g$ denotes the element

$$\sum_{\substack{\tau(x_i)=\sigma(\xi) \\ \sigma(y_j)=\tau(\xi)}} k_i \ell_j x_i [\xi] y_j$$

of $F \cdot X \cdot F$. In particular, if $f = \sigma(\xi)$ (resp. $g = \tau(\xi)$), this element is simply written $[\xi]g$ (resp. $f[\xi]$).

Let $\succ$ be a well-order on the set $\Sigma^* X \Sigma^*$. We assume that it is compatible, that is, for any $f = x[\xi]y \in \Sigma^*_{\sigma(\xi)} \times {}_{\tau(\xi)}\Sigma^*$ and $f' = x'[\xi']y' \in \Sigma^*_{\sigma(\xi')} \times {}_{\tau(\xi')}\Sigma^*$ such that $\sigma(x) = \sigma(x')$ and $\tau(y) = \tau(y')$ and for any $a \in \Sigma^*_{\sigma(x)}$, and $b \in {}_{\tau(y)}\Sigma^*$, $f \succ f'$ implies $a \cdot f \cdot b \succ a \cdot f' \cdot b$ and $a \succ a'$ in $\Sigma^*_{\sigma(x)}$ implies $a \cdot f \succ a' \cdot f$ and $b \succ b'$ in $\Sigma^*_{\tau(y)}$ implies $f \cdot b \succ f \cdot b'$ (the order $\succ$ on Σ^* is previously given and fixed). The order $\succ$ on $\Sigma^* X \Sigma^*$ can be extended to a partial order $\succ$ on $F \cdot X \cdot F$ in a similar manner as we did on F in Section 2.

The *leading term* $\mathrm{lt}(f)$ of f written as (3.1) is the term $k_i x_i[\xi_i]y_i$ such that $x_i[\xi_i]y_i \succ x_j[\xi_j]y_j$ for all $j \neq i$. The element f is *monic* if the coefficient k_i of the leading term $k_i x_i[\xi_i]y_i$ is 1. If moreover $x_i = \sigma(\xi_i)$, f is called *left very monic.* f is *uniform* if $\sigma(x_i) = \sigma(x_j) = v$ and $\tau(y_i) = \tau(y_j) = v'$ for all i, j. For this uniform f we define $\sigma(f) = v$ and $\tau(f) = v'$.

A *rewriting rule* is a pair (s, t) with $s \in \Sigma^* X \Sigma^*$ and $t \in F \cdot X \cdot F$ such that $s \succ t$ and $s - t$ is uniform. A *rewriting system* T on $F \cdot X \cdot F$ is a set of rewriting rules. If $f \in F \cdot X \cdot F$ has a term $k \cdot x[\xi]y$, $x = x'u$, $y = vy'$ and $s = u[\xi]v \to t \in T$, then $f \to_T f - k \cdot x'(u[\xi]v - t)y'$ by an application of the rule $s \to t$.

A rule $u \to v$ $(u - v \in G)$ in R_G can also be applied to a term $k \cdot x[\xi]y$ of f, if x or y are G-reducible, that is, $x = x'ux''$ or $y = y'uy''$. In the former case, $f \to_G f - k \cdot x'(u-v)x''[\xi]y$, and in the latter, $f \to_G f - k \cdot x[\xi]y'(u-v)y''$. The relation $\to_G$ on $F \cdot X \cdot F$ is complete, because $\to_G$ is complete on F. So, any $f \in F \cdot X \cdot F$ has the unique normal form $\hat{f}$ with respect to $\to_G$. An element f written as (3.1) is G-irreducible, if and only if every x_i and y_i are G-irreducible. Thus, we have

$$\hat{f} = \sum k_i \hat{x}_i[\xi_i]\hat{y}_i.$$

Let $\to_{T,G} = \to_T \cup \to_G$, then $\to_{T,G}$ is a noetherian relation on $F \cdot X \cdot F$ because $f \to_{T,G} g$ implies $f \succ g$ by the compatibility of $\succ$. Let $\to^*_{T,G}$ and $\leftrightarrow^*_{T,G}$ be the reflexive transitive closure and the reflexive symmetric transitive closure of $\to_{T,G}$, respectively. Set $I_X = F \cdot X \cdot I + I \cdot X \cdot F$, where

$$I \cdot X \cdot F = \{\sum f_i[\xi_i]g_i \mid f_i \in I \cap F_{\sigma(\xi_i)},\ g_i \in {}_{\tau(\xi_i)}F,\ \xi_i \in X\}$$

and

$$F \cdot X \cdot I = \{\sum f_i[\xi_i]g_i \mid f_i \in F_{\sigma(\xi_i)},\ g_i \in I \cap {}_{\tau(\xi_i)}F,\ \xi_i \in X\}.$$

Then I_X is the F-subbimodule of $F \cdot X \cdot F$ generated by $G \cdot \Sigma^* \cdot X \cup X \cdot \Sigma^* \cdot G$. Let $L(H)$ be the F-subbimodule of $F \cdot X \cdot F$ generated by $H = H_T = \{s - t \mid s \to t \in T\}$ and let $L(T,G)$ be the F-subbimodule of $F \cdot X \cdot F$ generated by $H \cup G \cdot \Sigma^* \cdot X \cup X \cdot \Sigma^* \cdot G$, then, $L(T,G) = L(H) + I_X$. The relation $\leftrightarrow^*_{T,G}$ is equal to the F-bimodule congruence of $F \cdot X \cdot F$ modulo the subbimodule $L(T,G)$;

$$f \leftrightarrow^*_{T,G} g \;\Leftrightarrow\; f \equiv g \pmod{L(T,G)}.$$

The quotient $M = M(T,G) = (F \cdot X \cdot)F/\leftrightarrow^*_{T,G} = (F \cdot X \cdot F)/L(T,G)$ is an F-bimodule, and actually, it is an A-bimodule in a natural way. Let $\eta : F \cdot X \cdot F \to M$ be the natural surjection.

Let $A \cdot X \cdot A$ be the projective A-bimodule generated by X. An element f of $A \cdot X \cdot A$ is written as a finite sum $f = \sum x_i[\xi_i]y_i$ with $\xi \in X$, $x_i \in A_{\sigma(\xi)}$ and $y_i \in A_{\tau(\xi)}$. We have a morphism $\rho_X : F \cdot X \cdot F \to A \cdot X \cdot A$ of K-modules defined by

$$\rho_X(x[\xi]y) = \rho(x)[\xi]\rho(y)$$

for $x \in \Sigma^*_{\sigma(\xi)}$, $y \in {}_{\tau(\xi)}\Sigma^*$ and $\xi \in X$. In fact, ρ_X is a morphism of F-bimodules. Since

$$\rho_X(f) = 0 \Leftrightarrow \hat{f} = 0 \Leftrightarrow f \leftrightarrow^*_G 0 \Leftrightarrow f \equiv 0 \pmod{I_X},$$

$\mathrm{Ker}(\rho_X)$ is equal to the F-subbimodule $I_X = I \cdot X \cdot F + F \cdot X \cdot I$, that is, $(F \cdot X \cdot F)/I_X \cong A \cdot X \cdot A$. Since M is an A-bimodule, we have a surjection $\overline{\eta} : A \cdot X \cdot A \to M$ with $\eta = \overline{\eta} \circ \rho_X$. Hence, $\mathrm{Ker}(\overline{\eta}) = \rho_X(L(T,G))$, which is denoted by $L_A(H)$, is the A-subbimodule of $A \cdot X \cdot A$ generated by $\rho_X(H)$ and we have

$$M \cong (A \cdot X \cdot A)/L_A(H).$$

If the rewriting system $\to_{T,G}$ is complete on $F \cdot X \cdot F$, we say T is *complete modulo* G. An element $f \in F \cdot X \cdot F$ is (T,G)-*irreducible*, if no rule from $T \cup R_G$ can be applied to f, otherwise f is (T,G)-*reducible*.

Similar to Proposition 2.1, we have

Proposition 3.1. *If a rewriting system T on $F \cdot X \cdot F$ is complete modulo G, then for any $f \in F \cdot X \cdot F$, there is a unique (T, G)-irreducible element (the normal form of f) $\tilde{f} \in F \cdot X \cdot F$ such that $f \to^*_{T,G} \tilde{f}$, and for any $f, g \in F \cdot X \cdot F$, we have*

$$\tilde{f} = \tilde{g} \Leftrightarrow f \leftrightarrow^*_{T,G} g \Leftrightarrow f - g \to^*_{T,G} 0 \Leftrightarrow f \equiv g \pmod{L(T, G)}.$$

A subset H of $F \cdot X \cdot F$ is a *Gröbner basis* (modulo G), if every element of H is monic and uniform and the associated system $T_H = \{\mathrm{lt}(f) \to -\mathrm{rt}(f) \mid f \in H\}$ is a complete rewriting system on $F \cdot X \cdot F$ modulo G. For an F-subbimodule L of $F \cdot X \cdot F$, if H is a Gröbner basis such that $L = L(H, G)$, H is said to be a Gröbner basis of L. It is also called a Gröbner basis for the A-subbimodule $\rho_X(L)$ of $A \cdot X \cdot A$. We write $\to_{H,G}$ and $\to^*_{H,G}$ for $\to_{T_H,G}$ and $\to^*_{T_H,G}$ respectively. A $(\to_{H,G})$-(ir)reducible element is called (H, G)-(ir)reducible. The quotient $M(H, G) = (F \cdot X \cdot F)/L(H, G) = (A \cdot X \cdot A)/L_A(H)$ is called the A-bimodule defined by a pair (G, H) of Gröbner bases.

A rewriting system T on $F \cdot X \cdot F$ or the set $H = H_T$ is *normalized modulo G* if for any $s \to t \in T$, t is (H, G)-irreducible and s is $(H \setminus \{s - t\}, G)$-irreducible. We have a similar result to Proposition 2.4.

Proposition 3.2. *If an F-subbimodule L of $F \cdot X \cdot F$ has a Gröbner basis H modulo G, it has a normalized Gröbner basis H' modulo G. If H is finite, we can choose H' as finite.*

Let T be a normalized rewriting system on $F \cdot X \cdot F$. We consider three rules $x[\xi]y \to t, x'[\xi]y' \to t' \in T$ ($t, t' \in F \cdot X \cdot F$, $\xi \in X$, $x, x' \in \Sigma^*_{\sigma(\xi)}$ and $y, y' \in {}_{\tau(\xi)}\Sigma^*$) and $u - v \in G$.

(i) First, suppose that y overlaps with u, that is, $y = y_1 z$, $u = z u_1$ with $z \in \Sigma^+$. We can apply the rules on $x[\xi]yu_1 = x[\xi]y_1 u$ in two ways, and obtain a *critical pair*

$$(x[\xi]yu_1 \to_T tu_1,\ x[\xi]x_1 u \to_G x[\xi]y_1 v) \tag{3.2}$$

of reductions and a critical pair $(tu_1, x[\xi]y_1 v)$ of elements.

(ii) Next, suppose that x overlaps with u, that is, $x = zx_1$, $u = u_1 z$ with $z \in \Sigma^+$. Then we obtain a *critical pair*

$$(ux_1[\xi]y \to_G vx_1[\xi]y,\ u_1 x[\xi]y \to_T u_1 t)$$

of reductions and a critical pair $(vx_1[\xi]y, u_1 t)$ of elements.

(iii) Lastly, suppose that $x[\xi]y$ overlaps with $x'[\xi]y'$, that is, $x = x_1 x'$ and $y' = yy_1$ with $x_1, y_1 \in \Sigma^+$. Then we obtain a *critical pair*

$$(x[\xi]yy_1 \to_T ty_1,\ x_1 x'[\xi]y' \to_T x_1 t')$$

of reductions and a critical pair $(ty_1, x_1 t')$ of elements.

A critical pair (s, t) of elements is *resolvable* if $s \downarrow_{T,G} t$, that is, there is $f \in F \cdot X \cdot F$ such that $s \to^*_{T,G} f$ and $t \to^*_{T,G} f$. We have the following so-called critical pair theorem.

Proposition 3.3. *A normalized system T on $F \cdot X \cdot F$ is complete modulo G if and only if all the critical pairs are resolvable.*

A rewriting system T on $F \cdot X \cdot F$ (and $H = H_T$) is *left very monic* if the left-hand side of each rule of T is left very monic, that is, every rule of T is of the form $[\xi]x \to t$ with $\xi \in X$, $x \in {}_{\tau(\xi)}\Sigma^*$ and $t \in F$.

Very monic systems are very special, but they suffice to construct our resolutions in Section 5.

Example 3.4. Let V be a subset of Σ^0 satisfying the condition that any $x \in \mathrm{Left}(G)$, which passes through some $u \in V$ (that is, $\tau(u') \in V$ for some prefix u' of u), ends in V (that is, $\tau(x) \in V$). Then, the (two-sided) ideal J of A generated by V admits a left very monic Gröbner basis as an A-bimodule. In fact, the set

$$H = \{[\sigma(x)]x - x[\tau(x)] \,|\, x \in X\}$$

is a left very monic Gröbner basis on the projective A-bimodule $A \cdot V \cdot A$ such that

$$(A \cdot V \cdot A)/L_A(H) \cong J,$$

where X is the set of G-irreducible words x such that $\sigma(x), \tau(x) \in V$ and x has no proper prefix y with $\tau(y) \in V$. In particular, the bimodule A is defined by the Gröbner basis

$$\{[\sigma(a)]a - a[\tau(a)] \,|\, a \in \Sigma^1\}.$$

If T is normalized and left very monic, only critical pairs of type (i) above can appear. Moreover, we need to consider only *proper* critical pairs. A critical pair of type (i) is *proper*, if yu_1 is a minimal G-reducible word, that is, any proper prefix of yu_1 is G-irreducible in (3.2).

Proposition 3.5. *A normalized left very monic system T on $F \cdot X \cdot F$ is complete modulo G if and only if all the proper critical pairs of type* (i) *are resolvable.*

4 Standard reductions and the K-linear map β Let X be an edged set and T be a normalized (but not necessarily complete) left very monic rewriting system on the projective F-bimodule $F \cdot X \cdot F$. Set $H = H_T = \{s - t \,|\, s \to t \in T\}$.

A reduction

$$f_1 \to_{T,G} f_2 \to_{T,G} \cdots \to_{T,G} f_n \tag{4.1}$$

is called *standard*, if for every $i = 1, \ldots, n-1$,

(i) when f_i is G-reducible, the reduction $f_i \to_{T,G} f_{i+1}$ is an application of a rule from G, and

(ii) when f_i is G-irreducible, a rule from T is applied to the smallest T-reducible term in f_i with respect to $\succ$ in the reduction step $f_i \to_{T,G} f_{i+1}$.

If f_1 is reduced to f_n through a standard reduction as above, we write as $f_1 \Rightarrow^*_{T,G} f_n$. A standard one-step reduction by a rule from T is denoted by $\Rightarrow_T$, that is, $f \Rightarrow_T g$ if f is G-irreducible and g is obtained by applying a rule of T to the smallest T-reducible term of f.

Since $\to_G$ is complete, the standard reduction (4.1) can be rewritten as

$$f_1 = g_1 \to^*_G \hat{g}_1 \Rightarrow_T g_2 \to^*_G \hat{g}_2 \Rightarrow_T \cdots \Rightarrow_T g_m \to_G \hat{g}_m = f_n,$$

and since T is left very monic and normalized, in the step $\hat{g}_i \Rightarrow_T g_{i+1}$ in the above reduction sequence, only one rule from T is applicable to the smallest T-reducible term of $\hat{g}_i$. In this sense, a standard reduction from f to a (T,G)-irreducible element f' is unique. The unique element f' is called the *standard form* of f. If T is complete modulo G, f' coincides with the normal form $\tilde{f}$ of f.

The set H is considered to be an edged set because every element h in H is uniform and $\sigma(h)$ and $\tau(h)$ are defined. We consider the projective F-bimodule $F \cdot H \cdot F$ generated by

H. For $h \in H$, $[h]$ denotes the formal generator of $F \cdot H \cdot F$ corresponding to $h \in H$. Let $\delta = \delta_H : F \cdot H \cdot F \to F \cdot X \cdot F$ be a morphism of F-bimodules defined by

$$\delta([h]) = h.$$

Let $A \cdot H \cdot A$ be the projective A-bimodule generated by H and we consider a morphism $\partial = \partial_H : A \cdot H \cdot A \to A \cdot X \cdot A$ of A-bimodules defined by $\partial([h]) = \rho_X(h)$. Then we have a commutative diagram

$$\begin{array}{ccc} F \cdot H \cdot F & \xrightarrow{\delta} & F \cdot X \cdot F \\ \big\downarrow \rho_H & & \big\downarrow \rho_X \\ A \cdot H \cdot A & \xrightarrow{\partial} & A \cdot X \cdot A \end{array}$$

where ρ_X and ρ_H are the surjections. Clearly we have $\mathrm{Im}(\delta) = L(H)$ and $\mathrm{Im}(\partial) = L_A(H)$.

Now we define a K-linear map $\boldsymbol{\beta} = \boldsymbol{\beta}_H : F \cdot X \cdot F \to F \cdot H \cdot F$, which will play a key role in the rest of this paper. For $f \in F \cdot X \cdot F$ let f' be the standard form of f. Then we have a unique standard reduction

$$f = f_1 \to_G^* \hat{f}_1 \Rightarrow_H f_2 \to_G^* \hat{f}_2 \Rightarrow_H \cdots \Rightarrow_H f_n \to_G^* \hat{f}_n = f', \tag{4.2}$$

here in every step $\hat{f}_i \Rightarrow_H f_{i+1}$, a rule corresponding to $h_i = s_i - t_i \in H$ is applied, that is, $k_i x_i s_i y_i$ with $k_i \in K \setminus \{0\}$, $x_i \in A_{\sigma(s_i)}$ and $y_i \in {}_{\tau(s_i)}A$ is the least H-reducible term of $\hat{f}_i$ and $f_{i+1} = k_i x_i (t_i - s_i)_i y_i + \hat{f}_i$. Now define

$$\boldsymbol{\beta}(f) = \sum_{i=1}^{n-1} k_i x_i [h_i] y_i.$$

Thus, $\boldsymbol{\beta}(f)$ depicts the H-reductions in the standard reduction (4.2) of f.

Lemma 4.1. *(1) $\boldsymbol{\beta}(f) = \boldsymbol{\beta}(\hat{f})$ for $f \in F \cdot X \cdot F$, where $\hat{f}$ is the normal form of f with respect to G.*

(2) $\boldsymbol{\beta}(x \cdot f) = (x \cdot \boldsymbol{\beta}(f))\hat{}$ for $x \in \Sigma^*$ and $f \in F \cdot X \cdot F$.

(3) $\boldsymbol{\beta}$ is uniform, that is, $\sigma(\boldsymbol{\beta}(f)) = \sigma(f)$ and $\tau(\boldsymbol{\beta}(f)) = \tau(f)$ for any uniform element $f \in F \cdot X \cdot F$.

(4) $\boldsymbol{\beta}$ is a morphism of K-modules, that is,

$$\boldsymbol{\beta}(kf + \ell g) = k\boldsymbol{\beta}(f) + \ell\boldsymbol{\beta}(g)$$

for $k, \ell \in K$ and $f, g \in F \cdot X \cdot F$.

Proof. (1) and (2) are immediate from the definition of $\boldsymbol{\beta}$. (3) follows from the fact that in the standard reduction (4.2), f_i and $\hat{f}_i$ are all parallel to each other.

Since $\boldsymbol{\beta}$ is defined here in a little different way from [10], we give a proof of (4). First we note that $\boldsymbol{\beta}(k \cdot f) = k \cdot \boldsymbol{\beta}(f)$ holds for $k \in K$ and $f \in F \cdot X \cdot F$. Let $f, g \in F \cdot X \cdot F$, and we shall show $\boldsymbol{\beta}(f+g) = \boldsymbol{\beta}(f) + \boldsymbol{\beta}(g)$ by induction with respect to the order $\succ$. We may assume that f and g are G-irreducible due to (1). If f and g are H-irreducible, then $f+g$ is also H-irreducible and we have $\boldsymbol{\beta}(f+g) = \boldsymbol{\beta}(f) + \boldsymbol{\beta}(g) = 0$. Suppose that f is H-reducible and $kxsy$ with $k \in K \setminus \{0\}$, $x, y \in \Sigma^*$ is its least H-reducible term, where $h = s - t \in H$.

Then, $f \Rightarrow_H f' = kx(t-s)y + f$ and $\boldsymbol{\beta}(f) = kx[h]y + \boldsymbol{\beta}(f')$. If g is H-irreducible or the least H-reducible term of g is greater than $xsty$, then $kxsy$ is the least H-reducible term of $f+g$ too. Since $f \succ f'$, by induction hypothesis we have

$$\boldsymbol{\beta}(f+g) = kx[h]y + \boldsymbol{\beta}(f'+g) = kx[h]y + \boldsymbol{\beta}(f') + \boldsymbol{\beta}(g) = \boldsymbol{\beta}(f) + \boldsymbol{\beta}(g).$$

The case where g has an H-reducible term less than $kxsy$ is symmetric. Suppose that g has the least H-reducible term $k'xsy$ with $k' \in K \setminus \{0\}$, and set $g' = k'x(t-s)y + g$. Then, $\beta(g) = k'x[h]y + \beta(g')$. If $k + k' \neq 0$, then $(k+k')xsy$ is the least H-reducible term of $f+g$, and we have

$$\begin{aligned}\boldsymbol{\beta}(f+g) &= (k+k')x[h]y + \boldsymbol{\beta}(f'+g') \\ &= (k+k')x[h]y + \boldsymbol{\beta}(f') + \boldsymbol{\beta}(g') \\ &= \boldsymbol{\beta}(f) + \boldsymbol{\beta}(g).\end{aligned}$$

If $k + k' = 0$, then $f + g = f' + g'$ and we have

$$\boldsymbol{\beta}(f+g) = \boldsymbol{\beta}(f'+g') = \boldsymbol{\beta}(f') + \boldsymbol{\beta}(g') = \boldsymbol{\beta}(f) + \boldsymbol{\beta}(g).$$

□

The assertion (1) in Lemma 4.1 means that $\boldsymbol{\beta}(f) = \boldsymbol{\beta}(g)$ follows from $\rho_X(g) = \rho_X(f)$. Thus, $\boldsymbol{\beta}$ induces a K-linear map $\boldsymbol{\beta}' : A \cdot X \cdot A \to F \cdot H \cdot F$ such that $\boldsymbol{\beta} = \boldsymbol{\beta}' \circ \rho_X$. The composition $\overline{\boldsymbol{\beta}} = \overline{\boldsymbol{\beta}}_H = \rho_H \circ \boldsymbol{\beta}' : A \cdot X \cdot A \to A \cdot H \cdot A$ with the surjection ρ_H is a K-linear map, but due to Lemma 4.1,(2), we see that $\overline{\boldsymbol{\beta}}$ is a morphism of left A-modules. Thus,

Lemma 4.2. *The K-linear map $\boldsymbol{\beta}$ induces a morphism $\overline{\boldsymbol{\beta}}$ of left A-modules and we have a commutative diagram*

$$\begin{array}{ccc} F \cdot X \cdot F & \xrightarrow{\boldsymbol{\beta}} & F \cdot H \cdot F \\ \downarrow{\scriptstyle \rho_X} & & \downarrow{\scriptstyle \rho_H} \\ A \cdot X \cdot A & \xrightarrow{\overline{\boldsymbol{\beta}}} & A \cdot H \cdot A. \end{array}$$

Lemma 4.3. *For $f \in F \cdot X \cdot F$ we have*

$$\delta \circ \boldsymbol{\beta}(f) \equiv f - f' \pmod{I_X},$$

where f' is the standard form of f.

Proof. Consider the standard reduction (4.2). We have

$$\delta \circ \boldsymbol{\beta}(f) = \delta \left(\sum_{i=1}^{n-1} k_i x_i [h_i] y_i \right) = \sum_{i=1}^{n-1} k_i x_i h_i y_i. \tag{4.3}$$

Since $k_i x_i h_i y_i = \hat{f}_i - f_{i+1}$, the righthand side of (4.3) is equal to

$$f_1 - \hat{f}_n + \sum_{i=1}^{n} (\hat{f}_i - f_i) \equiv f - f' \pmod{I_X}.$$

□

Since $\rho_X(f) = \rho_X(g)$ if and only if $\hat{f} = \hat{g}$, we sometimes regard a G-irreducible element of $F \cdot X \cdot F$ as an element of $A \cdot X \cdot A$. Thus, a G-irreducible element f and its standard form f', which is also G-irreducible, are considered to be an element of $A \cdot X \cdot A$. With this convention, Lemma 4.3 means

Lemma 4.4. *For $f \in A \cdot X \cdot A$ we have*

$$\partial \circ \overline{\boldsymbol{\beta}}(f) = f - f'.$$

Lemma 4.5. *The mapping sending $f \in F \cdot X \cdot F$ to its standard form f' is K-linear, that is,*

$$(k \cdot f + \ell \cdot g)' = k \cdot f' + \ell \cdot g' \tag{4.4}$$

for any $k, \ell \in K$ and $f, g \in F \cdot X \cdot F$.

Proof. By Lemmas 4.1 and 4.3,

$$\begin{aligned} kf + \ell g - (kf + \ell g)' &\equiv \delta \circ \boldsymbol{\beta}(kf + \ell g) \pmod{I_X} \\ &= k \cdot \delta \circ \boldsymbol{\beta}(f) + \ell \cdot \delta \circ \boldsymbol{\beta}(g) \\ &\equiv k(f - f') + \ell(g - g') \pmod{I_X}. \end{aligned}$$

Hence, we see

$$(k \cdot f + \ell \cdot g)' \equiv k \cdot f' + \ell \cdot g' \pmod{I_X},$$

but the elements in the both sides in the above congruence are G-irreducible, we have the equality (4.4) in $F \cdot X \cdot F$. ◻

5 Construction of Gröbner bases and exact sequences of projective bimodules

Let X be an edged set and let H be a normalized left very monic Gröbner basis on the projective F-bimodule $F \cdot X \cdot F$ generated by X. Let M be the A-bimodule defined by (H, G), that is, $M = (F \cdot X \cdot F)/L(H, G)$. Via the surjection $\rho_X : F \cdot X \cdot F \to A \cdot X \cdot A$, M is isomorphic to the quotient $(A \cdot X \cdot A)/L_A(H)$ as stated in the previous section.

Now we are going to construct a Gröbner basis on $F \cdot H \cdot F$ under the situation above. We need a compatible well-order on $F \cdot H \cdot F$ suitable for our purpose. We define an order $\succ$ on $\Sigma^* H \Sigma^* = \bigcup_{h \in H} \Sigma^*_{\sigma(h)} \times {}_{\tau(h)}\Sigma^*$ under the condition that a compatible well-order $\succ$ is already given on $\Sigma^* X \Sigma^* = \bigcup_{\xi \in X} \Sigma^*_{\sigma(\xi)} \times {}_{\tau(\xi)}\Sigma^*$. For $f = x[h]y$ and $g = x'[h']y'$ in $\Sigma^* H \Sigma^*$ with $h, h' \in H$, $x \in \Sigma^*_{\sigma(h)}$, $x' \in \Sigma^*_{\sigma(h')}$, $y \in {}_{\tau(h)}\Sigma^*$ and $y' \in {}_{\tau(h')}\Sigma^*$, $f \succ g$ if and only if

(i) $x \cdot \mathrm{lt}(h) \cdot y \succ x' \cdot \mathrm{lt}(h') \cdot y'$ in $\Sigma^* X \Sigma^*$, or

(ii) $x \cdot \mathrm{lt}(h) \cdot y = x' \cdot \mathrm{lt}(h') \cdot y'$ and $|y| > |y'|$, or

(iii) $x \cdot \mathrm{lt}(h) \cdot y = x' \cdot \mathrm{lt}(h') \cdot y', |y| = |y'|$ and $|x| < |x'|$.

It is easy to see that $\succ$ is a compatible well-order on $\Sigma^* H \Sigma^*$. It can be extended to the partial order $\succ$ on $F \cdot H \cdot F$ as we did on F in Section 2.

Let $h = [\xi]x - t \in H$ and $u'' \in \Sigma^*$ such that $x = x'u'$, $u = u'u''$, $u - v \in G$ and xu'' is a minimal G-reducible word. We have a proper critical pair $([\xi]xu'' \to_H tu'', [\xi]x'u \to_G [\xi]x'v)$ of reductions. We consider an element c of $F \cdot H \cdot F$ corresponding to this critical pair defined by

$$c = [h]u'' + \boldsymbol{\beta}(tu'') - \boldsymbol{\beta}([\xi]x'v). \tag{5.1}$$

From the definition of $\succ$ above, we see that $[h]u'' \succ -\boldsymbol{\beta}([\xi]x'v)+\boldsymbol{\beta}(tu'')$ and so $\mathrm{lt}(c) = [h]u''$ and c is a left very monic element of $F \cdot H \cdot F$. Moreover, we see that c is uniform because the elements $[h]u''$, $[\xi]x'v$ and tu'' are uniform and parallel and the mapping $\boldsymbol{\beta}$ is uniform.

Let C be the set of the elements c given as (5.1) for all proper critical pairs of reductions. Accordingly, the rewriting system T_C on $F \cdot H \cdot F$ associated with C is the set of all rules

$$[h]u'' \to \boldsymbol{\beta}([\xi]x'v) - \boldsymbol{\beta}(tu'')$$

corresponding to proper critical pairs.

The following lemmas can be proved in a similar way to the previous paper [10, Lemmas 5.2 and 5.3] and we omit the proofs. Our key result (Theorem 5.3), which asserts that the set C is a Gröbner basis on $F \cdot H \cdot F$, also can be proved using these lemmas in a similar manner to [10].

Lemma 5.1. *The element $\boldsymbol{\beta}(f)$ is (C, G)-irreducible for any $f \in F \cdot X \cdot F$.*

Lemma 5.2. *For $f \in F \cdot X \cdot F$ and $x \in F$ we have a standard reduction*

$$\boldsymbol{\beta}(f) \cdot x \Rightarrow^*_{C,G} \boldsymbol{\beta}(f \cdot x) - \boldsymbol{\beta}(\tilde{f} \cdot x),$$

where $\tilde{f}$ is the normal form of f with respect to $G \cup H$.

Theorem 5.3. *The set C is a normalized left very monic Gröbner basis on $F \cdot H \cdot F$.*

We call C the *Gröbner basis made from critical pairs of reductions* for H. We consider the projective F-bimodule $F \cdot C \cdot F$ and the projective A-bimodule $A \cdot C \cdot A$ generated by C. As before, $[c]$ denotes the generator corresponding to $c \in C$. We have a morphism $\delta_C : F \cdot C \cdot F \to F \cdot H \cdot F$ of F-bimodules and a morphism $\partial_C : A \cdot C \cdot A \to A \cdot H \cdot A$ of A-bimodules defined by $\delta_C([c]) = c$, and $\partial_C([c]) = \rho_H(c)$, for $c \in C$. With these morphisms we have a commutative diagram

$$\begin{tikzcd}
F \cdot C \cdot F \arrow[r, "\delta_C"] \arrow[d, "\rho_C"] & F \cdot H \cdot F \arrow[r, "\delta_H"] \arrow[d, "\rho_H"] & F \cdot X \cdot F \arrow[d, "\rho_X"] \\
A \cdot C \cdot A \arrow[r, "\partial_C"] & A \cdot H \cdot A \arrow[r, "\partial_H"] & A \cdot X \cdot A
\end{tikzcd}$$

We also have a commutative diagram of K-modules

$$\begin{tikzcd}
F \cdot C \cdot F \arrow[d, "\rho_C"] & F \cdot H \cdot F \arrow[l, "\boldsymbol{\beta}_C"'] \arrow[d, "\rho_H"] & F \cdot X \cdot F \arrow[l, "\boldsymbol{\beta}_H"'] \arrow[d, "\rho_X"] \\
A \cdot C \cdot A & A \cdot H \cdot A \arrow[l, "\overline{\boldsymbol{\beta}}_C"'] & A \cdot X \cdot A, \arrow[l, "\overline{\boldsymbol{\beta}}_H"']
\end{tikzcd} \tag{5.2}$$

where $\boldsymbol{\beta}_C$ is the K-linear map obtained through standard reduction on $F \cdot H \cdot F$ with respect to the Gröbner basis C and $\overline{\boldsymbol{\beta}}_C$ is the induced mapping from $\boldsymbol{\beta}_C$ on $A \cdot H \cdot A$.

Lemma 5.4. *The equality $\boldsymbol{\beta}_H \circ \delta_H(f) = \tilde{f}$ holds for any $f \in F \cdot H \cdot F$, where $\tilde{f}$ is the normal form with respect to (C, G).*

Proof. Let $f = \sum k_i x_i [h_i] y_i \in F \cdot H \cdot F$, then

$$\boldsymbol{\beta}_H \circ \delta_H(f) = \sum k_i \cdot \boldsymbol{\beta}_H(x_i h_i y_i).$$

By Lemma 5.2, we have

$$\boldsymbol{\beta}_H(x_ih_i)y_i \to^*_{C,G} \boldsymbol{\beta}_H(x_ih_iy_i) - \boldsymbol{\beta}_H((xh_i)\tilde{}y_i),$$

where $(xh_i)\tilde{}$ is the normal form of xh_i with respect to $G \cup H$, and equals 0 because $h_i \in H$. On the other hand, $\boldsymbol{\beta}_H(x_ih_i) = \hat{x}_i[h_i]$ by the definition of $\boldsymbol{\beta}_H$. Thus, we have

$$f \to^*_G \sum k_i \boldsymbol{\beta}_H(x_ih_i)y_i \to^*_{C,G} \sum k_i \boldsymbol{\beta}_H(x_ih_iy_i) = \boldsymbol{\beta}_H \circ \delta_H(f).$$

Since $\boldsymbol{\beta}_H \circ \delta_H(f)$ is (C,G)-irreducible by Lemma 5.1, and C is a Gröbner basis by Theorem 5.3, $\boldsymbol{\beta}_H \circ \delta_H(f)$ must be equal to the unique normal form $\tilde{f}$ of f. $\square$

Since $\delta_C \circ \boldsymbol{\beta}_C(f) \equiv f - \tilde{f} \pmod{I_H}$ for $f \in F \cdot H \cdot F$ by Lemma 4.3, we have

Proposition 5.5. *For any $f \in F \cdot H \cdot F$,*

$$\delta_C \circ \boldsymbol{\beta}_C(f) + \boldsymbol{\beta}_H \circ \delta_H(f) \equiv f \pmod{I_H}.$$

Corollary 5.6. *we have*

$$\partial_C \circ \overline{\boldsymbol{\beta}}_C + \overline{\boldsymbol{\beta}}_H \circ \partial_H = \mathrm{id}_{A \cdot H \cdot A}.$$

By Lemma 4.3, for $c \in C$ given as (5.1) we have

$$\begin{aligned}\delta_H \circ \delta_C([c]) = \delta_H(c) &= \delta_H([h]u'') - \delta_H \circ \boldsymbol{\beta}_H([\xi]x'v) + \delta_H \circ \boldsymbol{\beta}_H(tu'') \\ &\equiv hu'' - [\xi]x'v + ([\xi]x'v)\tilde{} + tu'' - (tu'')\tilde{} \pmod{I_X} \\ &= 0 \pmod{I_X}.\end{aligned}$$

This identity and Proposition 5.5 show that

$$\mathrm{Im}(\delta_C) + I_H = \mathrm{Ker}(\delta_H) + I_H.$$

Thus, C is a normalized left very monic Gröbner basis of $\mathrm{Ker}(\delta_H) + I_H$ on $F \cdot H \cdot F$ modulo G, that is, C is a Gröbner basis for $\mathrm{Ker}(\partial)$. Moreover, Corollary 5.6 means that the $\boldsymbol{\beta}$-mappings are contracting homotopy mappings and

$$\mathrm{Im}(\partial_C) = \mathrm{Ker}(\partial_H).$$

So the lower sequence in (5.2) is exact, and we have

Theorem 5.7. *Let M be defined by a normalized left very monic Gröbner basis H on $F \cdot X \cdot F$ modulo G. Then, C is a normalized left very monic Gröbner basis for* $\mathrm{Ker}(\partial)$*, and we have an exact sequence of A-bimodules:*

$$A \cdot C \cdot A \xrightarrow{\partial_C} A \cdot H \cdot A \xrightarrow{\partial_H} A \cdot X \cdot A \xrightarrow{\overline{\eta}} M \longrightarrow 0,$$

where $\overline{\eta}$ is the natural surjection. Moreover, we have morphisms $\overline{\boldsymbol{\beta}}_H : A \cdot X \cdot A \to A \cdot H \cdot A$ and $\overline{\boldsymbol{\beta}}_C : A \cdot H \cdot A \to A \cdot C \cdot A$ of left A-modules such that $\partial_C \circ \overline{\boldsymbol{\beta}}_C + \overline{\boldsymbol{\beta}}_H \circ \partial_H = \mathrm{id}_{A \cdot H \cdot A}$.

Let M be an A-bimodule admitting a normalized left very monic Gröbner basis X_1 on the projective F-bimodule $F \cdot X_0 \cdot F$ generated by an edged set X_0, that is, $M \cong M(X_1, G)$. An *n-chain* with respect to X_1 is a sequence $(\xi, u_0, u_1, \ldots, u_{n-1})$ such that $\xi \in X_0$, $[\xi]u_0 \in \mathrm{Left}(X_1)$ and u_i is nonempty proper suffixes of words in $\mathrm{Left}(G)$ and $u_{i-1}u_i$ are minimal G-reducible words for $i = 1, \ldots, i-1$. Let X_n be the set of n-chains. Because a 0-chain is an element of X_0 and to a 1-chain (ξ, u_0) uniquely corresponds an element of the Gröbner basis X_1 with leading term $[\xi]u_0$, there is no inconsistency in using the symbols X_0 and X_1 for the sets of 0-chains and of 1-chains, respectively.

Starting with the initial exact sequence

$$A \cdot X_1 \cdot A \xrightarrow{\partial_1} A \cdot X_0 \cdot A \xrightarrow{\overline{\eta}} M \longrightarrow 0$$

and applying Theorem 5.7 inductively, we have

Theorem 5.8. *Let M be an A-bimodule defined by a left very monic normalized Gröbner basis X_1 on the projective F-bimodule $F \cdot X_0 \cdot F$ generated by an edged set X_0. With the set X_n of n-chains we have a projective A-bimodule resolution of M:*

$$\mathbf{X} :\to A \cdot X_n \cdot A \xrightarrow{\partial_n} A \cdot X_{n-1} \cdot A \to \cdots \to A \cdot X_1 \cdot A \xrightarrow{\partial_1} A \cdot X_0 \cdot A \xrightarrow{\overline{\eta}} M. \tag{5.3}$$

Here, $\mathrm{Ker}(\partial_{n-1})$ *has a left very monic normalized Gröbner basis $\{h_c \mid c \in X_n\}$ on $F \cdot X_{n-1} \cdot F$ parameterized with X_n such that* $\mathrm{lt}(h_c) = [c']u_{n-1}$ *and $\partial_n([c]) = \rho_{X_{n-1}}(h_c)$, where c is an n-chain $(\xi, u_0, u_1, \ldots, u_{n-1})$ and c' is the $(n-1)$-chain $(\xi, u_0, u_1, \ldots, u_{n-2})$. Moreover, we have morphisms $\boldsymbol{\beta}_n : A \cdot X_{n-1}A \to A \cdot X_n \cdot A$ of left A-modules such that $\partial_{n+1} \circ \boldsymbol{\beta}_{n+1} + \boldsymbol{\beta}_n \circ \partial_n = \mathrm{id}_{A \cdot X_n \cdot A}$ for $n \geq 1$.*

Suppose A is supplemented with augmentation (K-algebra morphism) $\epsilon : A \to K$. Then taking the functor $K \otimes_A$ on the resolution (5.3), we have a complex $\mathbf{X}^r = K \otimes_A \mathbf{X}$:

$$\to X_n \cdot A \xrightarrow{\partial_n^r} X_{n-1} \cdot A \to \cdots \to X_1 \cdot A \xrightarrow{\partial_1^r} X_0 \cdot A \xrightarrow{\overline{\eta}^r} M^r, \tag{5.4}$$

where $M^r = K \otimes_A M$ is the right A-module induced from M with trivial left action via ϵ and $\partial_n^r = 1 \otimes \delta_n$, $\overline{\eta}^r = 1 \otimes \overline{\eta}$. Moreover, the morphism $\boldsymbol{\beta}_n : A \cdot X_{n-1} \cdot A \to A \cdot X_n \cdot A$ of left A-modules induces a K-linear map: $\boldsymbol{\beta}_n^r = 1 \otimes \boldsymbol{\beta}_n : X_{n-1} \cdot A \to X_n \cdot A$, and we have $\partial_{n+1}^r \circ \boldsymbol{\beta}_{n+1}^r + \boldsymbol{\beta}_n^r \circ \partial_n^r = \mathrm{id}_{X_n \cdot A}$ for $n \geq 1$. Hence (5.4) remains exact and forms a projective resolution of the right A-module M^r. On the other hand, let M be a right A-module. By tensoring with A on the left we have an A-bimodule $A \otimes_K M$. If $A \otimes_K M$ admits a left very monic Gröbner basis as an A-bimodule (when K is a field, M always admits a (possibly infinite) *right* Gröbner basis H and this H gives rise to a left very monic Gröbner basis of $A \otimes_K M$ as an A-bimodule), we have the projective A-bimodule resolution $\mathbf{X}$ of $A \otimes_K M$ in Theorem 5.4. But, since $(A \otimes_K M)^r \cong M$, $\mathbf{X}^r$ gives a projective resolution of M as a right A-module. This, in particular, yields the Anick-Green resolution given in [1]. In this sense our construction is a generalization of theirs.

The algebra A is itself an A-bimodule. Consider the projective F-bimodule $F \cdot \Sigma^0 \cdot F$ and the projective A-bimodule generated by the set Σ^0 of vertices of Σ, where Σ^0 is considered to be an edged set such that $\sigma(v) = \tau(v) = v$ for $v \in \Sigma^0$. Let $\eta : F \cdot \Sigma^0 \cdot F \to F$ and $\overline{\eta} : A \cdot \Sigma^0 \cdot A \to A$ be the augmentation map, which is an F-bimodule morphism and an A-bimodule morphism, respectively, given by

$$\eta([v]) = \overline{\eta}([v]) = v$$

for $v \in \Sigma^0$.

Let $X_1 = \{[\sigma(a)]a - a[\tau(a)] \mid a \in \Sigma^1\}$. Then, X_1 is a left very monic normalized Gröbner basis on $F \cdot \Sigma^0 \cdot F$ for $\mathrm{Ker}(\overline{\eta})$ as stated in Example 3.4. Since X_1 is bijective to Σ^1, we have an exact sequence

$$A \cdot \Sigma^1 \cdot A \xrightarrow{\partial_1} A \cdot \Sigma^0 \cdot A \xrightarrow{\overline{\eta}} A \longrightarrow 0$$

of A-bimodules, where $\partial_1([a]) = [\sigma(a)]a - a[\tau(a)]$ for $a \in \Sigma^1$ and $\eta([v]) = v$ for $v \in \Sigma^0$.

Based on this initial exact sequence, we can construct a projective bimodule resolution of A. For $n \geq 1$, an n-chain for A is a sequence $(a, u_1, \dots, u_{n-1})$ such that $a \in \Sigma^1$, $au_1 \in G$ and $u_{i-1}u_i$ is a minimal G-reducible word for $i = 1, \dots, n-2$. Let C_n be the set of n-chains for A. As above, $C_0 = \Sigma^0$ and $C_1 = \Sigma^1$. C_2 is bijective to G and $\{[g]_1 \,|\, g \in G\}$ forms a Gröbner basis of $\mathrm{Ker}(\partial_1)$, where $[g]_1$ is defined as follows. For words $x = a_1a_2 \cdots a_m$, $[x]_1 = [a_1]a_2 \cdots a_m + a_1[a_2] \cdots a_m + \cdots + a_1a_2 \cdots [a_m]$ with $a_i \in \Sigma^1$, and for $g = \sum_i k_i x_i$ with $k_i \in K$ and $x_i \in \Sigma^*$, $[g]_1 = \sum_i k_i [x_i]_1$.

Theorem 5.9. *We have a projective A-bimodule resolution* $\mathbf{H}$*:*

$$(5.5)\quad \cdots \to A\cdot C_n\cdot A \xrightarrow{\partial_n} A\cdot C_{n-1}\cdot A \to \cdots \to A\cdot C_3\cdot A \xrightarrow{\partial_3} A\cdot G\cdot A \xrightarrow{\partial_2} A\cdot\Sigma^1\cdot A \xrightarrow{\partial_1} A\cdot\Sigma^0\cdot A \xrightarrow{\overline{\eta}} A$$

of A. Here, $\partial_1([a]) = [\sigma(a)]a - a[\tau(a)]$ for $a \in \Sigma^1$ and $\partial_2[g] = [g]_1$ for $g \in G$, and in general for $n \geq 3$, $\mathrm{Ker}(\partial_{n-1})$ has a left very monic normalized Gröbner basis $\{h_c \,|\, c \in C_n\}$ on $F \cdot C_{n-1} \cdot F$ such that $\mathrm{lt}(h_c) = [c']u_n$ and $\partial_n([c]) = \rho_{C_{n-1}}(h_c)$, where $c = (a, u_1, \dots, u_{n-1})$ and $c' = (a, u_1, \dots, u_{n-2})$.

For two A-bimodules M and N, let $\mathrm{Hom}_{A,A}(M, N)$ be the K-modules consisting of all bimodules morphisms from M to N. Taking the functor $\mathrm{Hom}_{A,A}(., N)$ with the resolution (5.5), we have a complex

$$\mathrm{Hom}_{A,A}(\mathbf{H}, N) : \bigoplus_{v \in \Sigma^0} {}_vM_v \xrightarrow{\partial_1^*} \bigoplus_{a \in \Sigma^0} {}_{\sigma(a)}M_{\tau(a)} \longrightarrow \cdots \longrightarrow$$

$$\bigoplus_{c \in C_{n-1}} {}_{\sigma(c)}M_{\tau(c)} \xrightarrow{\partial_n^*} \bigoplus_{c \in C_n} {}_{\sigma(c)}M_{\tau(c)} \longrightarrow \cdots .$$

The cohomology group $\mathrm{Ker}(\partial_{n+1}^*)/\mathrm{Im}(\partial_n^*)$ is equal to the *Hochschild cohomology* $H^n(A, M)$ of dimension n with coefficients in M. In particular, letting $M = A$ we have the *Hochschild cohomology* $H^n(A)$ ([7]).

Let U be a subset of $\Sigma^{\geq 2}$ such that any element of U is not a subword of another word in U. Then U is a normalized Gröbner basis of the ideal $I = I(U)$ of F generated by U, and we have a monomial algebra $A = F/I$. The resolution for A constructed on this basis in our method is essentially a resolution given by Bardzell [2]. The resolution is used to calculate the Hochschild cohomology groups for a certain type of monomial algebras in [11] (the algebra structure of the cohomology is given for more special monomial algebras in [3]).

6 Diagonal maps and products The Hochschild cohomology group $H(A)$ has a ring structure with the cup product (the Yoneda product). In this section we describe the ring structure of $H(A)$ in terms of our resolution constructed in the previous section. Again, our K-linear map $\boldsymbol{\beta}$ will play an important role.

Let X be an edged set, and H be a normalized left very monic Gröbner basis on $F \cdot X \cdot F$. We have the K-linear mappings $\boldsymbol{\beta}_H : F \cdot X \cdot F \to F \cdot H \cdot F$ and $\overline{\boldsymbol{\beta}}_H : A \cdot X \cdot A \to A \cdot H \cdot A$.

Let Z be another edged set. Let $F \cdot Z \cdot F \cdot X \cdot F$ denotes the tensor product $F \cdot Z \cdot F \otimes_F F \cdot X \cdot F$, which is the projective F-bimodule generated by $\bigcup_{\zeta \in Z, \xi \in X} {}_{\tau(\zeta)}\Sigma^*_{\sigma(\xi)}$. It is the free K-module generated by

$$\bigcup_{\zeta \in Z, \xi \in X} \Sigma^*_{\sigma(\zeta)} \times {}_{\tau(\zeta)}\Sigma^*_{\sigma(\xi)} \times {}_{\tau(\xi)}\Sigma^*.$$

So, an element f of $F \cdot Z \cdot F \cdot X \cdot F$ is uniquely written as a finite sum

$$f = \sum k_i x_i [\zeta_i] y_i [\xi_i] z_i$$

where $k_i \in K \setminus \{0\}$, $\zeta_i \in Z$, $\xi_i \in X$, $x_i \in \Sigma^*_{\sigma(\zeta_i)}$, $y_i \in {}_{\tau(\zeta_i)}\Sigma^*_{\sigma(\xi_i)}$, $z_i \in {}_{\tau(\xi_i)}\Sigma^*$ and $(x_i, \zeta_i, y_i, \xi_i, z_i)$ are all different for i. We also consider the projective A-bimodule $A \cdot Z \cdot A \cdot X \cdot A = A \cdot Z \cdot A \otimes_A A \cdot X \cdot A$ generated by $\bigcup_{\zeta \in Z, \xi \in X} {}_{\tau(\zeta)}A_{\sigma(\xi)}$. An element f of $A \cdot Z \cdot A \cdot X \cdot A$ is written as

$$f = \sum k_i \hat{x}_i [\zeta_i] \hat{y}_i [\xi_i] \hat{z}_i$$

with $k_i \in K \setminus \{0\}$, $\zeta_i \in Z$, $\xi_i \in X$, $x_i \in \Sigma^*_{\sigma(\zeta_i)}$, $y_i \in {}_{\tau(\zeta_i)}\Sigma^*_{\sigma(\xi_i)}$, $z_i \in {}_{\tau(\xi_i)}\Sigma^*$. This expression is unique if $(\hat{x}_i, \zeta_i, \hat{y}_i, \xi_i, \hat{z}_i)$ are all different for i.

We define K-linear mappings

$${}_Z\boldsymbol{\beta}_H : F \cdot Z \cdot F \cdot X \cdot F \to F \cdot Z \cdot F \cdot H \cdot F$$

and

$${}_Z\overline{\boldsymbol{\beta}}_H : A \cdot Z \cdot A \cdot X \cdot A \to A \cdot Z \cdot A \cdot H \cdot A$$

by

$${}_Z\boldsymbol{\beta}_H(x[\zeta]y[\xi]z) = x[\zeta]\boldsymbol{\beta}_H(y[\xi]z)$$

and

$${}_Z\overline{\boldsymbol{\beta}}_H(\hat{x}[\zeta]\hat{y}[\xi]\hat{z}) = \hat{x}[\zeta]\overline{\boldsymbol{\beta}}_H(\hat{y}[\xi]\hat{z})$$

for $\zeta \in Z$, $\xi \in X$, $x \in \Sigma^*_{\sigma(\zeta)}$, $y \in {}_{\tau(\zeta)}\Sigma^*_{\sigma(\xi)}$ and $z \in {}_{\tau(\xi)}\Sigma^*$. Since $\overline{\boldsymbol{\beta}}_H$ is a morphism of left A-modules, we have

$$(6.1) \qquad {}_Z\overline{\boldsymbol{\beta}}_H = \mathrm{id}_{A \cdot Z \cdot A} \otimes_A \overline{\boldsymbol{\beta}}_H.$$

Clearly ${}_Z\boldsymbol{\beta}_H$ and ${}_Z\overline{\boldsymbol{\beta}}_H$ are uniform.

By Corollary 5.6 we have

Proposition 6.1. *We have*

$$(\mathrm{id}_{A \cdot Z \cdot A} \otimes \partial_C) \circ {}_Z\overline{\boldsymbol{\beta}}_C + {}_Z\overline{\boldsymbol{\beta}}_H \circ (\mathrm{id}_{A \cdot Z \cdot A} \otimes \partial_H) = \mathrm{id}_{A \cdot Z \cdot A \cdot H \cdot A}.$$

By Theorem 5.9 we have the projective A-bimodule resolution $\mathbf{H}$ in (5.5) of A. Let 1_r denote the identity mapping on $A \cdot C_r \cdot A$, in particular $1_0 = \mathrm{id}_{A \cdot \Sigma^0 \cdot A}$ and $1_1 = \mathrm{id}_{A \cdot \Sigma^1 \cdot A}$. For $r \geq 0$ and $s > 0$, let ${}_r\boldsymbol{\beta}_s$ denote the K-linear mapping

$$(-1)^r \cdot {}_{C_r}\overline{\boldsymbol{\beta}}_{C_s} = (-1)^r \cdot 1_r \otimes \overline{\boldsymbol{\beta}}_{C_s} : A \cdot C_r \cdot A \cdot C_{s-1} \cdot A \to A \cdot C_r \cdot A \cdot C_s \cdot A,$$

that is,

$${}_r\boldsymbol{\beta}_s(x[c]y[c']z) = (-1)^r x[c]\overline{\boldsymbol{\beta}}_{C_s}(y[c']z) = (-1)^r x[c]y\overline{\boldsymbol{\beta}}_{C_s}([c']z)$$

for $c \in C_r$, $c' \in C_{s-1}$ and $x, y, z \in A$.

Now, for $r, s \geq 0$ we define a morphism

$$\Delta_{r,s} : A \cdot C_{r+s} \cdot A \to A \cdot C_r \cdot A \cdot C_s \cdot A$$

of A-bimodules by induction on s. First, define $\Delta_{r,0}$ by

$$(6.2) \qquad \Delta_{r,0}([c]) = [c][\tau(c)]$$

for $c \in C_r$ and extend it A-bilinearly. Let $s > 0$ and assuming that $\Delta_{r,s-1}$ is already defined, define $\Delta_{r,s}$ by

$$(6.3) \qquad \Delta_{r,s}([c]) = {}_r\boldsymbol{\beta}_s \circ \Delta_{r,s-1} \circ \partial_{r+s}([c]).$$

for $c \in C_{r+s}$ and extend it A-bilinearly. Note that $\Delta_{r,s}$ is not necessarily equal to the composition ${}_r\boldsymbol{\beta}_s \circ \Delta_{r,s-1} \circ \partial_{r+s}$ because ${}_r\boldsymbol{\beta}_s$ is not a morphism of A-bimodules.

As before, for a word $x = a_1 \cdots a_n \in \Sigma^*$ with $a_i \in \Sigma^1$, $[x]_1$ denotes the element $[a_1]a_2 \cdots a_n + \cdots + a_1 \cdots a_{n-1}[a_n]$ of $A \cdot \Sigma^1 \cdot A$. We know that

$$\overline{\boldsymbol{\beta}}_{C_1}([\sigma(x)]x) = [x]_1$$

for any $x \in \Sigma^*$.

Lemma 6.2. *Let $c \in C_{r+1}$ and suppose that $\partial_{r+1}([c])$ is written as*

$$\partial_{r+1}([c]) = \sum k_i x_i [c_i] y_i \tag{6.4}$$

with $k_i \in K$, $x_i \in \Sigma^*_{v_i}$, $y_i \in {}_{v_i'}\Sigma^*$ and $c_i \in C_r$, where $v_i = \sigma(c_i)$, $v_i' = \tau(c_i)$. Then,

$$\Delta_{r,1}([c]) = (-1)^r \sum k_i x_i [c_i][y_i]_1.$$

Proof. By definition

$$\begin{aligned} \Delta_{r,1}([c]) &= {}_r\boldsymbol{\beta}_1 \circ \Delta_{r,0}(\sum k_i x_i [c_i] y_i) \\ &= \sum k_i \cdot {}_r\boldsymbol{\beta}_1(x_i[c_i][v_i'] y_i) \\ &= (-1)^r \sum k_i x_i [c_i][y_i]_1. \end{aligned}$$

□

Theorem 6.3. *The following identities among our mappings hold: First,*

$$\eta = (\eta \otimes \eta) \circ \Delta_{0,0}, \tag{6.5}$$

holds, that is, the diagram

$$\begin{tikzcd} A \cdot \Sigma^0 \cdot A \arrow[r, "\Delta_{0,0}"] \arrow[dr, "\eta"'] & A \cdot \Sigma^0 \cdot A \cdot \Sigma^0 \cdot A \arrow[d, "\eta\otimes\eta"] \\ & A = A \otimes_A A \end{tikzcd}$$

commutes. In general, for $r, s \geq 0$

$$\Delta_{r,s} \circ \partial_{r+s+1} = (-1)^r (1_r \otimes \partial_{s+1}) \circ \Delta_{r,s+1} + (\partial_{r+1} \otimes 1_s) \circ \Delta_{r+1,s}, \tag{6.6}$$

holds, that is, the diagram

$$\begin{tikzcd} A \cdot C_{r+s+1} \cdot A \arrow[r, "\Delta_{r,s+1}\oplus\Delta_{r+1,s}"] \arrow[d, "\partial_{r+s+1}"] & A \cdot C_r \cdot A \cdot C_{s+1} \cdot A \oplus A \cdot C_{r+1} \cdot A \cdot C_s \cdot A \arrow[d, "(-1)^r 1_r\otimes\partial_{s+1}+\partial_{r+1}\otimes 1_s"] \\ A \cdot C_{r+s} \cdot A \arrow[r, "\Delta_{r,s}"] & A \cdot C_r \cdot A \cdot C_s \cdot A \end{tikzcd}$$

commutes.

Proof. First (6.5) is easily checked by (6.2) as

$$(\eta \otimes \eta) \circ \Delta_{0,0}([v]) = (\eta \otimes \eta)([v][v]) = v \cdot v = v = \eta([v]).$$

Now, we prove (6.6) by induction on s. Let $r \geq 0$ and $c \in C_{r+1}$, and let $\partial_{r+1}([c])$ be given as (6.4). By Lemma 6.2 we have

$$\begin{aligned}(-1)^r(1_r \otimes \partial_1)\circ&\Delta_{r,1}([c]) + (\partial_{r+1} \otimes 1_0) \circ \Delta_{r+1,0}([c]) \\ &= (1_r \otimes \partial_1)\left(\sum k_i x_i [c_i][y_i]_1\right) + (\partial_{r+1} \otimes 1_0)([c][\tau(c)]) \\ &= \sum k_i x_i [c_i]([\sigma(y_i)]y_i - y_i[\tau(y_i)]) + \sum k_i x_i [c_i] y_i [\tau(c)] \\ &= \sum k_i x_i [c_i][\sigma(y_i)]y_i = \Delta_{r,0} \circ \partial_{r+1}([c]\}.\end{aligned}$$

This implies that (6.6) holds for $s = 0$.

Next, suppose $s \geq 1$ and let $c \in C_{r+s+1}$. By our definition (6.3) we see

$$\Delta_{r,s+1}([c]) = {}_r\boldsymbol{\beta}_{s+1} \circ \Delta_{r,s} \circ \partial_{r+s+1}([c]). \tag{6.7}$$

By induction hypothesis we have

$$\Delta_{r,s-1} \circ \partial_{r+s} = (-1)^r(1_r \otimes \partial_s) \circ \Delta_{r,s} + (\partial_{r+1} \otimes 1_{s-1}) \circ \Delta_{r+1,s-1}. \tag{6.8}$$

By Proposition 6.1 we have

$$(1_r \otimes \partial_{s+1}) \circ {}_r\boldsymbol{\beta}_{s+1} + {}_r\boldsymbol{\beta}_s \circ (1_r \otimes \partial_s) = (-1)^r \cdot \mathrm{id}_{A \cdot C_r \cdot A \cdot C_s \cdot A}. \tag{6.9}$$

By (6.7), (6.8) and (6.9), we get

$$\begin{aligned}(-1)^r&(1_r \otimes \partial_{s+1}) \circ \Delta_{r,s+1}([c]) \\ &= \Delta_{r,s} \circ \partial_{r+s+1}([c]) - {}_r\boldsymbol{\beta}_s \circ \Delta_{r,s-1} \circ \partial_{r+s} \circ \partial_{r+s+1}([c]) \\ &\qquad + {}_r\boldsymbol{\beta}_s \circ (\partial_{r+1} \otimes 1_{s-1}) \circ \Delta_{r+1,s-1} \circ \partial_{r+s+1}([c]).\end{aligned} \tag{6.10}$$

Due to (6.1) we have,

$${}_r\boldsymbol{\beta}_s \circ (\partial_{r+1} \otimes 1_{s-1}) = (-1)^r(\partial_{r+1} \otimes \overline{\boldsymbol{\beta}}_{C_s}) = -(\partial_{r+1} \otimes 1_s) \circ {}_{r+1}\boldsymbol{\beta}_s.$$

Thus, ${}_r\boldsymbol{\beta}_s \circ (\partial_{r+1} \otimes 1_{s-1}) \circ \Delta_{r+1,s-1} \circ \partial_{r+s+1}([c])$ is equal to $-(\partial_{r+1} \otimes 1_s) \circ \Delta_{r+1,s}([c])$ by the inductive definition of $\Delta_{r+1,s}$. Moreover, since $\partial_{r+s} \circ \partial_{r+s+1} = 0$, the righthand side of (6.10) is equal to

$$\Delta_{r,s}\partial_{r+s+1}([c]) - (\partial_{r+1} \otimes 1_s) \circ \Delta_{r+1,s}([c]),$$

as desired. □

Let M and N be A-bimodules. For $f \in \mathrm{Hom}_{A,A}(A \cdot C_r \cdot A, M)$ and $g \in \mathrm{Hom}_{A,A}(A \cdot C_s \cdot A, N)$ define the cup product $f \cup g \in \mathrm{Hom}_{A,A}(A \cdot C_{r+s} \cdot A, M \otimes_A N)$ by

$$f \cup g = (f \otimes g) \circ \Delta_{r,s}. \tag{6.11}$$

This product $\cup$ induces the product on the cohomology, that is, it induces the product $\cup\colon H(A, M) \times H(A, N) \to H(A, M \otimes_A N)$. By the uniqueness of the product (see [13]), this is actually the Yoneda product. In particular, we have an algebra structure on the Hochschild cohomology $H(A)$. Thus, we have

Theorem 6.4. *The Yoneda product in the graded algebra $H(A) = \bigoplus H^n(A)$ over the center $Z(A) = H^0(A)$ of A is given by the product $\cup$ in (6.11) above.*

7 Examples In this section we calculate the Hochschild cohomology algebras of two special example algebras. Only these simple examples will be illustrative enough to show how our methods can be applied effectively.

Example 7.1. Let Σ be a quiver with two vertices u, v and two arrows a, b given as

$$u \underset{b}{\overset{a}{\rightleftarrows}} v.$$

Let

$$G = \{ababa + aba, babab + bab\}$$

and I be the ideal generated by G of the path algebra $\mathrm{F} = K \cdot \Sigma^*$ over K. Then, G is a Gröbner basis of I. The quotient algebra $A = F/I$ is spanned by

$$\mathrm{Irr}(G) = \{u, v, a, b, ab, ba, aba, bab, abab, baba\},$$

over K and decomposed as

$$A = {}_uA_u \oplus {}_vA_v \oplus {}_uA_v \oplus {}_vA_u,$$

where

$${}_uA_u = K \cdot \{u, ab, abab\}, {}_uA_v = K \cdot \{v, ba, baba\}, {}_uA_v = K \cdot \{a, aba\}, {}_vA_u = K \cdot \{b, bab\}.$$

Let $V = \{\alpha, \beta\}$ and $\overline{V} = \{\overline{\alpha}, \overline{\beta}\}$ be edged sets such that

$$\sigma(\alpha) = u,\ \tau(\alpha) = v,\ \sigma(\beta) = v,\ \tau(\beta) = u, \sigma(\overline{\alpha}) = \tau(\overline{\alpha}) = u,\ \sigma(\overline{\beta}) = \tau(\overline{\beta}) = v,$$

and let $\partial_1 : A \cdot V \cdot A \to A \cdot \overline{V} \cdot A$, $\partial_2 : A \cdot V \cdot A \to A \cdot V \cdot A$, $\partial_3 : A \cdot \overline{V} \cdot A \to A \cdot V \cdot A$, and $\partial_4 : A \cdot \overline{V} \cdot A \to A \cdot \overline{V} \cdot A$ be morphisms of A-bimodules defined by

$$\begin{aligned}
&\partial_1([\alpha]) = [\overline{\alpha}]a - a[\overline{\beta}],\ \partial_1([\beta]) = [\overline{\beta}]b - b[\overline{\alpha}],\\
&\partial_2([\alpha]) = [\alpha]baba + a[\beta]aba + ab[\alpha]ba + aba[\beta]a + abab[\alpha] + [\alpha]ba + a[\beta]a + ab[\alpha],\\
&\partial_2([\beta]) = [\beta]abab + b[\alpha]bab + ba[\beta]ab + bab[\alpha]b + baba[\beta] + [\beta]ab + b[\alpha]b + ba[\beta],\\
&\partial_3([\overline{\alpha}]) = [\alpha]b - a[\beta],\ \partial_3([\overline{\beta}]) = [\beta]a - b[\alpha],\\
&\partial_4([\overline{\alpha}]) = [\overline{\alpha}]abab + a[\overline{\beta}]bab + ab[\overline{\alpha}]ab + aba[\overline{\beta}]b + abab[\overline{\alpha}] + [\overline{\alpha}]ab + a[\overline{\beta}]b + ab[\overline{\alpha}],\\
&\partial_4([\overline{\beta}]) = [\overline{\beta}]baba + b[\overline{\alpha}]aba + ba[\overline{\beta}]ba + bab[\overline{\alpha}]a + baba[\overline{\beta}] + [\overline{\beta}]ba + b[\overline{\alpha}]a + ba[\overline{\beta}].
\end{aligned}$$

With these morphisms we have a projective bimodule resolution of A:

$$\cdots \longrightarrow A \cdot C_n \cdot A \xrightarrow{\partial_n} A \cdot C_{n-1} \cdot A \longrightarrow \ldots \xrightarrow{\partial_2} A \cdot C_1 \cdot A \xrightarrow{\partial_1} A \cdot C_0 \cdot A \xrightarrow{\epsilon} A, \tag{7.1}$$

where

$$C_n = \begin{cases} V & \text{if} \quad n \equiv 1, 2 \pmod 4 \\ \overline{V} & \text{if} \quad n \equiv 0, 3 \pmod 4), \end{cases}$$

and

$$\epsilon([\overline{\alpha}]) = u,\ \epsilon([\overline{\beta}]) = v,$$

$$\partial_n = \partial_r \quad \text{if} \quad n \equiv r \pmod 4).$$

As we saw in Section 5, (7.1) is exact up to dimension 2 and $H = \partial_2(C_2) = \{h_\alpha, h_\beta\}$ is a Gröbner basis for $\mathrm{Ker}(\partial_1)$, where

$$h_\alpha = [\alpha]baba + a[\beta]aba + ab[\alpha]ba + aba[\beta]a + abab[\alpha] + [\alpha]ba + a[\beta]a + ab[\alpha],$$
$$h_\beta = [\beta]abab + b[\alpha]bab + ba[\beta]ab + bab[\alpha]b + baba[\beta] + [\beta]ab + b[\alpha]b + ba[\beta].$$

We have two critical pairs of reduction

$$[\alpha]babab \begin{array}{l} \nearrow_G \ -[\alpha]bab \\ \searrow_H \ -a[\beta]abab - ab[\alpha]bab - aba[\beta]ab - abab[\alpha]b - [\alpha]bab - a[\beta]ab - ab[\alpha]b \end{array}$$

and

$$[\beta]ababa \begin{array}{l} \nearrow_G \ -[\beta]aba \\ \searrow_H \ -b[\alpha]baba - ba[\beta]aba - bab[\alpha]ba - baba[\beta]a - [\beta]aba - b[\alpha]ba - ba[\beta]a \end{array}$$

with respect to H and G. Corresponding to these critical pairs we have two elements

$$\begin{aligned}[t][h_\alpha]b - \boldsymbol{\beta}_H([\alpha]bab) \\ + \boldsymbol{\beta}_H(-a[\beta]abab - ab[\alpha]bab - aba[\beta]ab - abab[\alpha]b - [\alpha]bab - a[\beta]ab - ab[\alpha]b) \\ = [h_\alpha]b - a[h_\beta]\end{aligned}$$

and

$$[h_\beta]a - b[h_\alpha],$$

which constitute a Gröbner basis C_3 for $\mathrm{Ker}(\partial_2)$.

In general for all $n \geq 3$ we can check that $\partial_n(C_n)$ forms a Gröbner basis for $\mathrm{Ker}(\partial_{n-1})$ and we see that (7.1) is actually exact.

Taking the functor $\mathrm{Hom}_{A,A}(.,A)$ on (7.1) we have a complex:

$${}_uA_u \oplus {}_vA_v \xrightarrow{\partial_1^*} {}_uA_v \oplus {}_vA_u \xrightarrow{\partial_2^*} {}_uA_v \oplus {}_vA_u \xrightarrow{\partial_3^*} {}_uA_u \oplus {}_vA_v \xrightarrow{\partial_4^*} {}_uA_u \oplus {}_vA_v,$$

where

$$\partial_1^*(x,y) = (xa - ay, yb - bx),$$
$$\begin{aligned}\partial_4^*(x,y) = (&xabab + aybab + abxab + abayb + ababx + xab + ayb + abx, \\ &ybaba + bxaba + bayba + babxa + babay + yba + bxa + bay)\end{aligned}$$

for $(x,y) \in {}_uA_u \times {}_vA_v$, and

$$\begin{aligned}\partial_2^*(x,y) = (&xbaba + ayaba + abxba + abaya + ababx + xba + aya + abx, \\ &yabab + bxbab + bayab + babxb + babay + yab + bxb + bay),\end{aligned}$$
$$\partial_3^*(x,y) = (xb - ay, ya - bx)$$

for $(x,y) \in {}_uA_v \times {}_vA_u$.

Note that all the calculations can be done in A because ${}_uA_u \oplus {}_vA_v$ and ${}_uA_v \oplus {}_vA_u$ are contained in A. From here we assume that K is a field of characteristic p.

Elements $x \in {}_uA_u$ and $y \in {}_vA_v$ are uniquely written as

$$x = ku + \ell ab + m\,abab, \;\; y = k'v + \ell' ba + m' baba \tag{7.2}$$

with $k, \ell, m, k', \ell', m' \in K$, respectively. Since

$$xa - ay = (k - k')a + (\ell - m - \ell' + m')aba,$$

$$yb - bx = -(k - k')b - (\ell - m - \ell' + m')bab,$$

we have

$$\mathrm{Ker}(\partial_1^*) = K \oplus K \cdot (ab + ba) \oplus K \cdot (ab + abab) \oplus K \cdot (abab + baba)$$

and

$$\mathrm{Im}(\partial_1^*) = K \cdot (a - b) \oplus K \cdot (aba - bab).$$

For $x \in {}_uA_v$ and $y \in {}_vA_u$ given as

$$x = ka + \ell aba, \;\; y = k'b + \ell' bab \tag{7.3}$$

with $k, \ell, k', \ell' \in K$, we have

$$\partial_2^*(x, y) = ((-k + \ell - k' + \ell')aba, \;\; (-k' + \ell' - k + \ell)bab),$$

$$\partial_3^*(x, y) = ((k - k')ab + (\ell - \ell')abab, \;\; (k' - k)ba + (\ell' - \ell)baba).$$

Thus, we have

$$\mathrm{Ker}(\partial_2^*) = K \cdot (a - b) \oplus K \cdot (aba - bab) \oplus K \cdot (a + aba)$$

$$\mathrm{Im}(\partial_2^*) = K \cdot (aba + bab),$$

$$\mathrm{Ker}(\partial_3^*) = K \cdot (a + b) \oplus K \cdot (aba + bab),$$

$$\mathrm{Im}(\partial_3^*) = K \cdot (ab - ba) \oplus K \cdot (abab - baba).$$

Therefore, we obtain

$$H^1(A) = \mathrm{Ker}(\partial_2^*)/\mathrm{Im}(\partial_1^*) \cong K \cdot (a + aba),$$

$$H^2(A) = \mathrm{Ker}(\partial_3^*)/\mathrm{Im}(\partial_2^*) \cong K \cdot (a + b).$$

For elements $x \in {}_uA_u$ and $y \in {}_vA_v$ given as (7.2) we have

$$\begin{aligned} \partial_4^*(x, y) \;&= ((2k + k')ab + (3k - \ell + m + 2k' - \ell' + m')abab, \\ &\quad (2k' + k)ba + (3k' - \ell' + m' + 2k - \ell + m)baba). \end{aligned}$$

Hence, $\partial_4^*(x, y) = 0$, if and only if

$$\begin{cases} k = k' = 0, \; \ell - m + \ell' - m' = 0 & \text{if} \quad p \neq 3 \\ k = k', \; 2k = \ell - m + \ell' - m' & \text{if} \quad p = 3, \end{cases}$$

and so we obtain

$$\mathrm{Ker}(\partial_4^*) = \begin{cases} K \cdot (ab - ba) \oplus K \cdot (abab - baba) \oplus K \cdot (ab + abab) & \text{if } p \neq 3 \\ K \cdot (ab - ba) \oplus K \cdot (abab - baba) \oplus K \cdot (ab + abab) \oplus K \cdot (1 + ab + ba) & \text{if } p = 3, \end{cases}$$

and

$$\mathrm{Im}(\partial_4^*) = \begin{cases} K\cdot(ab+ba)\oplus K\cdot(ab+abab)\oplus K\cdot(abab+baba) & \text{if} \quad p\neq 3 \\ K\cdot(ab-ba-abab)\oplus K\cdot(abab+baba) & \text{if} \quad p=3. \end{cases}$$

Therefore, we have

$$H^3(A) = \frac{\mathrm{Ker}(\partial_4^*)}{\mathrm{Im}(\partial_3^*)} \cong \begin{cases} K\cdot(ab+abab) & \text{if} \quad p\neq 3 \\ K\cdot(ab+abab)\oplus K\cdot(1+ab+ba) & \text{if} \quad p=3, \end{cases}$$

and

$$H^4(A) = \frac{\mathrm{Ker}(\partial_1^*)}{\mathrm{Im}(\partial_4^*)} \cong \begin{cases} K & \text{if} \quad p\neq 3 \\ K\oplus K\cdot(ab+ba) & \text{if} \quad p=3. \end{cases}$$

Summarizing,

$$\begin{aligned} &H^0(A) = Z(A) = K\oplus K\cdot(ab+ba)\oplus K\cdot(ab+abab)\oplus K\cdot(abab+baba), \\ &H^1(A) = K\cdot(a+aba), \\ &H^2(A) = K\cdot(a+b), \\ &H^3(A) = \begin{cases} K\cdot(ab+abab) & \text{if} \quad p\neq 3 \\ K\cdot(ab+abab)\oplus K\cdot(1+ab+ba) & \text{if} \quad p=3, \end{cases} \\ &H^4(A) = \begin{cases} K & \text{if} \quad p\neq 3 \\ K\oplus K\cdot(ab+ba) & \text{if} \quad p=3, \end{cases} \end{aligned}$$

and for $n\geq 5$,

$$H^n(A) = H^r(A) \quad \text{if} \quad n\equiv r \pmod 4,\ 1\leq r\leq 4.$$

Next, we calculate the ring structure of $H(A)$. First, setting

$$\lambda = ab+ba,\ \mu = ab+abab$$

in $H^0(A)$, we have

$$\lambda^2 = abab+baba,\ \lambda^3 = -\lambda^2,\ \lambda\mu = \mu^2 = 0.$$

Hence, $H^0(A)$ is the quotient of a polynomial ring:

$$K[\lambda,\mu]/(\lambda^3+\lambda^2,\ \mu^2,\ \lambda\mu).$$

Now, assuming that $p = \mathrm{char}(K)\neq 3$, set

$$\sigma = a+aba,\ \tau = a+b,\ \theta = 1$$

in $H^1(A), H^2(A), H^4(A)$, respectively. It is well-known that $H(A)$ is graded commutative.

We have

$$\sigma\cup\lambda = (a+aba)(ab+ba) = aba+ababa = 0$$

and

$$\sigma\cup\mu = (a+aba)(ab+abab) = 0.$$

in $H^1(A)$. Similarly we have

$$\tau \cup \lambda = \tau \cup \mu = 0$$

in $H^2(A)$. Moreover we have

$$\theta \cup \lambda = ab + ba \tag{7.4}$$

and

$$\theta \cup \mu = ab + abab, \tag{7.5}$$

in $H^4(A)$, which are equal to 0 when $p \neq 0$

By Lemma 6.2, we have

$$\begin{aligned}
\Delta_{1,1}([\alpha]) = & -\{[\alpha][\beta]aba + [\alpha]b[\alpha]ba + [\alpha]ba[\beta]a + [\alpha]bab[\alpha] + a[\beta][\alpha]ba + a[\beta]a[\beta]a \\
& + a[\beta]ab[\alpha] + ab[\alpha][\beta]a + ab[\alpha]b[\alpha] + aba[\beta][\alpha] + [\alpha][\beta]a + [\alpha]b[\alpha] + a[\beta][\alpha]\}, \\
\Delta_{1,1}([\beta]) = & -\{[\beta][\alpha]bab + [\beta]a[\beta]ab + [\beta]ab[\alpha]b + [\beta]aba[\beta] + b[\alpha][\beta]ab + b[\alpha]b[\alpha]b \\
& + b[\alpha]ba[\beta] + ba[\beta][\alpha]b + ba[\beta]a[\beta] + bab[\alpha][\beta] + [\beta][\alpha]b + [\beta]a[\beta] + b[\alpha][\beta]\},
\end{aligned}$$

from which we find

$$(\sigma \otimes \sigma) \circ \Delta_{1,1}([\alpha]) = 0, \ (\sigma \otimes \sigma) \circ \Delta_{1,1}([\beta]) = 0,$$

meaning

$$\sigma \cup \sigma = 0.$$

Since

$$\Delta_{2,1}([\overline{\alpha}]) = [\alpha][\beta], \ \Delta_{2,1}([\overline{\beta}]) = [\beta][\alpha],$$

we have

$$(\tau \otimes \sigma) \circ \Delta_{2,1}([\alpha]) = 0, \ (\tau \otimes \sigma) \circ \Delta_{2,1}([\beta]) = ba + baba.$$

Thus, we find that

$$\tau \cup \sigma = ab + abab \tag{7.6}$$

is the generator in $H^3(A)$. Since

$$\Delta_{2,2}([\overline{\alpha}]) = {}_2\boldsymbol{\beta}_2 \circ \Delta_{2,1} \circ \partial_4([\overline{\alpha}]) = [\alpha][\beta],$$

$$\Delta_{2,2}([\overline{\beta}]) = {}_2\boldsymbol{\beta}_2 \circ \Delta_{2,1} \circ \partial_4([\overline{\beta}]) = [\beta][\alpha],$$

we have

$$(\tau \otimes \tau) \circ \Delta_{2,2}([\overline{\alpha}]) = ab, \ (\tau \otimes \tau) \circ \Delta_{2,2}([\overline{\beta}]) = ba.$$

Hence,

$$\tau \cup \tau = ab + ba, \tag{7.7}$$

which equals 0 in $H^4(A)$ in case $p \neq 3$. Since

$$\Delta_{4,1}([\alpha]) = [\overline{\alpha}][\alpha], \ \Delta_{4,1}([\beta]) = [\overline{\beta}][\beta],$$

we have

$$(\theta \otimes \sigma) \circ \Delta_{4,1}([\alpha]) = a + aba, \ (\theta \otimes \sigma) \circ \Delta_{4,1}([\beta]) = 0,$$

and hence, $\theta \cup \sigma$ is equal to the generator $a + aba$ in $H^5(A)$ ($\cong H^1(A)$);

$$\theta \cup \sigma = a + aba \tag{7.8}$$

Similarly, we see that $\theta \cup \tau$ equals the generator $a + b$ in $H^6(A)$, and $\theta \cup \theta$ equals the generator 1 in $H^8(A)$.

Summarizing, when $p \neq 3$, $H(A)$ is isomorphic to the graded commutative algebra

$$K[\lambda, \mu, \sigma, \tau, \theta]/(\lambda^3 + \lambda^2, \mu^2, \sigma^2, \tau^2, \lambda\mu, \sigma\lambda, \sigma\mu, \tau\lambda, \tau\mu, \theta\lambda, \theta\mu)$$

over K with

$$\deg(\lambda) = \deg(\mu) = 0, \deg(\sigma) = 1, \deg(\tau) = 2, \deg(\theta) = 4.$$

Next, we consider the case where $p = 3$. In this case, the element $ab + ba$ in $H^4(A)$ is equal to $\theta \cup \lambda$ by (7.4). The element $1 + ab + ba$ in $H^3(A)$ cannot be expressed by the other elements, and we need a new variable $\zeta = 1 + ab + ba$ of degree 3.

Again an easy (but tedious) calculation shows that

$$\zeta \cup \lambda = -(ab + abab), \ \zeta \cup \mu = ab + abab \tag{7.9}$$

in $H^3(A)$,

$$\zeta \cup \sigma = 0$$

in $H^4(A)$,

$$\zeta \cup \tau = -(a + aba) \tag{7.10}$$

in $H^5(A)$,

$$\zeta \cup \zeta = 0,$$

in $H^6(A)$, and

$$\theta \cup \zeta = 1 + ab + ba$$

in $H^7(A)$. By (7.4), (7.5) and (7.7) we see

$$\theta \cup \lambda = -(\theta \cup \mu) = \tau \cup \tau,$$

By (7.8) and (7.10) we have

$$\zeta \cup \tau = -(\theta \cup \sigma),$$

and by (7.6) and (7.9) we have

$$-\zeta \cup \lambda = \zeta \cup \mu = \tau \cup \sigma.$$

Now, summarizing the above calculations, when $p = 3$, we find that $H(A)$ is isomorphic to the graded commutative algebra

$$K[\lambda, \mu, \sigma, \tau, \theta, \zeta]/J$$

over K with

$$\deg(\lambda) = \deg(\mu) = 0, \deg(\sigma) = 1, \deg(\tau) = 2, \deg(\theta) = 4, \deg(\zeta) = 3,$$

where J is the ideal

$$(\lambda^3 + \lambda^2, \mu^2, \sigma^2, \tau^2 - \theta\lambda, \zeta^2, \lambda\mu, \sigma\lambda, \sigma\mu, \tau\lambda, \tau\mu, \theta\lambda + \theta\mu, \zeta\sigma, \zeta\lambda - \tau\sigma, \zeta\mu + \tau\sigma, \zeta\tau + \theta\sigma).$$

Next we consider another simple example but this time we treat it over the integer ring $\mathbb{Z}$.

Example 7.2. Let $q(\neq 0) \in \mathbb{Z}$ be fixed and consider the quiver

$$\Sigma = \left\{ c \circlearrowleft u \quad \underset{b}{\overset{a}{\underset{\longleftarrow}{\longrightarrow}}} \quad v \circlearrowleft d \right\}.$$

Let

$$G = \{ca - ad, db - bc, ab - qc, ba - qd\},$$

and let I be the ideal of $F = \mathbb{Z} \cdot \Sigma^*$ generated by G. Then, G is a Gröbner basis of I, and we have a resolution of the quotient algebra $A = F/I$:

$$\to A\Sigma^1 A \overset{\partial_n}{\to} A\Sigma^1 A \to \cdots \to A\Sigma^1 A \overset{\partial_1}{\to} A\Sigma^0 A \overset{\epsilon}{\to} A \tag{7.11}$$

where

$$\epsilon([u]) = u,\ \epsilon([v]) = v,$$

$$\partial_1([a]) = [u]a - a[v],\ \partial_1([b]) = [v]b - b[u],\ \partial_1([c]) = [u]c - c[u],\ \partial_1([d]) = [v]d - d[v],$$

and for $n \geq 2$

$$\partial_n([a]) = [c]a - a[d] + c[a] - [a]d,\ \partial_n([b]) = [d]b - b[c] + d[b] - [b]c,$$
$$\partial_n([c]) = [a]b + a[b] - q[c],\ \partial_n([d]) = [b]a + b[a] - q[d]$$

if n is even, and

$$\partial_n([a]) = [c]a - a[d] + q[a],\ \partial_n([b]) = [d]b - b[c] + q[b],$$
$$\partial_n([c]) = [a]b + a[b] - c[c] + [c]c,\ \partial_n([d]) = [b]a + b[a] - d[d] + [d]d,$$

if n is odd.

Taking the functor $\mathrm{Hom}(., A)$ on (7.11) we have a complex:

$${}_uA_u \oplus {}_vA_v \overset{\partial_1^*}{\to} {}_uA_v \oplus {}_vA_u \oplus {}_uA_u \oplus {}_vA_v \to \cdots \to {}_uA_v \oplus {}_vA_u \oplus {}_uA_u \oplus {}_vA_v \overset{\partial_n^*}{\to} {}_uA_v \oplus {}_vA_u \oplus {}_uA_u \oplus {}_vA_v \to \ldots$$

with

$$\partial_1^*(z, w) = (za - aw, wb - bz, zc - cz, wd - dw),$$

and

$$\partial_n^*(x, y, z, w) = (za - aw + cx - xd, wb - bz + dy - yc, xb + ay - qz, ya + bx - qw)$$

for even $n \geq 2$, and

$$\partial_n^*(x, y, z, w) = (za - aw + qx, wb - bz + qy, xb + ay - cz + zc, ya + bx - dw + wd)$$

for odd $n \geq 3$, where $x \in {}_uA_v$, $y \in {}_vA_u$, $z \in {}_uA_u$ and $w \in {}_vA_v$.

For $z = \sum_{i\geq 0} m_i c^i \in {}_uA_u = \mathbb{Z} \cdot c^*$ and $w = \sum_{i\geq 0} n_i d^i \in {}_vA_v = \mathbb{Z} \cdot d^*$ with $m_i, n_i \in \mathbb{Z}$, where only finitely many m_i and n_i are nonzero, we have

$$\partial_1^*(z, w) = (\sum_{i\geq 0}(m_i - n_i)ad^i,\ \sum_{i\geq 0}(n_i - m_i)bc^i,\ 0,\ 0).$$

It follows that

$$\operatorname{Ker}(\partial_1^*) = \bigoplus_{i\geq 0} \mathbb{Z}(c^i + d^i),\ \operatorname{Im}(\partial_1^*) = \bigoplus_{i\geq 0} \mathbb{Z}(ad^i - bc^i).$$

For even $n \geq 2$ we can obtain (we omit the calculation)

$$\operatorname{Ker}(\partial_n^*) = \bigoplus_{i\geq 0} \mathbb{Z}(ad^i - bc^i) \ \bigoplus_{i\geq 0} \mathbb{Z}(ad^i + c^{i+1} + d^{i+1}),$$

$$\operatorname{Im}(\partial_n^*) = \bigoplus_{i\geq 0} \mathbb{Z}(ad^i - bc^i - qc^i) \ \bigoplus_{i\geq 0} q\mathbb{Z}(d^i + c^i).$$

For odd $n \geq 3$ we obtain

$$\operatorname{Ker}(\partial_n^*) = \bigoplus_{i\geq 0} \mathbb{Z}(c^i + d^i) \ \bigoplus_{i\geq 0} \mathbb{Z}(ad^i - bc^i - qc^i),$$

$$\operatorname{Im}(\partial_n^*) = \bigoplus_{i\geq 0} \mathbb{Z}(ad^i - bc^i) \ \bigoplus_{i\geq 0} q\mathbb{Z}(ad^i + c^{i+1} + d^{i+1}).$$

Consequently we have

$$H^0(A) = \bigoplus_{i\geq 0} \mathbb{Z}(c^i + d^i),$$

$$H^1(A) = \bigoplus_{i\geq 0} \mathbb{Z}(ad^i + c^{i+1} + d^{i+1}),$$

$$H^n(A) = \bigoplus_{i\geq 0} \mathbb{Z}_q(c^i + d^i)$$

for even $n \geq 2$, and

$$H^n(A) = \bigoplus_{i\geq 0} \mathbb{Z}_q(ad^i + c^{i+1} + d^{i+1})$$

for odd $n \geq 3$.

Set $\lambda = c + d$ in $H^0(A)$. Since $\lambda^i = c^i + d^i$, we have

$$H^0(A) = \mathbb{Z}[\lambda].$$

Set $\mu_n = a + c + d$ in $H^n(A)$ for odd $n \geq 1$. Since

$$\mu_n \cup \lambda^i = ad^i + c^{i+1} + d^{i+1}$$

in $H^n(A)$, we have

$$H^1(A) = \mathbb{Z}[\lambda] \cdot \mu_1$$

and

$$H^n(A) = \mathbb{Z}_q[\lambda] \cdot \mu_n.$$

for odd $n \geq 3$. Similarly, setting $\nu_n = 1$ in $H^n(A)$ for even $n \geq 2.$, we have

$$H^n(A) = \mathbb{Z}_q[\lambda] \cdot \nu_n.$$

Since

$$(\mu_1 \otimes \mu_1) \circ \Delta_{1,1}([a] + [b] + [c] + [d]) = ca - ad = 0$$

in $H^2(A)$, we see

$$\mu_1 \cup \mu_1 = 0.$$

Moreover, by an easy calculation we obtain

$$\nu_n \cup \mu_1 = \mu_{n+1}$$

and

$$\nu_n \cup \nu_2 = \nu_{n+2}.$$

Therefore, letting $\mu = \mu_1$ and $\nu = \nu_2$, we have

$$H(A) = \mathbb{Z}[\lambda, \mu, \nu]/(\mu^2,\, q\nu)$$

with $\deg(\lambda) = 0, \deg(\mu) = 1$ and $\deg(\nu) = 2$. In particular, when $q = \pm 1$,

$$H(A) = \mathbb{Z}[\lambda, \mu]/(\mu^2).$$

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