SEMIPRIME SKEW POLYNOMIAL RINGS

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Abstract. A ring R with a monomorphism α and an α-derivation δ with $\alpha\delta = \delta\alpha$ is called (α,δ)-*quasi Baer* (resp. *quasi Baer*) if the right annihilator of every (α,δ)-ideal (resp. ideal) of R is generated by an idempotent of R. In this paper we show that a semiprime ring $R[x;\alpha,\delta]$ is α-quasi Baer if and only if $S = R[x;\alpha,\delta]$ is $(\alpha,\overline{\delta})$-quasi Baer for every extended α-derivation $\overline{\delta}$ on S of δ if and only if R is (α,δ)-quasi Baer.

Throughout this paper R denotes an associative ring with unity, $\alpha : R \to R$ is a monomorphism which is not assumed to be surjective and δ is an α-derivation of R, that is, δ is an additive map such that $\delta(ab) = \delta(a)b + \alpha(a)\delta(b)$, for all $a, b \in R$. We denote $S = R[x;\alpha,\delta]$ the Ore extension whose elements are the polynomials $\Sigma_{i=0}^{n} r_i x^i$, $r_i \in R$, where the addition is defined as usual and the multiplication by $xb = \alpha(b)x + \delta(b)$, for each $b \in R$. An ideal I of R is called an α-*ideal* (resp. δ-*ideal*) if $\alpha(I) \subseteq I$ (resp. $\delta(I) \subseteq I$). If $\alpha^{-1}(I) = I$, then it is called $\alpha - invariant$. If I is both an α-ideal (resp. α-invariant ideal) and δ-ideal, then it is called an (α,δ)-*ideal* (resp. (α,δ)-*invariant ideal*).

In [5] Clark defines a ring to be *quasi Baer* if the right annihilator of every ideal is generated, as a right ideal, by an idempotent. He then uses the quasi Baer concept to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. Pollingher and Zaks [13] show that the quasi Baer condition is a Morita invariant property. Further work on quasi Baer rings appears in [3-4], [6-8] and [13]. According to Hirano [7 and 8], a ring R is called δ-*quasi Baer* (resp. α-*quasi Baer*) if the right annihilator of every δ-ideal (resp. α-ideal) of R is generated by an idempotent. A ring R is called (α,δ)-*quasi Baer* if the right annihilator of every (α,δ)-ideal of R is generated by an idempotent. A ring R is called *semiprime* (resp. δ-*semiprime*) if for any ideal (resp. δ-ideal) I of R, $I^2 = 0$ implies $I = 0$.

There are examples which show that the Baer condition is not preserved by various polynomial extensions (see [1 and 4]). However all is not lost for, in spite of the examples, some "Baerness" remains. Following [1,4 and 7-10], in this paper we study some Baerness property of the skew polynomial ring $R[x;\alpha,\delta]$. We first prove that if $S = R[x;\alpha,\delta]$ is a semiprime ring, then R is (α,δ)-quasi Baer if and only if $r_S(J)$ is generated by an idempotent as a right ideal of S, where J is an ideal of S such that $\alpha(a_n)x^n + \cdots + \alpha(a_0) \in J$ for each $a_nx^n + \cdots + a_0 \in J$. As a corollary we obtain [9, Theorem 11]. We also prove that, a semiprime ring $R[x;\alpha,\delta]$ is α-quasi Baer if and only if $S = R[x;\alpha,\delta]$ is $(\alpha,\overline{\delta})$-quasi Baer for every extended α-derivation $\overline{\delta}$ on S of δ if and only if R is (α,δ)-quasi Baer. This is a generalization of [7] to the more general setting.

Recall from [2] that, an idempotent $e \in R$ is *left* (resp. *right*) *semicentral* if $ere = re$ (resp. $ere = er$) for each $r \in R$. Equivalently, an idempotent $e \in R$ is left (resp. right) semicentral if Re (resp. eR) is an ideal of R.

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Lemma 1. *Let $S = R[x;\alpha,\delta]$ be a semiprime ring and $e(x) = e_nx^n + \cdots + e_0$ be a central idempotent of S. If I is an (α,δ)-ideal of R and $r_S(IS) = eS$, then $e(x) = e_0$.*

Proof. Since $x(e_nx^n+\cdots+e_0) = (e_nx^n+\cdots+e_0)x$, we have $\delta(e_0) = 0$, $\alpha(e_0)+\delta(e_1) = e_0$, $\cdots$, $\alpha(e_{n-1}) + \delta(e_n) = e_{n-1}$ and $\alpha(e_n) = e_n$. Thus $e_nI = 0$ and that $\delta(e_n)I = 0$. But $(e_nx^n + \cdots + e_0)I = 0$ and I is an (α,δ)-ideal of R, so $e_{n-1}\alpha^{n-1}(I) = 0$. Since $(\alpha(e_{n-1}) + \delta(e_n))\alpha^{n-1}(I) = e_{n-1}\alpha^{n-1}(I) = 0$, we have $e_{n-1}\alpha^{n-2}(I) = 0$. Similarly $e_{n-1}\alpha^{n-3}(I) = \cdots = e_{n-1}I = 0$. Thus $\alpha(e_{n-1})I = \delta(e_{n-1})I = 0$. Continuing in this way we see that $e_iI = 0$ for $0 \le i \le n$. Observe that $e_0 = e_0e(x) = e(x)e_0$ and $e_n = e(x)e_n = e_ne(x)$ since S is semiprime and $e_0, e_n \in r_R(I)$. Thus $e_0 = e_0^2$, $e_n = e_ne_0$ and $e_n\alpha^n(e_0) = e_0e_n = 0$. Since $\alpha(e_n) = e_n$ and α is injective, we have $e_n = 0$. Therefore $e(x) = e_0$.

Lemma 2. *If $S = R[x;\alpha,\delta]$ is semiprime and I is an (α,δ)-ideal of R, then $r_R(I) = \ell_R(I)$.*

Proof. Let $a \in \ell_R(I)$. It is clear that $aSI = 0$ and so $IaSIaS = 0$. Since S is semiprime $IaS = 0$ so $a \in r_R(I)$. Next assume $a \in r_R(I)$. Then $a \in r_S(SI)$. Since S is semiprime, by [2] we have $r_S(SI) = \ell_S(SI)$. Thus $aSI = 0$ and so $aI = 0$.

Theorem 3. *Let $S = R[x;\alpha,\delta]$ be a semiprime ring. Then the following are equivalent:*

(1) R is (α,δ)-quasi Baer.

(2) $r_S(J)$ is generated by an idempotent as a right ideal of $R[x;\alpha,\delta]$, where J is an ideal of S such that, $\alpha(a_n)x^n + \cdots + \alpha(a_0) \in J$ for each $a_nx^n + \cdots + a_0 \in J$.

Proof. (1) $\to$ (2) Let J be an ideal of S such that, $\alpha(a_n)x^n + \cdots + \alpha(a_1)x + \alpha(a_0) \in J$ for each $a_nx^n + \cdots + a_1x + a_0 \in J$. Consider the set J_0 of leading coefficients of polynomials in J. Clearly J_0 is an α-ideal of R. We have $xf(x)$-$(\alpha(a_n)x^n + \cdots + \alpha(a_1)x + \alpha(a_0))x = \delta(a_n)x^n$+(terms of lower degrees) for each $f(x) = a_nx^n + \cdots + a_1x + a_0 \in J$. Thus $\delta(a_n) \in J_0$, and that J_0 is an (α,δ)-ideal of R. Hence there exists a left semicentral idempotent $e \in R$ such that $r_R(J_0) = eR$. We show that $r_S(J) = eS$. Take $f(x) = a_nx^n + \cdots + a_1x + a_0 \in r_S(J)$ and $g(x) = b_mx^m + \cdots + b_1x + b_0 \in J$. Since $\alpha^m(b_m)x^m + \cdots + \alpha^m(b_1)x + \alpha^m(b_0) \in J$, we have $\alpha^m(b_m)\alpha^m(a_n) = 0$ and so $b_ma_n = 0$. Thus $a_n \in r_R(J_0)$. Observe that $b_mx^ma_nx^n = 0$ since $r_R(J_0) = \ell_R(J_0)$ and S is semiprime. But $e \in r_R(J_0)$ and hence $eg(x) = eb_{m-1}x^{m-1} + \cdots + eb_1x + eb_0$, so $eb_{m-1} \in J_0$ and $eb_{m-1} = 0$. Since e is left semicentral and $a_n \in r_R(J_0)$, $b_{m-1}x^{m-1}a_nx^n = 0$. Continuing in this way, we have $b_ix^ia_jx^j = 0$ for $0 \le i \le m$ and $0 \le j \le n$. Thus $a_j \in r_R(J_0) = eR$ for $0 \le j \le n$. Hence $r_S(J) \subseteq eS$. Since $e \in r_R(J_0)$ and S is semiprime, $eS \subseteq r_S(J)$. Therefore $r_S(J) = eS$.

(2) $\to$ (1). Let I be an (α,δ)-ideal of R. Then IS is an ideal of S such that, $\alpha(a_n)x^n + \cdots + \alpha(a_0) \in I$ for each $a_nx^n + \cdots + a_0 \in I$. Since S is semiprime, By [2] and Lemma 1, there exists a central idempotent $e_0 \in R$ such that $r_S(IS) = e_0S$. Since $r_R(I) = \ell_R(I)$ and S is semiprime, $r_R(I) = r_S(IS)\bigcap R = e_0R$. Therefore R is (α,δ)-quasi Baer.

According to Krempa [12], an endomorphism α of a ring R is called to be *rigid* if $a\alpha(a) = 0$ implies $a = 0$ for $a \in R$. A ring R is said to be *α-rigid* if there exists a rigid endomorphism α of R. Note that any rigid endomorphism of a ring is injective and α-rigid rings are reduced by [9]. Now we show that Theorem 2 implies the following:

Corollary 4. (Hong et. al [9, Theorem 11]). *Let R be an α-rigid ring. Then the following are equivalent:*

(1) R is a quasi Baer ring.

(2) $S = R[x;\alpha,\delta]$ is a quasi Baer ring.

Proof. Since R is an α-rigid ring, R is (α,δ)-quasi Baer if and only if R is quasi Baer. Let I be an ideal of S. Consider the set I_0 of all coefficients of elements of I. Then I_0 is a left ideal of R. Let $I_{0(\alpha,\delta)}$ be the (α,δ)-ideal of R generated by I_0. Then $I_{0(\alpha,\delta)}S$ is an ideal of S such that $\alpha(a_n)x^n+\cdots+\alpha(a_1)x+\alpha(a_0) \in I_{0(\alpha,\delta)}S$ for each $a_nx^n+\cdots+a_1x+a_0 \in I_{0(\alpha,\delta)}S$. Since R is α-rigid, by a direct calculation one can show that, $r_S(I) = r_S(I_{0(\alpha,\delta)}S)$. Clearly S is semiprime and so the rest of proof follows from Theorem 2.

The following example [7, Example 9] shows that there exists a commutative α-quasi Baer ring R such that $R[x;\alpha]$ is semiprime quasiBaer, but R is neither α-rigid nor quasi Baer.

Example 5. Let Z be the ring of integers and consider the ring $Z \bigoplus Z$ with the usual addition and multiplication. Then the subring
R=$\{(a,b) \in Z \bigoplus Z \mid a \equiv b \pmod 2\}$ of $Z \bigoplus Z$ is a commutative reduced ring. Note that the only idempotents of R are $(0,0)$ and $(1,1)$. For $(2,0) \in R$, we note that $r_R((2,0))$=$\{(0,2n) \mid n \in Z\}$. So $r_R((2,0))$ does not contain a non-zero idempotent of R. Hence R is not quasi Baer. Now let $\alpha : R \to R$ be defined by $\alpha((a,b))$=(b,a). Then α is an automorphism of R. Note that R is not α-rigid and $R[x;\alpha]$ is quasi Baer. Since R is commutative and $R[x;\alpha]$ is semiprime quasi Baer, so by Theorem 4, R is α-quasi Baer.

For a ring R with an α-derivation δ, if $\alpha\delta = \delta\alpha$ then we can extend α to $S = R[x;\alpha,\delta]$, by $\alpha(f(x)) = \alpha(a_n)x^n + \alpha(a_1)x + \cdots + \alpha(a_0)$ for all $f(x) = a_nx^n + \cdots + a_1x + a_0 \in S$. Also there exists α-derivation $\overline{\delta}$ on S which extends δ. For example, consider α-derivation $\overline{\delta}$ on S defined by $\overline{\delta}(f(x))$=$\delta(a_0)+\delta(a_1)x+\cdots+\delta(a_n)x^n$ for all $f(x) = a_0+a_1x+\cdots+a_nx^n \in S$.

Theorem 6. *If $S = R[x;\alpha,\delta]$ is a semiprime ring and $\alpha\delta = \delta\alpha$, then the following are equivalent:*
(1) R is (α,δ)-quasi Baer;
(2) S is α-quasi Baer;
(3) S is $(\alpha,\overline{\delta})$-quasi Baer for every extended α-derivation $\overline{\delta}$ on S of δ.

Proof. The equivalence (1) $\leftrightarrow$ (2) follows from Theorem 2.
(2) $\to$ (3). It is clear.
(3) $\to$ (1). Suppose that S is $(\alpha,\overline{\delta})$-quasi Baer for every extended α-derivation $\overline{\delta}$ on S of δ. Let I be any (α,δ)-ideal of R. It is clear that IS is an α-ideal of S. Let $f(x) \in IS$. Then $f(x) = \Sigma_{i=1}^{n} t_i f_i$, where $t_i \in I$ and $f_i \in S$ for $1 \leq i \leq n$. Since $\overline{\delta}(t_i f_i) = \overline{\delta}(t_i)f_i + \alpha(t_i)\overline{\delta}(f_i) \in IS$, so IS is an $(\alpha,\overline{\delta})$-ideal of S. Since S is semiprime $(\alpha,\overline{\delta})$-quasi Baer, $r_S(IS) = e_0S$ for some central idempotent $e_0 \in R$. The rest of proof is similar to (2) $\to$ (1) of Theorem 2.

Corollary 7. (Han et al. [7, Theorem 8]). *Let R be a δ-semiprime ring and let $S = R[x;\delta]$. Then the following are equivalent:*
(1) R is δ-quasi Baer;
(2) S is quasi Baer;
(3) $S = R[x;\delta]$ is $\overline{\delta}$-quasi Baer for every extended derivation $\overline{\delta}$ on S of δ.

Proof. This is a special case of theorem 5 by taking $\delta = 0$.

The following example [8, Example 2] shows that there exists a ring R such that $\alpha\delta = \delta\alpha$, $R[x;\alpha,\delta]$ is semiprime and R is (α,δ)-quasi Baer but it is not quasi Baer.

Example 8. Let K be a field, let A=K[s,t] be a commutative polynomial ring, and R=A/(st). Then R is reduced. Let $\overline{s}$=s+(st) and $\overline{t}$=t+(st) in R=A/(st). Define an automorphism α of R by $\alpha(\overline{s}) = \overline{t}$ and $\alpha(\overline{t}) = \overline{s}$. Since $\overline{s}\alpha(\overline{s})= 0$ and $\overline{s} \neq \overline{0}$, R is not α-rigid. Now, define $\delta : R \longrightarrow R$ by setting $\delta(\overline{r})$=$\overline{r}$-$\alpha(\overline{r})$. Clearly δ is an α-derivation of R, $\alpha\delta$=$\delta\alpha$ and $R[x;\alpha,\delta]$ is semiprime. We have $r_R(\overline{s})$=$\overline{t}R$. Since this ideal is not generated by any idempotent of R, R is not quasi Baer. However it is easily seen that any non-zero α-ideal I of R, is essential in R, and so $r_R(I)$= 0. Therefore R is (α,δ)-quasi Baer.

Lemma 9. *Let e be a left semicentral idempotent of R and let $\alpha(e) \in eR$. Then e is a left semicentral idempotent of $R[x;\alpha,\delta]$.*

Proof. We will proceed by induction on the degree of polynomials in $R[x;\alpha,\delta]$. Let $f(x) = a_0 + a_1x$. Then $f(x)e = a_0e + a_1\delta(e) + a_1\alpha(e)x$. Since $\alpha(e) \in eR$, we have $\alpha(e) = e\alpha(e)$ and hence $\delta(e) = \delta(e^2) = \alpha(e)\delta(e) + \delta(e)e = e(\alpha(e)\delta(e) + \delta(e)e)$. Thus $\delta(e) \in eR$ and $\delta(e) = e\delta(e)$. Therefore $f(x)e = a_0e + a_1e\delta(e) + a_1e\alpha(e)x = ea_0e + ea_1e\delta(e) + ea_1e\alpha(e)x = ea_0e + ea_1(\alpha(e)x + \delta(e)) = ea_0e + ea_1xe = e(a_0 + a_1x)e = ef(x)e$. Now suppose the statement is true for polynomials of degree less than n. Let $f(x) = ax^n + h(x)$, with deg $h(x) < n$. Then $f(x)e = a(\alpha^n(e)x^n + g(x))e + h(x)e$, with deg $g(x) < n$. Now $f(x)e = a\alpha^n(e)x^ne + ag(x)e + h(x)e$. Since $\alpha(e) = e\alpha(e)$, so $\alpha^n(e) = e\alpha^n(e)$. Therefore $f(x)e = eae\alpha^n(e)x^ne + eag(x)e + eh(x)e = ea(x^ne)e + eh(x)e = e(ax^n + h(x))e = ef(x)e$.

Now we turn our attention to the case where α is assumed to be an automorphism and δ an α-derivation of the ring R.

Lemma 10. Let I be an (α,δ)-invariant ideal of R and $t \in R$. If $It = 0$, then $Ix^nt = 0$ for each $n \geq 1$.

Proof. We will proceed by induction on n. For n=1, it implies that $\alpha(I)\alpha(t) = 0$. Since I is α-invariant, $I\alpha(t) = 0$ and $I\delta(t) = 0$. Thus $Ixt = I(\alpha(t)x+\delta(t)) \subseteq I\alpha(t)x+I\delta(t) = 0$. Now suppose that $Ix^nt = 0$. Then we have $Ix^{n+1}t = Ix^n(\alpha(t)x + \delta(t)) \subseteq Ix^n\alpha(t)x + Ix^n\delta(t)$, and $I\alpha(t) = 0 = I\delta(t) = 0$. Therefore by induction $Ix^{n+1}t = 0$.

The next theorem is also a generalization of [7, Theorem 8], to the skew polynomial ring $S = R[x;\alpha,\delta]$.

Theorem 11. Let α be an automorphism and δ be an α-derivation of R with $\alpha\delta = \delta\alpha$. Let $S = R[x;\alpha,\delta]$ be a semiprime ring. If $\alpha(e) \in eR$ for each semicentral idempotent $e \in R$, then the following are equivalent :

(1) the right annihilator of every (α,δ)-invariant ideal of R is generated by some idempotent as a right ideal of R.

(2) the right annihilator of every α-invariant ideal of S is generated by some idempotent as a right ideal of S.

(3) for every extended α-derivation $\overline{\delta}$ on S of δ, the right annihilator of every $(\alpha,\overline{\delta})$-invariant ideal of S is generated by some idempotent as a right ideal of S.

Proof. We will mention some notes in the proof of (1) $\rightarrow$ (2) and the remaining parts are similar to those of Theorem 5.

(1) $\rightarrow$ (2). Let I be an α-invariant ideal of S and let I_0 denotes the set of leading coefficients of polynomials in I. Clearly I_0 is an (α,δ)-ideal of R. Hence there exists a left semicentral idempotent $e \in R$ such that $r_R(I_0) = eR$. By Lemma 8, e is a semicentral idempotent of S. Since S is semiprime, e is central by [2]. Hence $\alpha(e) = e$ and $\delta(e) = 0$. For each $f(x) = a_n x^n + \cdots + a_1 x + a_0 \in r_S(I)$ and each $g(x) = b_m x^m + \cdots + b_1 x + b_0 \in I$, we have $b_i a_j = 0$ for $0 \leq i \leq m$ and $0 \leq j \leq n$. Thus $b_i \in r_R(I_0) = eR$ for $0 \leq i \leq m$. Using the fact that $\alpha(e) = e$ and $\delta(e) = 0$, it is clear that $f(x) \in eS$ and that $r_S(I) = eS$.

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