Scientiae Mathematicae Japonicae Online, **e-2006**, 375–393

OPTIMAL STOPPING PROBLEM WITH QUITTING OFFERS AND SEARCH COST

MONG SHAN EE AND SEIZO IKUTA

Received November 29, 2005; revised January 13, 2006

Abstract. This paper considers a discrete time optimal stopping problem with a finite planning horizon. In addition to an offer that may appear randomly, at each point in time over the entire planning horizon a quitting offer is assumed to be available. By accepting this, a decision maker can terminate the process. This paper assumes that the probability that an offer will appear if a search cost is paid is higher than if it is not paid. Thus, decisions must be made as to whether or not to accept the quitting offer, to accept an appearing offer, and to conduct a search for an offer. The main purpose of this paper is to clarify the properties of the optimal decision rules. One of our main findings is that the quitting offer must either be accepted at the starting point of the process or not be accepted prior to the end of the planning horizon, the deadline.

1 Introduction

In everyday life, we often encounter situations which require us to choose the best from chances subsequently appearing within a given planning horizon. Examples of these situations include finding an apartment with the lowest rent, selling an asset at the highest price, accepting a business proposal with the highest profit potential, and so on. In each of the above situations, at each point in time up to the end of the planning horizon, the deadline, the decision maker has to decide whether to stop the search process by accepting a currently available offer or to continue the process. The above problem is usually called the *optimal stopping problem* [2]-[7][11][12][15]-[17] where the decision maker is usually referred to as the *searcher*.

Since the seminal works by Stigler [19] and McCall [12], over the years many models of the optimal stopping problem have been proposed and examined. In the majority of these, for example, [1][3]-[8][10]-[12][15][16], it is assumed that a search cost must be paid to find an offer. The search cost can be represented by, for example, the advertising cost paid by businesses to attract more customers. The adoption of the search cost necessitates the introduction of the search skipping option. This is because with an excessively large search cost, it may become optimal to skip the search if the time period remaining up to the deadline is sufficiently long. Although literature on the optimal stopping problem with search cost is abundant, we found only one article [1] in which the search skipping option is taken into consideration. In the articles on the optimal stopping problem with search cost such as those cited above except [1], each search at a point in time is assumed to produce no more than one offer at the next point in time. Contrary to this, Assaf and Levikson [1] study the optimal stopping problem with an infinite planning horizon where the effect of the search effort (advertising) may last more than one period, implying that searching may be skipped if the search effort is still effective in producing offer at the next point in

2000 *Mathematics Subject Classification.* 90C39, 90B40, 91B70.
Key words and phrases. Optimal stopping problem, dynamic programming, quitting offer.

time. In addition to [1], two articles on the dynamic pricing problem [20] and admission control problem [18], which are related to the optimal stopping problem, also take the search skipping option into account.

In the conventional models with search cost, it has been implicitly assumed so far that an offer will definitely appear when the search cost is paid. However, a scenario where an offer may not appear even if a search cost is invested seems to be more natural from a practical viewpoint. In this scenario, a state that no offer will appear up to the deadline may become a possibility. This leads to a violation of the absolute requirement of the optimal stopping problem with finite planning horizon that an offer must be necessarily accepted up to the deadline. To avoid the occurrence of this violation, in this paper, we introduce a quitting offer at the deadline, which by accepting, the searcher can quit the process. This quitting offer is similar to the salvaging offer which is normally assumed to exist at the deadline in the newsboy problem [9][13][14]. The introduction of quitting offer is not needed in [1] because the planning horizon is assumed to be infinite in their model.

The introduction of a quitting offer only at the deadline is rather restrictive since in reality we often encounter situations where a quitting offer is also available at any point in time prior to the deadline. Here, for explanatory convenience in the subsequent discussions, we shall provide the definitions of some terms which will be referred to throughout this paper. First, let a randomly appearing offer be referred to as the *random offer*, and one which is readily available at each point in time over the entire planning horizon be the *quitting offer*. Next, the term *stopping* means the act of terminating the search process by accepting a randomly appearing offer, and the term *quitting* is the act of terminating the search process by accepting the quitting offer. Below let us provide three examples in which the above two types of offer are available.

1. Consider a short term traveler to a foreign country who has bought a car with a buyback agreement (quitting offer) from a dealer and he plans to sell it before returning to his home country. At any time before his departure date, he can sell the car back to the dealer or search for other buyers (random offer) who may offer a price higher than the one stated in the buyback agreement.
2. Consider a company which plans to divest itself from its wholly-owned trucking subsidiary by the year end in an attempt to refocus on its core business. Before the year end, the company can close down this subsidiary by selling the trucks to a salvage dealer (quitting offer) who has agreed to purchase them at any time or find another firm which may make a buying offer (random offer) for the subsidiary as a whole.
3. Consider a company which plans to launch a new product after say five years. The company can invest money to find a new product idea with high profit potential (random offer) or undertake the readily available product idea created by the previous product development project team (quitting offer).

Furthermore, it is assumed in [18] and [20] that no offer will appear if the search is skipped. In reality, however, even if a search is skipped, it is possible that an offer will appear with some probability, and this will be normally lower than one if a search is conducted. So far, this possibility has not been considered in any model proposed on the optimal stopping problem with search cost. Below, let us illustrate our viewpoint with an example. Consider a seller who has a piece of land for sale. Even if the seller does not spend any money to advertise the sale except to put up a 'for sale' sign on the land, a buyer who coincidentally passes by and becomes interested in buying may also approach the seller.

Taking the above into consideration, in this paper we propose a model of the optimal stopping problem with the following four assumptions:

1. A quitting offer is available at every point in time throughout the entire planning horizon;

2. Searching for a random offer can be skipped if the searcher so wishes.
3. If the search is skipped, a random offer may appear with a probability lower than that when the search is conducted.
4. If the search is conducted, a search cost must be paid. The search cost paid at the beginning of a period produces no more than one offer at the next point in time, which implies that an offer may not appear.

In this problem, the searcher has to made decisions as to whether or not to accept the quitting offer, to accept an appearing offer, and to conduct a search for an offer. The objective of this paper is to clarify the properties of the optimal decision rules, which consist of the three rules: The optimal quitting rule, the optimal stopping rule, and the optimal search rule.

Based on our review of the previous research, a model which takes the above four assumptions into consideration has never been proposed in the literature on the optimal stopping problem. However, we notice that research on the inventory problem with salvage option [14] possesses a structure which is related to ours in terms of the introduction of the quitting offers. In [14], Petruzzi and Monahan deal with a problem of determining when a retailer should terminate the selling season in the primary market by selling the remaining inventory in a secondary market. Our model differs from theirs in the following three major respects. Firstly, they assume that it is the searcher who offers a price for the assets on sale whereas in our model the searcher does not make an offer; he instead weighs the randomly appearing offer against his reservation value in determining whether or not to accept it. Secondly, they assume that a single fixed price is offered by the seller throughout the selling period so long as he does not terminate the selling process while in our model the offer's value is a random variable. Thirdly, they do not take the search cost of finding an offer into consideration.

Our major finding is that in the optimal decision rules a quitting offer must either be accepted at the starting point of the process or not be accepted prior to the deadline. However, it may be accepted at the deadline. Besides, our model yields the result that there may exist a *searching time threshold*, which implies that the searcher should skip the search from the starting point of the process up to this time threshold. In other words, the time period between the searching time threshold and the deadline can be called the *searching period*.

The rest of the paper is organized as follows. Section 2 provides a strict definition of our model. Section 3 defines several functions and examines their properties, which will be used in the subsequent analysis. In Section 4 we derive the optimality equations of the model, and in Section 5 we clarify the properties of the optimal decision rules. In Section 6 we extend the discussion to the case where the planning horizon is infinite. In Section 7 we provide some numerical examples that ascertain the properties of the optimal decision rules. In Section 8 we present the overall conclusions of our research and suggest some further works.

2 Model

Consider the following discrete-time stochastic decision process where points in time are numbered backward from the final point in time of the planning horizon, time 0 (the deadline) as $0, 1, \cdots$ and so on. Accordingly, if time t is a present point in time, the two adjacent times $t+1$ and $t-1$ are the previous and next points in time, respectively. Let the time interval between times t and $t-1$ be called the period t. This is small enough that no more than one offer may appear.

It is assumed that the searcher must necessarily accept one of the offers subsequently appearing up to the deadline. If a fixed cost $s \in [0, \infty)$ (search cost) is paid at the beginning of a period, an offer can be found at the next point in time with a probability $\lambda_1 \in (0, 1]$ and if the search cost is not paid, an offer may appear with a probability $\lambda_0 \in [0, 1)$ where $\lambda_1 > \lambda_0$; for convenience let us define $\lambda = \lambda_1 - \lambda_0$ where $0 < \lambda \le 1$. In the discussion that follows, the value w of an offer appearing randomly will be referred to as a *random offer* w. Random offers appearing at successive points in time, $w,\ w', \cdots$, are independent identically distributed random variables having a known continuous distribution function $F(w)$ with a finite expectation μ; let $f(w)$ denote its probability density function, which is truncated on both sides. More precisely, $F(w)$ and $f(w)$ are defined as follows. For certain given numbers a and b such that $0 < a < b < \infty$

$$F(w) = 0,\ w \le a, \qquad 0 < F(w) < 1, \quad a < w < b, \qquad F(w) = 1,\ b \le w, \tag{2.1}$$

$$f(w) = 0,\ \ w < a, \qquad f(w) > 0, \quad a \le w \le b, \qquad f(w) = 0,\ \ b < w, \tag{2.2}$$

where clearly $0 < a < \mu < b$.

In addition to the random offer defined above, a fixed *quitting offer* $\rho \in (-\infty, \infty)$ is assumed to be also available at each point in time where $\rho < 0$ implies the disposal cost to discard the unsold asset if the decision process is an asset selling process. Let us refer to the quitting offer at each point in time except the deadline as the *intervening quitting offer* and to the one on the deadline as the *terminal quitting offer*. By $\beta \in (0, 1]$ let us denote the discount factor, implying that the monetary value of one unit a period hence is equivalent to that of β units at the present point in time.

The decision rules of the model consist of the following three rules:

1. *Quitting rule* prescribing whether or not to quit the process by accepting the quitting offer ρ.
2. *Stopping rule* prescribing whether or not to stop the process by accepting a random offer w.
3. *Search rule* prescribing whether or not to search for a random offer at the beginning of every period.

The objective here is to find the optimal decision rules to maximize the total expected present discounted net profit over the planning horizon, i.e., the expected present discounted revenue from accepting an offer whether a random or quitting offer *minus* the total expected present discounted search cost paid up to a time when the process terminates by accepting an offer.

3 Preliminaries

This section defines the functions that will be used to describe the optimality equations of the model in Section 4. The properties of the functions verified in this section will be applied to the analysis of the model in the sections that follow. For any x let us define the following function

$$T(x) \quad = \quad \mathbf{E}[\max\{w - x, 0\}] \tag{3.1}$$

where $\mathbf{E}$ represents the taking of expectation with respects to w. Then, using the function, let us define:

$$K_0(x) \quad = \quad \lambda_0 \beta T(x) - (1 - \beta)x, \tag{3.2}$$

$$K_1(x) \quad = \quad \lambda_1 \beta T(x) - (1 - \beta)x - s, \tag{3.3}$$

$$L(x) \quad = \quad \lambda \beta T(x) - s \tag{3.4}$$

where

$$L(x) + K_0(x) = K_1(x) \tag{3.5}$$

due to the definition of $\lambda = \lambda_1 - \lambda_0$. By x_{K_0}, x_{K_1}, and x_L let us denote the solutions of the equations $K_0(x) = 0$, $K_1(x) = 0$, and $L(x) = 0$, respectively, if they exist, i.e.,

$$K_0(x_{K_0}) = 0, \qquad K_1(x_{K_1}) = 0, \qquad L(x_L) = 0. \tag{3.6}$$

If these equations have multiple solutions, then let us define the *minimum* of them by x_{K_0}, x_{K_1}, and x_L, respectively. In addition, for technical reason, if $\lambda_0 = 0$, let us define

$$x_{K_0} = 0. \tag{3.7}$$

Lemma 3.1 below will be used to examine the properties of the functions $K_0(x)$, $K_1(x)$, and $L(x)$ and their solutions x_{K_0}, x_{K_1}, and x_L stated in the two lemmas that follows.

Lemma 3.1

(a) *$T(x)$ is continuous and nonincreasing on $(-\infty, \infty)$.*
(b) *$T(x)$ is strictly decreasing on $(-\infty, b]$.*
(c) *$T(x) = 0$ on $[b, \infty)$ and $T(x) > 0$ on $(-\infty, b)$.*

Proof. See Appendix A. ▮

Lemma 3.2

(a) *$K_1(x)$ and $K_0(x)$ are continuous and strictly decreasing on $(-\infty, \infty)$ if $\beta < 1$.*
(b) *$K_1(x) + x$ and $K_0(x) + x$ are nondecreasing on $(-\infty, \infty)$.*
(c) *$|K_1(x)+x-K_1(y)-y| \le \beta|x-y|$ and $|K_0(x)+x-K_0(y)-y| \le \beta|x-y|$ for any x and y.*
(d) *Let $(1-\beta)^2 + s^2 = 0$. Then $x_{K_1} = b$ where $x < (\ge)\ x_{K_1} \Leftrightarrow K_1(x) > (=)\ 0 \Rightarrow K_1(x) > (\le)\ 0$.*
(e) *Let $(1-\beta)^2 + s^2 \ne 0$. Then there uniquely exists $x_{K_1} < b$ where $x < (= (>))\ x_{K_1} \Leftrightarrow K_1(x) > (= (<))\ 0$.*
(f) *Let $\lambda_0 > 0$.*
  1 *Let $\beta = 1$. Then $x_{K_0} = b$ where $x < (\ge)\ x_{K_0} \Leftrightarrow K_0(x) > (=)\ 0 \Rightarrow K_0(x) > (\le)\ 0$.*
  2 *Let $\beta < 1$. Then x_{K_0} uniquely exists with $0 < x_{K_0} < b$ where $x < (= (>))\ x_{K_0} \Leftrightarrow K_0(x) > (= (<))\ 0$.*
(g) *If $s > 0$, there uniquely exists $x_L < b$ where $x < (= (>))\ x_L \Leftrightarrow L(x) > (= (<))\ 0$.*

Proof. See Appendix B. ▮

Next, let us examine the relationship among x_{K_0}, x_{K_1}, and x_L. It will be seen later on that this relationship plays a key role in determining whether or not to conduct the search for a random offer.

Lemma 3.3 *Let $s > 0$. Then:*

(a) *Let $\lambda_0 = 0$. Then:*
  1 *If $\beta = 1$, then $x_{K_1} = x_L$.*
  2 *If $\beta < 1$, then $x_{K_0} < (= (>))\ x_{K_1} \Leftrightarrow x_{K_1} < (= (>))\ x_L$.*

(b) *Let* $\lambda_0 > 0$. *Then*:

1 *If* $\beta = 1$, *then* $x_{K_1} > x_L$.

2 *If* $\beta < 1$, *then* $x_{K_0} < (= (>))\ x_{K_1} \Leftrightarrow x_{K_1} < (= (>))\ x_L$.

Proof. See Appendix C. ∎

From Lemma 3.3 we immediately obtain the following corollary.

Corollary 3.1 *Let* $s > 0$. *Then*:

(a) *Let* $\beta = 1$. *If* $\lambda_0 = 0$, *then* $x_{K_1} = x_L$, *or else* $x_{K_1} > x_L$.

(b) *Let* $\beta < 1$. *Then* $\lambda\beta T(x_{K_0}) > (= (<))\ s \Leftrightarrow x_{K_1} < (= (>))\ x_L$.

Proof. (a) Immediate from Lemma 3.3(a1,b1).

(b) Since $K_0(x_{K_0}) = 0$ by definition, we have $L(x_{K_0}) = K_1(x_{K_0})$ from Eq. (3.5). Thus from Lemmas 3.3(a2,b2), 3.2(e) and Eq. (3.4) we see that $x_{K_1} < (= (>))\ x_L \Leftrightarrow x_{K_0} < (= (>))\ x_{K_1} \Leftrightarrow K_1(x_{K_0}) > (= (<))\ 0 \Leftrightarrow L(x_{K_0}) > (= (<))\ 0 \Leftrightarrow \lambda\beta T(x_{K_0}) > (= (<))\ s$. ∎

4 Optimality Equations

In this section, we provide the optimality equation that satisfies the objective function of the model. Let u_t and $r_t(w)$ be the maximum total expected present discounted profits, respectively, with no random offer and with a random offer w. Then we have

$$u_0 = \rho, \tag{4.1}$$

$$u_t = \max\{\rho, U_t\}, \qquad t \geq 1, \tag{4.2}$$

$$r_0(w) = \max\{w, \rho\}, \tag{4.3}$$

$$r_t(w) = \max\{w, \rho, U_t\} = \max\{w, u_t\}, \qquad t \geq 1 \tag{4.4}$$

where U_t is the maximum total expected present discounted profits from rejecting both random offer w and intervening quitting offer ρ, expressed as follows.

$$U_t = \max \left\{ \begin{array}{l} \mathtt{K} : \beta(\lambda_0\, \mathbf{E}[r_{t-1}(\xi)] + (1-\lambda_0)u_{t-1}), \\ \\ \mathtt{C} : \beta(\lambda_1\, \mathbf{E}[r_{t-1}(\xi)] + (1-\lambda_1)u_{t-1}) - s \end{array} \right\}, \qquad t \geq 1 \tag{4.5}$$

where ξ is the random offer appears at time $t-1$, the next point in time and where the symbols K and C represent, respectively, the decision to, respectively, skip and conduct the search for a random offer†. Thus the first and second terms inside the braces of the right-hand side of Eq. (4.5) are the maximum total expected present discounted profits from, respectively, *skipping* and *conducting* the search for a random offer. Now, for convenience, let

$$U_0 = \rho. \tag{4.6}$$

Then Eq. (4.2) holds for $t \geq 0$ instead of $t \geq 1$. Thus

$$u_t = \max\{\rho, U_t\}, \quad t \geq 0, \quad \text{with} \quad u_0 = \rho. \tag{4.7}$$

$$r_t(w) = \max\{w, u_t\}, \quad t \geq 0. \tag{4.8}$$

†We use K instead of S as a symbol representing "skip the search" because the symbol S is normally used to represent the decision of "stopping the process" in conventional optimal stopping problems.

Since $\mathbf{E}[r_{t-1}(\xi)] = \mathbf{E}[\max\{\xi, u_{t-1}\}] = \mathbf{E}[\max\{\xi - u_{t-1}, 0\}] + u_{t-1} = T(u_{t-1}) + u_{t-1}$ for $t \geq 1$, noting Eqs. (3.2) and (3.3), we can rewrite Eq. (4.5) as follows.

$$
\begin{aligned}
U_t &= \max\left\{\begin{array}{l} \mathtt{K}: \lambda_0\beta T(u_{t-1}) + \beta u_{t-1}, \\ \mathtt{C}: \lambda_1\beta T(u_{t-1}) + \beta u_{t-1} - s \end{array}\right\} \\
&= \max\{K_0(u_{t-1}) + u_{t-1}, K_1(u_{t-1}) + u_{t-1}\} && (4.9) \\
&= \max\{K_0(u_{t-1}), K_1(u_{t-1})\} + u_{t-1}, \quad t \geq 1. && (4.10)
\end{aligned}
$$

Since $\lambda = \lambda_1 - \lambda_0$, from Eq. (3.4) we have

$$K_1(u_{t-1}) + u_{t-1} - K_0(u_{t-1}) - u_{t-1} = \lambda\beta T(u_{t-1}) - s = L(u_{t-1}), \qquad t \geq 1. \tag{4.11}$$

Accordingly, Eq. (4.9) can be rewritten as

$$U_t = \max\{0, L(u_{t-1})\} + K_0(u_{t-1}) + u_{t-1}, \quad t \geq 1. \tag{4.12}$$

From all the above the optimal decision rule of the model can be prescribed as follows.

Optimal Decision Rule 4.1

(a) *Let* $t = 0$.

1 *Suppose no random offer exists. Then quit the process by accepting the terminal quitting offer* ρ (*see* Eq. (4.1)).

2 *Suppose a random offer* w *appears. Then, if* $w \geq \rho$, *accept the offer* w, *or else accept the terminal quitting offer* ρ (*see* Eq. (4.3)).

(b) *Let* $t \geq 1$.

1 *Suppose no random offer exists.*

i. *If* $\rho \geq (\leq)\ U_t$, *quit the process by accepting the intervening quitting offer* ρ (*continue the search process*) (*see* Eq. (4.2)).

ii. *Assume that the process continues. If* $L(u_t) \geq (\leq)\ 0$, *conduct the search for a random offer by paying the search cost* s (*skip the search for a random offer*) (*see* Eq. (4.12)), *and then the process proceeds to time* $t - 1$; *go to* (a) *if* $t = 1$ *and to* (b) *if* $t \geq 2$.

2 *Suppose a random offer* w *appears.*

i. *If* $w \geq u_t$, *accept the random offer* w, *or else do not* (*i.e.*, u_t *becomes the searcher's* optimal reservation value) (*see* Eq. (4.8)).

ii. *If the searcher rejects the random offer* w, *either quit the process by accepting the intervening quitting offer* ρ *or continue the process* (*see* Eq. (4.4)). *Then the decision rule is the same as in* (b1i).

iii. *Assume that the process continues. Then the decision rule is the same as in* (b1ii).

5 Analysis

This section is devoted to examining the properties of the optimal quitting rule and optimal search rule. The optimal stopping rule is prescribed by comparing the random offer w against u_t for $t \geq 0$ as stated in Optimal Decision Rule 4.1(a2,b2i).

5.1 Optimal quitting rule

This subsection examines the searcher's optimal quitting rule where he has to decide at each point in time whether to accept the quitting offer or continue the process. Here, let $u = \lim_{t\to\infty} u_t$ if it exists.

Theorem 5.1

(a) u_t *is nondecreasing in* $t \geq 0$.

(b) *Let* $\rho \geq \max\{x_{K_0}, x_{K_1}\}$. *Then* $U_t \leq \rho$ *and* $u_t = \rho$ *for* $t \geq 0$.

(c) *Let* $\rho < \max\{x_{K_0}, x_{K_1}\}$. *Then*:

1 $\rho \leq U_t = u_t \leq b$ *for* $t \geq 0$ *where*

$$\begin{aligned}
(5.1)\qquad u_t &= \max\{K_0(u_{t-1}) + u_{t-1}, K_1(u_{t-1}) + u_{t-1}\},\\
(5.2)\qquad &= \max\{K_0(u_{t-1}), K_1(u_{t-1})\} + u_{t-1},\\
(5.3)\qquad &= \max\{0, L(u_{t-1})\} + K_0(u_{t-1}) + u_{t-1}, \quad t \geq 1.
\end{aligned}$$

2 u_t *converges to a finite* u *as* $t \to \infty$ *where* $\max\{K_0(u), K_1(u)\} = 0$.

Proof. Note that $U_1 - \rho = \max\{K_0(\rho), K_1(\rho)\} \cdots (1^*)$ from Eqs. (4.10) and (4.7), and that if $\lambda_0 = 0$, then $x_{K_0} = 0$ by definition (Eq. (3.7)) and $K_0(x) = -(1-\beta)x \cdots (2^*)$ from Eq. (3.2).

(a) From Eq. (4.7) with $t = 1$ we have $u_1 \geq \rho = u_0$. Let $u_{t-1} \geq u_{t-2}$. Then from Eq. (4.9) and Lemma 3.2(b) we obtain $U_t \geq \max\{K_0(u_{t-2})+u_{t-2}, K_1(u_{t-2})+u_{t-2}\} = U_{t-1}$, so that $u_t \geq \max\{\rho, U_{t-1}\} = u_{t-1}$ due to Eq. (4.7). Therefore, by induction we get $u_t \geq u_{t-1}$ for $t \geq 1$, implying that u_t is nondecreasing in $t \geq 0$.

(b) Let $\rho \geq \max\{x_{K_0}, x_{K_1}\}$. Then $\rho \geq x_{K_0}$ and $\rho \geq x_{K_1}$. If $\lambda_0 = 0$, then $\rho \geq x_{K_0} = 0$, hence $K_0(\rho) = -(1-\beta)\rho \leq 0$ from (2^*), and if $\lambda_0 > 0$, then $K_0(\rho) \leq 0$ from Lemma 3.2(f). Hence $K_0(\rho) \leq 0$ whether $\lambda_0 = 0$ or $\lambda_0 > 0$. Since $\rho \geq x_{K_1}$, we have $K_1(\rho) \leq 0$ from Lemma 3.2(d,e). Thus $U_1 - \rho \leq 0$, i.e., $U_1 \leq \rho$ from (1^*), so $u_1 = \rho$ due to Eq. (4.7). Suppose $U_{t-1} \leq \rho$. Then since $u_{t-1} = \rho$ from Eq. (4.7), we have $U_t = \max\{K_0(\rho) + \rho, K_1(\rho) + \rho\} = U_1 \leq \rho$ from Eq. (4.9), so $u_t = \rho$ due to Eq. (4.7) for $t \geq 1$. Accordingly, by induction it follows that $U_t \leq \rho = u_t$ for $t \geq 1$. From this result and the fact that $U_0 = \rho$ and $u_0 = \rho$ due to, respectively, Eqs. (4.6) and (4.7) we see that the assertion holds.

(c) Let $\rho < \max\{x_{K_0}, x_{K_1}\}$.

(c1) Note that $u_0 = U_0 = \rho$ due to Eqs. (4.6) and (4.7), so $u_0 = U_0 \geq \rho$. Let $x_{K_1} > x_{K_0}$. Then since $\rho < x_{K_1}$, we have $K_1(\rho) > 0$ from Lemma 3.2(d,e), so $K_1(\rho) \geq 0$. Hence $U_1 - \rho \geq K_1(\rho) \geq 0$ from (1^*), so $U_1 \geq \rho$. Let $x_{K_1} \leq x_{K_0}$. Then $\rho < x_{K_0}$, hence it follows that $\lambda_0 = 0$ leads to $\rho < x_{K_0} = 0$ by the definition of x_{K_0}, so $K_0(\rho) = -(1-\beta)\rho \geq 0$ from (2^*) and that $\lambda_0 > 0$ leads to $K_0(\rho) > 0$ from Lemma 3.2(f), hence $K_0(\rho) \geq 0$. Thus $U_1 - \rho \geq K_0(\rho) \geq 0$ from (1^*), so $U_1 \geq \rho$. Accordingly, we have $U_1 \geq \rho$ whether $x_{K_1} > x_{K_0}$ or $x_{K_1} \leq x_{K_0}$. Assume $U_{t-1} \geq \rho$, hence $u_{t-1} = U_{t-1}$ due to Eq. (4.7), so $u_{t-1} \geq \rho$. Then from Eq. (4.9) and Lemma 3.2(b) we get $U_t \geq \max\{K_0(\rho) + \rho, K_1(\rho) + \rho\} = U_1 \geq \rho$, so $u_t = U_t$ for $t \geq 1$ due to Eq. (4.7). Hence, by induction it follows that $u_t = U_t \geq \rho \cdots (3^*)$ for $t \geq 0$. Therefore, from Eqs. (4.9), (4.10), and (4.12) we have, respectively, Eqs. (5.1), (5.2), and (5.3). Here note that if $\lambda_0 > 0$, then $x_{K_0} \leq b$ due to Lemma 3.2(f) and if $\lambda_0 = 0$, then $x_{K_0} = 0 < b$ by the definition of x_{K_0}, so $x_{K_0} \leq b$. Accordingly, $x_{K_0} \leq b$ whether $\lambda_0 > 0$ or $\lambda_0 = 0$. In addition, since $x_{K_1} \leq b$ due to Lemma 3.2(d,e), it eventually follows that $\rho < \max\{x_{K_0}, x_{K_1}\} \leq b$, hence $u_0 = \rho < b$ from Eq. (4.7). Suppose $u_{t-1} \leq b$. Then from

Eq. (5.1), Lemma 3.2(b), Eqs. (3.2), (3.3), Lemma 3.1(c), and the assumption of $s \geq 0$ we have

$$\begin{aligned} u_t &\leq \max\{K_0(b)+b, K_1(b)+b\} \\ &= \max\{\beta(\lambda_0 T(b)+b), \beta(\lambda_1 T(b)+b)-s\} = \max\{\beta b, \beta b - s\} = \beta b \leq b. \end{aligned}$$

Accordingly, by induction $u_t \leq b$ for $t \geq 0$. Therefore, from (3*) it follows that $\rho \leq U_t = u_t \leq b$ for $t \geq 0$.

(c2) Since u_t is upper bounded in $t \geq 0$ from (c1), it follows from (a) that u_t converges to a finite u as $t \to \infty$. Then, noting Eq. (5.1) and Lemma 3.2(c) we have

$$\begin{aligned} &|u_t - \max\{K_0(u)+u, K_1(u)+u\}| \\ &\quad = |\max\{K_0(u_{t-1})+u_{t-1}, K_1(u_{t-1})+u_{t-1}\} - \max\{K_0(u)+u, K_1(u)+u\}| \\ &\quad \leq \max\{|K_0(u_{t-1})+u_{t-1}-K_0(u)-u|, |K_1(u_{t-1})+u_{t-1}-K_1(u)-u|\} \\ &\quad \leq \max\{\beta|u_{t-1}-u|, \beta|u_{t-1}-u|\} = \beta|u_{t-1}-u|, \end{aligned}$$

which converges to 0 as $t \to \infty$. Accordingly, u_t converges to $\max\{K_0(u)+u, K_1(u)+u\}$, hence $u = \max\{K_0(u)+u, K_1(u)+u\}$ or equivalently $\max\{K_0(u), K_1(u)\} = 0$. ∎

Here we shall discuss the practical implications of Theorem 5.1. Let $t \geq 1$. First, the assertion (b) implies that when ρ is large enough to be greater than or equal to $\max\{x_{K_0}, x_{K_1}\}$, since $U_t \leq \rho = u_t$ for $t \geq 0$, if no random offer w exists at that time, it is optimal to quit the process by accepting the quitting offer ρ for $t \geq 1$ (Optimal Decision Rule 4.1(b1i)), whereas, if a random offer w appears at the any $t \geq 1$, the searcher must decide to quit the process by accepting either the quitting offer ρ or random offer w (Optimal Decision Rule 4.1(b2i)). In other words, it follows that the search process starts and ends at the same time.

On the other hand, the assertion (c) implies that when ρ is small enough to be less than $\max\{x_{K_0}, x_{K_1}\}$, since $U_t \geq \rho$ for $t \geq 0$, it is optimal to continue the process by rejecting the intervening quitting offer for $t \geq 1$ if a random offer does not appears or appears but is rejected (Optimal Decision Rule 4.1(b1i,b2ii)). Thus from the above result, Eqs. (4.1), and (4.3) we see that the quitting offer is never accepted prior to the deadline; however, it may be accepted at the deadline. This implies that the process is reduced to the one with only the terminal quitting offer. Under this condition, at all points in time except the deadline, the searcher will make a decision only between accepting a current random offer or continue the process. Furthermore, since ρ is assumed to be small enough to be less than $\max\{x_{K_0}, x_{K_1}\}$, a searcher, in order to avoid being forced to accept ρ at the deadline, would be more motivated to accept the random offer. Consequently, he will tend to lower his optimal reservation value u_t as the remaining time periods up to the deadline t decreases. Therefore, it can be conjectured that u_t is nondecreasing in t. Theorem 5.1(a) affirms this conjecture.

5.2 Optimal search rule

In this subsection, we will discuss the optimal search rule on whether or not to invest a search cost to find a random offer. From Optimal Decision Rule 4.1(b1ii), we see that the sign of $L(u_t)$ determines whether or not the search should be conducted.

Theorem 5.2 *Let $\rho < \max\{x_{K_0}, x_{K_1}\}$.*

(a) *Let $s = 0$. Then conduct the search for $t \geq 1$.*

(b) *Let $s > 0$.*

1 *For any given $t \geq 1$, if $u_{t-1} \leq (\geq)\ x_L$, then conduct (skip) the search.*

2 *Let $\rho \geq x_L$. Then skip the search for $t \geq 1$.*

3 *Let $\rho < x_L$.*

i. *Let $\lambda_0 = 0$.*

1. *Let $\beta = 1$ or let $\beta < 1$ and $\lambda\beta T(x_{K_0}) \geq s$. Then conduct the search for $t \geq 1$.*
2. *Let $\beta < 1$ and $\lambda\beta T(x_{K_0}) < s$. Then there exists a $t^* \geq 1$ such that conduct the search if $1 \leq t \leq t^*$ and skip the search if $t^* < t$.*

ii. *Let $\lambda_0 > 0$.*

1. *Let $\beta < 1$ and $\lambda\beta T(x_{K_0}) \geq s$. Then conduct the search for $t \geq 1$ where $u_t \leq x_L$ for $t \geq 0$.*
2. *Let $\beta = 1$ or let $\beta < 1$ and $\lambda\beta T(x_{K_0}) < s$. Then there exists a $t^* \geq 1$ such that conduct the search if $1 \leq t \leq t^*$ and skip the search if $t^* < t$.*

Proof. Let $\rho < \max\{x_{K_0}, x_{K_1}\}$. Then $u_1 - u_0 = \max\{K_0(\rho), K_1(\rho)\}$ from Eqs. (5.2) and (4.7).

(a) Let $s = 0$. Since $\lambda_1 > \lambda_0$ by assumption and $T(u_{t-1}) \geq 0$ for $t \geq 1$ due to Lemma 3.1(c), we have $K_1(u_{t-1}) + u_{t-1} \geq K_0(u_{t-1}) + u_{t-1}$ for $t \geq 1$, hence $L(u_{t-1}) \geq 0$ for $t \geq 1$ from Eq. (4.11). Consequently, the assertion holds due to Optimal Decision Rule 4.1(b1ii).

(b) Let $s > 0$. Then $(1-\beta)^2 + s^2 \neq 0$.

(b1) If $u_{t-1} \leq (\geq)\ x_L$, then $L(u_{t-1}) \geq (\leq)\ 0$ from Lemma 3.2(g), hence the assertion holds due to Optimal Decision Rule 4.1(b1ii).

(b2) Let $\rho \geq x_L$. Then since $u_0 = \rho \geq x_L$ due to Eq. (4.7), we have $u_{t-1} \geq x_L$ for $t \geq 1$ from Theorem 5.1(a), hence skipping the search for $t \geq 1$ is optimal from (b1).

(b3) Let $\rho < x_L$. Then $u_0 = \rho < x_L \cdots (1^*)$ from Eq. (4.7).

(b3i) Let $\lambda_0 = 0$. Then $x_{K_0} = 0$ by definition (Eq. (3.7)) and $K_0(x) = -(1-\beta)x$ from Eq. (3.2).

(b3i1) Let $\beta = 1$. Then $x_{K_1} = x_L$ due to Corollary 3.1(a), hence $x_{K_1} \leq x_L$. Let $\beta < 1$ and $\lambda\beta T(x_{K_0}) \geq s$. Then $x_{K_1} \leq x_L$ due to Corollary 3.1(b). Accordingly, whether $\beta = 1$ or "$\beta < 1$ and $\lambda\beta T(x_{K_0}) \geq s$", we have $x_{K_1} \leq x_L$, so that $K_1(x_L) \leq 0$ from Lemma 3.2(e). Suppose $u_{t-1} \leq x_L$. Then, since $L(u_{t-1}) \geq 0$ due to Lemma 3.2(g), it follows from Eqs. (5.3), (3.5), and Lemma 3.2(b) that $u_t = L(u_{t-1}) + K_0(u_{t-1}) + u_{t-1} = K_1(u_{t-1}) + u_{t-1} \leq K_1(x_L) + x_L \leq x_L$. Accordingly, by induction $u_t \leq x_L$ for $t \geq 0$, so that $u_{t-1} \leq x_L$ for $t \geq 1$. Hence, the assertion holds from (b1).

(b3i2) Let $\beta < 1$ and $\lambda\beta T(x_{K_0}) < s$. Then $x_L < x_{K_1}$ due to Corollary 3.1(b). Furthermore, from Theorem 5.1(c2) we get $\max\{K_0(u), K_1(u)\} = 0$, implying that $K_1(u) \leq 0$, so $u \geq x_{K_1}$ due to Lemma 3.2(e). From (1^*) and the above we have $\rho = u_0 < x_L < x_{K_1} \leq u$. Accordingly, as $t \to \infty$, the u_t starts from $u_0 = \rho < x_L$, continues to increase, crosses through x_L, and converges to $u > x_L$ due to Theorem 5.1(a,c2), it follows that there exists a $t^* \geq 1$ such that $u_{t-1} < x_L$ for $1 \leq t \leq t^*$ and $x_L \leq u_{t-1}$ for $t^* < t$, implying that conduct the search if $1 \leq t \leq t^*$ and skip the search if $t^* < t$ due to (b1).

(b3ii) Let $\lambda_0 > 0$.

(b3ii1) Let $\beta < 1$ and $\lambda\beta T(x_{K_0}) \geq s$. Then, since $x_{K_1} \leq x_L$ due to Corollary 3.1(b), we have $K_1(x_L) \leq 0$ due to Lemma 3.2(e). Suppose $u_{t-1} \leq x_L$. Then in the same way as in the proof of (b3i1) we have $u_t \leq x_L$ for $t \geq 0$. Accordingly, the assertion holds for the same reason as in the proof of (b3i1).

(b3ii2) Let $\beta = 1$ or let $\beta < 1$ and $\lambda\beta T(x_{K_0}) < s$. Then $x_{K_1} > x_L$ due to Corollary 3.1(a,b). Furthermore, since $\max\{K_0(u), K_1(u)\} = 0$ from Theorem 5.1(c2), we have $K_1(u) \leq 0$, so $u \geq x_{K_1}$ due to Lemma 3.2(e). From (1*) and the above we have $\rho = u_0 < x_L < x_{K_1} \leq u$. Accordingly, the assertion holds for the same reason as in the proof of (b3i2). ∎

Suppose $\rho < \max\{x_{K_0}, x_{K_1}\}$ holds. Then the search process is reduced to the one with only the terminal quitting offer (see Theorem 5.1(c1)). Therefore, intuitively we might expect that if the terminal quitting offer ρ is sufficiently small or negative, the searcher would conduct the search to find a random offer w greater than ρ in order to avoid having to accept ρ at the deadline. This is particularly true for the case of $\rho < 0$ because accepting a negative terminal quitting offer will incur a cost for the searcher at the deadline. Now, with reference to Theorem 5.2, we can observe the followings.

1. Let $s = 0$. Clearly conducting the search for a random offer is optimal since no search cost is incurred.
2. Let $s > 0$, implying that a search cost is incurred if a search is conducted.

   i. Let $\rho \geq x_L$. Then it is optimal to skip the search for all $t \geq 1$. This result implies that instead of actively conducting the search, the searcher would wait passively for the random offer to appear.

   ii. Let $\rho < x_L$.

      1) Let $\lambda_0 = 0$ and $\beta = 1$ or let $\beta < 1$ and $\lambda\beta T(x_{K_0}) \geq s$. Then it is optimal to conduct the search for all $t \geq 1$. Note that if $\lambda_0 = 0$ and $\beta = 1$, we have $\rho < x_{K_1} = x_L$ from Corollary 3.1(a), and if $\beta < 1$ and $\lambda\beta T(x_{K_0}) \geq s$, we have $\rho < x_{K_1} \leq x_L$ from the fact that $x_{K_1} \leq x_L$ due to Corollary 3.1(b) and that $x_{K_0} \leq x_{K_1}$ due to Lemma 3.3(a2,b2). Thus in these two cases, we get $\rho < x_{K_1} \leq x_L$. Therefore, this result agrees with the above stated conjecture since ρ is sufficiently small.
      2) Let $\lambda_0 > 0$ and $\beta = 1$ or let $\beta < 1$ and $\lambda\beta T(x_{K_0}) < s$. Then there exists the searching time threshold $t^* \geq 1$ such that it is optimal to conduct the search if $1 \leq t \leq t^*$ and skip the search if $t^* < t$. This implies that the searching period exists. The occurrence of this phenomenon is plausible since a search cost $s > 0$ is incurred for every period if the searcher decides to conduct the search. Hence in order to save on the search cost, it becomes optimal to skip the search if the planning horizon is sufficiently long.

In Theorem 5.2 we show that under certain conditions, the searching time threshold t^* exists. Below, we shall investigate the monotonicity of t^* in ρ if it exists.

Theorem 5.3 *The searching time threshold t^* in* (b3i2) *and* (b3ii2) *of* Theorem 5.2 *is nonincreasing in ρ.*

Proof. Assume that $\rho < \rho'$. Then, by u'_t let us denote u_t for ρ'. First from Eq. (4.7) we have $u_0 - u'_0 = \rho - \rho' < 0$, hence $u_0 < u'_0$. Suppose $u_{t-1} \leq u'_{t-1}$. Then from Eq. (5.1) and Lemma 3.2(b) we obtain $u_t \leq \max\{K_0(u'_{t-1}) + u'_{t-1}, K_1(u'_{t-1}) + u'_{t-1}\} = u'_t$. Accordingly, by induction we obtain $u_t \leq u'_t$ for $t \geq 0$. Hence u_t is nondecreasing in ρ for $t \geq 0$. Furthermore, since the function $L(x)$ is independent of ρ (see Eq. (3.4)), it follows that x_L is also independent of ρ. From the above result and the fact that u_t is nondecreasing

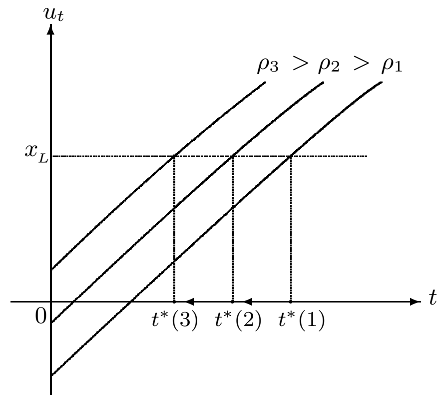


Figure 5.1: u_t in $t \geq 0$ for three different values of ρ.

in $t \geq 0$ due to Theorem 5.1(a) it can be immediately seen that the assertion holds (see Figure 5.1). ∎

In Theorem 5.3 we can successfully verify the monotonicity of the searching time threshold t^* in ρ because the functions $K_0(x)$, $K_1(x)$, and $L(x)$ are independent of ρ. However, since $K_0(x)$, $K_1(x)$, and $L(x)$ are dependent on the model's other parameters s, β, λ_0, and λ_1, it is very difficult to mathematically examine the monotonicity of t^* in these parameters. In Section 7.2 we will numerically investigate the monotonicities in s, β, λ_0, and λ_1.

6 Infinite Planning Horizon

Let us now extend the discussion into an infinite planning horizon. First, we need the following corollary which is obtained directly from Theorem 5.2, Eqs. (5.1), and (4.9).

Corollary 6.1

(a) *In* (a), (b3i1), *and* (b3ii1) *of* Theorem 5.2 *we have* $u_t = K_1(u_{t-1}) + u_{t-1}$ *for* $t \geq 1$ *with* $u_0 = \rho$.

(b) *In* (b2) *of* Theorem 5.2 *we have* $u_t = K_0(u_{t-1}) + u_{t-1}$ *for* $t \geq 1$ *with* $u_0 = \rho$.

(c) *In* (b3i2) *and* (b3ii2) *of* Theorem 5.2 *we have* $u_t = K_1(u_{t-1}) + u_{t-1}$ *for* $1 \leq t \leq t^*$ *and* $u_t = K_0(u_{t-1}) + u_{t-1}$ *for* $t^* < t$.

When the planning horizon is sufficiently long, the maximum total expected present discounted profit can be approximated by the u derived in Theorem 6.1 below.

Theorem 6.1

(a) *In* (a) *of* Corollary 6.1, *if* $(1-\beta)^2 + s^2 = 0$, *then* $u = b$, *or else* $u = x_{K_1}$.

(b) *Let* $\lambda_0 > 0$. *Then* $u = b$ *for* $\beta = 1$ *and* $u = x_{K_0}$ *for* $\beta < 1$ *in* (b) *and* (c) *of* Corollary 6.1.

(c) *Let* $\lambda_0 = 0$. *Then* $u = 0$ *for* $\beta < 1$ *and* $u = \rho$ *for* $\beta = 1$ *in* (b) *of* Corollary 6.1 *and* $u = 0$ *for* $\beta < 1$ *and* $u = u_{t^*}$ *for* $\beta = 1$ *in* (c) *of* Corollary 6.1.

Proof. (a) In this case we have $K_1(u) = 0 \cdots (1^*)$ from Theorem 5.1(c2). Let $(1-\beta)^2+s^2 = 0$, hence $\beta = 1$ and $s = 0$. Then $\lambda_1 T(u) = 0$ from (1^*) and Eq. (3.3), so $T(u) = 0$ due to the fact that $\lambda_1 > 0$. Thus from Lemma 3.1(c) we have $u \geq b$. Since $u \leq b$ from Theorem 5.1(c1), we get $u = b$, hence the former half of the assertion holds. The latter half is true due to Lemma 3.2(e), Eq. (3.6), and (1^*).

(b) Let $\lambda_0 > 0$. Then we have $K_0(u) = 0 \cdots (2^*)$ from Theorem 5.1(c2). Suppose $\beta = 1$. Then $\lambda_0 T(u) = 0$ from (2^*) and Eq. (3.2), so $T(u) = 0$ due to the fact that $\lambda_0 > 0$. Thus we obtain $u = b$ in almost the similar way as in the proof of (a). Suppose $\beta < 1$. Then $u = x_{K_0}$ due to Lemma 3.2(f2), Eq. (3.6), and (2^*).

(c) Let $\lambda_0 = 0$. Then $K_0(x) = -(1-\beta)x \cdots (3^*)$ from Eq. (3.2). Assume that (b) of Corollary 6.1 holds, so $u_t = K_0(u_{t-1}) + u_{t-1} = \beta u_{t-1}$ for $t \geq 1$ due to (3^*). Since $u_0 = \rho$ from Eq. (4.7), we get $u_t = \beta^t \rho$ for $t \geq 1$. Thus $u = 0$ if $\beta < 1$, or else $u_t = \rho$, so $u = \rho$. Assume that (c) of Corollary 6.1 holds. Since $t^* \geq 1$ exists, we obtain $u_t = K_0(u_{t-1}) + u_{t-1} = \beta u_{t-1}$ for $t^* < t$ due to Corollary 6.1(c) and (3^*). Thus $u_t = \beta^{t-t^*} u_{t^*}$ for $t^* < t$. Accordingly, we get $u = 0$ if $\beta < 1$, or else $u_t = u_{t^*}$, so $u = u_{t^*}$. ∎

7 Numerical Examples

In this section, through numerical experiments let us exemplify the properties of the optimal decision rules and the monotonicity of the searching time threshold t^* in four of the model's parameters s, λ_0, λ_1, and β.

7.1 Properties of optimal decision rules

Let $\beta = 0.99$, $\lambda_0 = 0.2$, $\lambda_1 = 0.8$, $s = 0.07$, $\rho = 1.0$. Then $(1-\beta)^2+s^2 \neq 0$. Also, let $F(w)$ be the uniform distribution on $[1.5, 2.5]$, i.e., $a = 1.5$ and $b = 2.5$. Using Eqs. (3.2) to (3.4), we obtain $x_{K_0} \approx 2.0455$, $x_{K_1} \approx 2.0227$, and $x_L \approx 2.0145$, so $x_L < x_{K_1} < x_{K_0}$. In this case, since $1.0 = \rho < \max\{x_{K_0}, x_{K_1}\} = x_{K_0} = 2.0455$ (Theorem 5.1(c)), the search process reduces to the one with only the terminal quitting offer. Figure 7.2 depicts the monotonicity of u_t in t, in which u_t is nondecreasing in t (Theorem 5.1(a)). In addition, since $1.0 = \rho < x_L = 2.0145$ and since $\beta < 1$ and $\lambda\beta T(x_{K_0}) = (0.8 - 0.2) \times 0.99 \times 0.1033 = 0.0614$, i.e., $\lambda\beta T(x_{K_0}) < s$, the conditions in Theorem 5.2(b3,b3ii2) are also satisfied. Figure 7.2 demonstrates that the searching time threshold $t^* = 8$ exists, implying that it is optimal to conduct the search if $1 \leq t \leq t^*$ and to skip the search if $t^* < t$.

7.2 Monotonicity of t^* in s, λ_0, λ_1, and β

The monotonicity of t^* in ρ is successfully proven in Theorem 5.3; however, its monotonicity in other model's parameters s, λ_0, λ_1, and β are difficult to verify for the reason stated in the paragraph below the proof of Theorem 5.3. Accordingly, we shall numerically investigate the monotonicity in s, λ_0, λ_1, and β where $F(w)$ is the same as the one in Section 7.1.

1. Let the value of all the parameters be the same as those in Section 7.1 except s. Then t^* is nonincreasing in s (Figure 7.3(I)).
2. Let β and λ_1 be the same as those in Section 7.1, and let $s = 0.58$ and $\rho = -1.0$. Then t^* is nonincreasing in λ_0 (Figure 7.3(II)).
3. Let β be the same as that in Section 7.1 and let $s = 0.17$, $\rho = -1.0$, and $\lambda_0 = 0.1$. Then t^* is not always monotone in λ_1 (Figure 7.3(III)). From the graph we observe that t^* is nondecreasing on $\lambda_1 \in [0.16, 0.4242]$, nonincreasing on $\lambda_1 \in [0.4242, 0.8697]$, and nondecreasing on $\lambda_1 \in [0.8697, 1.0]$.

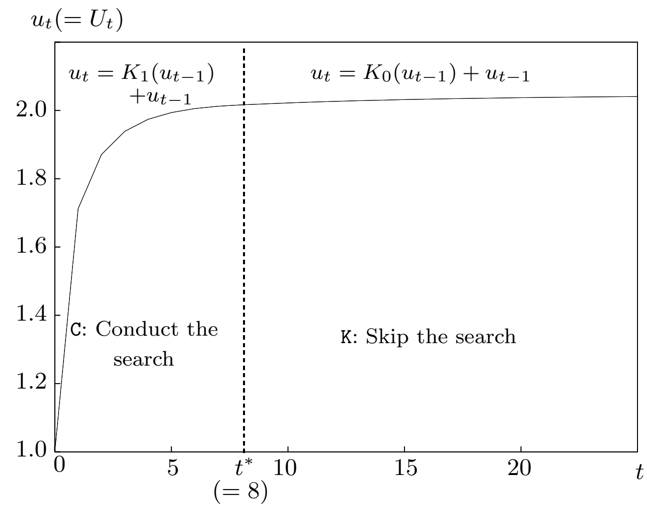


Figure 7.2: Time threshold t^* for conducting search.

4. Let λ_0 and λ_1 be the same as those in Section 7.1, and let $s = 0.58$ and $\rho = 0.01$. Then t^* is not always monotone in β (Figure 7.3(IV)). From the graph we observe that t^* is nondecreasing on $\beta \in [0.5, 0.7404]$ and nonincreasing on $\beta \in [0.7404, 1.0]$.

Through the numerical experiments, we notice that the t^* is nonincreasing in s and λ_0. These results are not surprising. In order to maximize profit, a searcher will try his best to save on the search cost as much as possible. Thus, with all else being the same, a higher search cost will decrease the searcher's incentive to search, thereby leading to a shorter searching period. Besides, a higher offer appearing probability λ_0 when searching is skipped will also decrease the searcher's incentive to search because a higher λ_0 reduces the need to search, thereby resulting in a shorter search period. Although we are able to interpret the implication of the above numerical results for the monotonicities in s and λ_0, we cannot provide a convincing explanation for the non-monotonicities of t^* in λ_1 and β. This type of non-monotonic property sometimes appears in the optimal stopping problem [3][6].

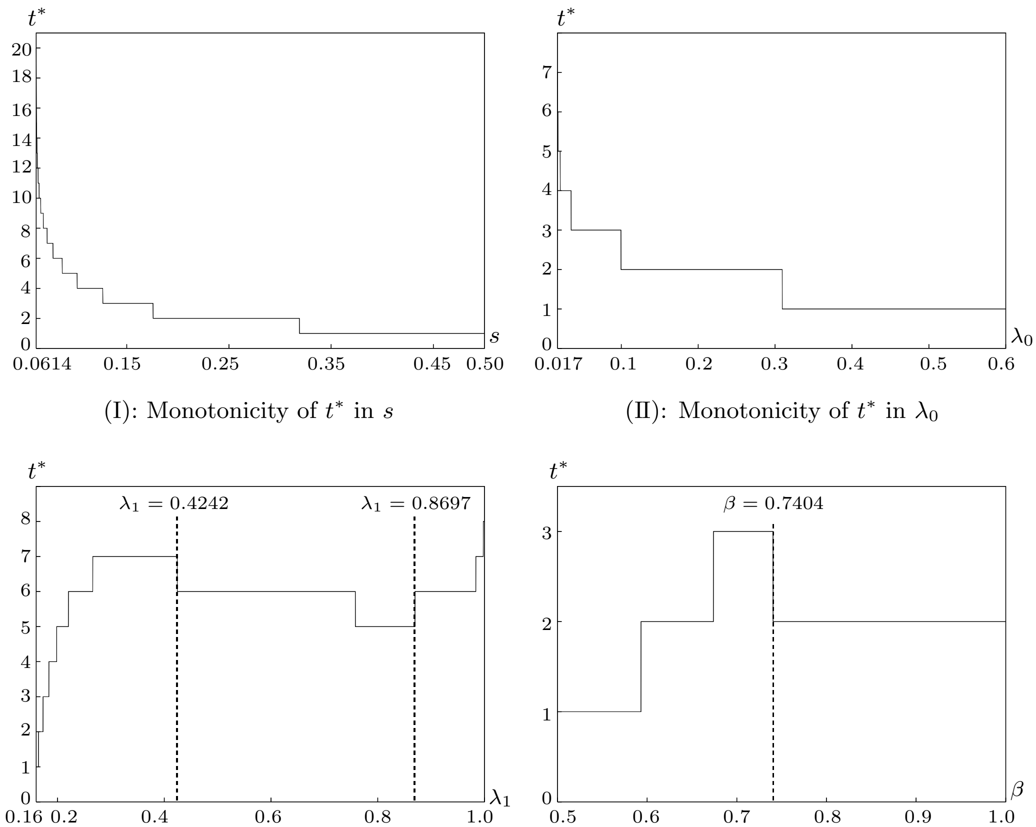
8 Conclusions and Suggested Future Studies

In this paper we have proposed a model of the optimal stopping problem where quitting offer is available at every point in time throughout the planning horizon and where a search cost is incurred to find a random offer. Below, we shall summarize some distinctive results derived from our analysis.

C1. Let $\rho \geq \max\{x_{K_0}, x_{K_1}\}$. Then it is optimal to quit the process by accepting either the intervening quitting offer ρ or the random offer w at the start of the process (Theorem 5.1(b)). In other words, the process starts and ends at the same time.

C2. Let $\rho < \max\{x_{K_0}, x_{K_1}\}$. Then:

1. It is not optimal to accept the intervening quitting offer at any point in time prior to the deadline; however, it may be accepted at the deadline. In other words, in this case the process reduces to one with only a terminal quitting offer (Theorem 5.1(c1)).
2. We obtained the conditions in which conducting the search is optimal for $t \geq 1$ (Theorem 5.2(a, b3i1, b3ii1)) and those on which a searching time threshold t^* exists (Theorem 5.2(b3i2, b3ii2)).



(I): Monotonicity of t^* in s

(II): Monotonicity of t^* in λ_0

(III): Monotonicity of t^* in λ_1

(IV): Monotonicity of t^* in β

Figure 7.3: Monotonicity of t^* in the model's parameters.

3. Let $s > 0$, $\rho < x_{_L}$, and $\beta = 1$. Then if $\lambda_0 = 0$, it is always optimal to conduct the search for $t \geq 1$ (Theorem 5.2(b3i1)). However, if $\lambda_0 > 0$, a searching time threshold t^* exists.
4. It was verified that the searching time threshold t^* is nonincreasing in ρ. In addition, from the numerical examples, we observe that t^* is nonincreasing in s and λ_0 while it is not always monotone in λ_1 and β.

Now, we conclude with a discussion of some directions in which our model could be extended to make it more practical. Suppose the optimal stopping problem is restricted to an asset selling problem. Then our model can be extended to deal with the sale of multiple homogeneous assets. In addition, an extension where ρ is t-dependent provides a useful generalization of our model. For example, ρ may be nondecreasing, nonincreasing, or may change in a cyclical fashion in time periods remaining up to the deadline. Moreover, another possible extension would be to consider the future availability of the offer once rejected; Representative articles include [3], [6], [7], [8], [10], and [15]. Finally, a model where a limited amount of budget [4] allocated to search for an offer at each point in time is also worth discussing.

Appendix : Proofs

A. Lemma 3.1

(a) Immediate from the fact that $\max\{w-x,0\}$ is continuous and nonincreasing in $x \in (-\infty,\infty)$ for any given w.

(b) Since $T(x) = \mathbf{E}[(w-x)I(w>x)] \geq \mathbf{E}[(w-x)I(w>y)]$‡ for any x and y, we have $T(x)-T(y) \geq \mathbf{E}[(w-x)I(w>y)] - \mathbf{E}[(w-y)I(w>y)] = -(x-y)\mathbf{E}[I(w>y)] = -(x-y)\mathbf{E}[1-I(w \leq y)] = -(x-y)(1-F(y))$. Similarly we get $T(x)-T(y) \leq -(x-y)(1-F(x))$. Hence

$$-(x-y)(1-F(y)) \leq T(x)-T(y) \leq -(x-y)(1-F(x)). \tag{A.1}$$

Let $y < x < b$, hence $F(x) < 1$ due to Eq. (2.1). Then $-(x-y)(1-F(x)) < 0$, so $T(x) < T(y)$ from Eq. (A.1), i.e., $T(x)$ is strictly decreasing on $(-\infty,b)$. Now, assume that $T(b) = T(x)$ for a certain $x < b$. Then $T(x') < T(x) = T(b)$ for $x < x' < b$ or equivalently $T(x') < T(b)$ for $x' < b$, which contradicts (a), thus it must be that $T(b) < T(x)$, implying that $T(x)$ is strictly decreasing on $(-\infty,b]$.

(c) Let $b \leq x$. If $w \leq b$, then $w \leq x$, hence $\max\{w-x,0\} = 0$, and if $b < w$, then $f(w) = 0$ due to Eq. (2.2). Accordingly, $T(x) = \mathbf{E}[\max\{w-x,0\}I(w \leq b)] + \mathbf{E}[\max\{w-x,0\}I(b<w)] = 0$, hence the former half is true. If $x < b$, then $T(x) > T(b) = 0$ from (b), hence the latter half is true. ∎

B. Lemma 3.2

Let $x \leq a$. If $a \leq w$, then $x \leq w$, hence $\max\{w-x,0\} = w-x$, and if $w < a$, then $f(w) = 0$ from Eq. (2.2). Thus

$$\begin{aligned} T(x) &= \mathbf{E}[\max\{w-x,0\}I(a \leq w)] + \mathbf{E}[\max\{w-x,0\}I(w<a)] \\ &= \mathbf{E}[(w-x)I(a \leq w)] + 0 \\ &= \mathbf{E}[(w-x)I(a \leq w)] + \mathbf{E}[(w-x)I(w<a)] = \mathbf{E}[w-x] = \mu - x. \end{aligned}$$

Therefore, $\lim_{x\to-\infty} T(x) = \infty$. From this result and the fact that $-(1-\beta)x$ is nonincreasing on $(-\infty,\infty)$, we immediately see that $\lim_{x\to-\infty} K_0(x) = \infty \cdots (1^*)$ if $\lambda_0 > 0$ or $\beta < 1$, $\lim_{x\to-\infty} K_1(x) = \infty \cdots (2^*)$, and $\lim_{x\to-\infty} L(x) = \infty \cdots (3^*)$.

(a) Evident from Eqs. (3.3), (3.2), Lemma 3.1(a), and the fact that $-(1-\beta)x$ is strictly decreasing on $(-\infty,\infty)$ if $\beta < 1$.

(b) From Eq. (A.1) we get

$$(x-y)F(y) \leq T(x)+x-T(y)-y \leq (x-y)F(x). \tag{B.1}$$

Let $y < x$. Then $(x-y)F(y) \geq 0$, thus $T(y)+y \leq T(x)+x$ from Eq. (B.1), hence $T(x)+x$ is nondecreasing on $(-\infty,\infty)$. From this result and the fact that $K_1(x)+x = \beta(\lambda_1 T(x)+x)-s = \beta(\lambda_1(T(x)+x)+(1-\lambda_1)x)-s$, it follows that $K_1(x)+x$ is nondecreasing on $(-\infty,\infty)$ since $(1-\lambda_1)x$ is nondecreasing on $(-\infty,\infty)$. Similarly we can show that $K_0(x)+x$ is nondecreasing on $(-\infty,\infty)$.

(c) From Eqs. (A.1) and (3.3), for any x and y we immediately get

$$-(x-y)\big(1-\beta(1-\lambda_1(1-F(y)))\big) \leq K_1(x)-K_1(y) \leq -(x-y)\big(1-\beta(1-\lambda_1(1-F(x)))\big),$$

from which

$$\beta(x-y)(1-\lambda_1(1-F(y))) \leq K_1(x)+x-K_1(y)-y \leq \beta(x-y)(1-\lambda_1(1-F(x))).$$

‡If a statement S is true, then $I(S) = 1$, or else $I(S) = 0$.

Since $\lambda_1 \le 1$ by assumption and since $0 \le 1-F(y) \le 1$ and $0 \le 1-F(x) \le 1$, the assertion for $K_1(x)$ clearly holds. Similarly we can obtain $|K_0(x)+x-K_0(y)-y| \le \beta|x-y|$ for any x and y.

(d) Let $(1-\beta)^2+s^2=0$, so $\beta=1$ and $s=0$. Then $K_1(x)=\lambda_1 T(x)$. Noting the assumption of $\lambda_1>0$, we see that $K_1(x)=0$ for $x \ge b$ and $K_1(x)>0$ for $x<b$ from Lemma 3.1(c), so that $x_{K_1}=b$ by the definition of x_{K_1}, hence $x<(\ge)\ x_{K_1} \Rightarrow K_1(x)>(=)$ 0. The inverse is true by contraposition.

(e) Let $(1-\beta)^2+s^2 \ne 0$. First, let $\beta<1$. Then $\lim_{x\to\infty} K_1(x) = -\infty \cdots (4^*)$ since $\lim_{x\to\infty} T(x)=0$ from Lemma 3.1(c) and since $-(1-\beta)x$ diverges to $-\infty$ as $x\to\infty$. Accordingly, there uniquely exists x_{K_1} from (a), (2^*), and (4^*). Since $K_1(b)=-(1-\beta)b-s<0$ due to Lemma 3.1(c), we have $x_{K_1}<b$ from (a). Next, let $\beta=1$, hence $s>0$. Then $K_1(x)=\lambda_1 T(x)-s$ for any $x\in(-\infty,\infty)$ from Eq. (3.3), which is nonincreasing on $(-\infty,\infty)$ due to Lemma 3.1(a) and strictly decreasing on $(-\infty,b]$ due to Lemma 3.1(b). In addition, we have $K_1(b)=\lambda_1 T(b)-s=-s<0$ due to Lemma 3.1(c) and the assumption of $\lambda_1>0$. Consequently, from (2^*) it follows that x_{K_1} uniquely exists with $x_{K_1}<b$. Thus, whether $\beta<1$ or $\beta=1$, there uniquely exists $x_{K_1}<b$; accordingly, the former half of the assertion holds. Since $K_1(x)$ is strictly decreasing on the neighborhood of $x=x_{K_1}<b$ due to Lemma 3.1(b) and the fact that $-(1-\beta)x$ is nonincreasing on $(-\infty,\infty)$, it follows that x $<(=(>))\ x_{K_1} \Rightarrow K_1(x)>(=(<))\ 0$. The inverse is true by contraposition.

(f) Let $\lambda_0>0$.

(f1) Let $\beta=1$. Then $K_0(x)=\lambda_0 T(x)=0$ for $x\ge b$ and $K_0(x)=\lambda_0 T(x)>0$ for $x<b$ from Eq. (3.2) and Lemma 3.1(c), hence $x_{K_0}=b$ by the definition of x_{K_0}, so $x<(\ge)$ $x_{K_0} \Rightarrow K_0(x)>(=)\ 0$. The inverse is true by contraposition.

(f2) Let $\beta<1$. Then $\lim_{x\to\infty} K_0(x)=-\infty \cdots (5^*)$ due to the fact that $\lim_{x\to\infty} T(x)=0$ from Lemma 3.1(c) and that $-(1-\beta)x$ diverges to $-\infty$ as $x\to\infty$. Therefore, x_{K_0} uniquely exists from (a), (1^*), and (5^*). Since $\beta>0$, $\mu>0$, and $b>0$ by assumptions, we have $K_0(0)=\lambda_0\beta T(0)=\lambda_0\beta\mu>0$ and $K_0(b)=\lambda_0\beta T(b)-(1-\beta)b=-(1-\beta)b<0$ due to Lemma 3.1(c), hence $0<x_{K_0}<b$ from (a). In addition, we have $x<(=(>))\ x_{K_0} \Rightarrow K_0(x)$ $>(=(<))\ 0$ from the definition of x_{K_0} and (a). The inverse is true by contraposition.

(g) First, note that $L(x)$ is continuous, nonincreasing on $(-\infty,\infty)$, and strictly decreasing on $(-\infty,b]$ due to Eq. (3.4) and Lemma 3.1(a,b). Let $s>0$. Then, since $L(b)=\lambda\beta T(b)-s=-s<0$ from Lemma 3.1(c) and since $L(x)>0$ for a certain sufficiently small $x<0$ from (3^*), it follows from the monotonicity of $L(x)$ that x_L uniquely exists. The inequality $x_L<b$ is immediate from the monotonicity of $L(x)$ and the inequality $L(b)<0$. The latter half is evident from the fact that $L(x)$ is strictly decreasing in the neighborhood of $x=x_L$ $(<b)$. ∎

C. Lemma 3.3

Let $s>0$, hence $(1-\beta)^2+s^2\ne 0$. Then, since $K_0(x_{K_0})=0$ and $K_1(x_{K_1})=0$ by the definitions of x_{K_0} and x_{K_1}, noting Eq. (3.5), we have

$$L(x_{K_0}) = K_1(x_{K_0}), \tag{C.1}$$

$$L(x_{K_1}) = -K_0(x_{K_1}). \tag{C.2}$$

(a) Let $\lambda_0=0$, hence $x_{K_0}=0$ by definition (Eq. (3.7)). Then $K_0(x_{K_1})=-(1-\beta)x_{K_1} \cdots (1^*)$ from Eq. (3.2).

(a1) Let $\beta=1$. Then $K_0(x_{K_1})=0$ from (1^*), hence $L(x_{K_1})=0$ from Eq. (C.2), so $x_L=x_{K_1}$ from Lemma 3.2(g). Thus the assertion holds.

(a2) Let $\beta<1$. If $x_{K_0}<(=(>))\ x_{K_1}$ or equivalently $0<(=(>))\ x_{K_1}$, then $K_0(x_{K_1})$ $<(=(>))\ 0$ from (1^*), hence $L(x_{K_1})>(=(<))\ 0$ from Eq. (C.2), so $x_{K_1}<(=(>))\ x_L$ due to Lemma 3.2(g). Thus $x_{K_0} < (= (>))\ x_{K_1} \Rightarrow x_{K_1} < (= (>))\ x_L$. The inverse is true by contraposition.

(b) Let $\lambda_0 > 0$.

(b1) Let $\beta = 1$. Then $x_{K_0} = b$ from Lemma 3.2(f1). Since $L(x_{K_0}) = L(b) = \lambda\beta T(b) - s = -s < 0$ from Lemma 3.1(c) and the assumption of $s > 0$, it follows that $K_1(x_{K_0}) < 0$ due to Eq. (C.1). Thus we get $x_{K_0} > x_{K_1}$ from Lemma 3.2(e). Hence $K_0(x_{K_1}) > 0$ from Lemma 3.2(f1), thus $L(x_{K_1}) < 0$ from Eq. (C.2), so $x_{K_1} > x_L$ from Lemma 3.2(g).

(b2) Let $\beta < 1$. Then, if $x_{K_0} < (= (>))\ x_{K_1}$, we have $K_0(x_{K_1}) < (= (>))\ 0$ from Lemma 3.2(f2), hence $L(x_{K_1}) > (= (<))\ 0$ from Eq. (C.2), so $x_{K_1} < (= (>))\ x_L$ from Lemma 3.2(g). Thus $x_{K_0} < (= (>))\ x_{K_1} \Rightarrow x_{K_1} < (= (>))\ x_L$. The inverse is true by contraposition. ∎

References

[1] D. Assaf and B. Levikson, *Optimal advertising policy for selling a single asset*, Probability in the Engineering and Informational Sciences, **5** (1991), 89-100.

[2] Y. S. Chow, H. Robbins, and D. Siegmund, *Great expectations: The theory of optimal stopping*, Houghton Mifflin Company, Boston (1971).

[3] S. Ikuta, *Optimal stopping problem with uncertain recall*, Journal of the Operations Research Society of Japan, **31** (1988), 145-170.

[4] S. Ikuta, *The optimal stopping problem in which the sum of the accepted offer's value and the remaining search budget is an objective function*, Journal of the Operations Research Society of Japan, **35** (1992), 172-193.

[5] S. Ikuta, *Optimal stopping problem with several search areas*, Journal of the Operations Research Society of Japan, **38** (1995), 89-106.

[6] B. K. Kang, *Optimal stopping problem with recall cost*, European Journal of Operational Research, **117** (1999), 222-238.

[7] B. K. Kang, *Optimal stopping problem with double reservation value property*, European Journal of Operational Research, **165** (2005), 765-785.

[8] E. Karni and A. Schwartz, *Search theory: the case of search with uncertain recall*, Journal of Economic Theory, **16** (1977), 38-52.

[9] M. Khouja, *The single-period (new-vendor) problem: literature review and suggestions for future research*, Omega, **27** (1999), 537-553.

[10] M. Landsberger and D. Peled, *Duration of offers, price structure, and the gain from search*, Journal of Economic Theory, **16** (1977), 17-37.

[11] S. A. Lippman and J. J. McCall, *The economics of job search: a survey*, Economic Inquiry, **14** (1976), 155-189.

[12] J. J. McCall, *The economics of information and optimal stopping rules*, Journal of Business, **38** (1965), 300-317.

[13] N. C. Petruzzi and M. Dada, *Pricing and the newsvendor problem: a review with extensions*, Operations Research, **47** (1999), 183-194.

[14] N. C. Petruzzi and G. E. Monahan, *Managing fashion goods inventories: dynamic recourse for retailers with outlet stores*, IIE Transactions, **35** (2003), 1033-1047.

[15] T. Saito, *Optimal stopping problem with controlled recall*, Probability in the Engineering and Informational Sciences, **12** (1998), 91-108.

[16] M. Sakaguchi, *Dynamic programming of some sequential sampling design*, Journal of Mathematical Analysis and Applications, **2** (1961), 446-466.

[17] M. Sakaguchi, *Optimal stopping problems for randomly arriving offers*, Mathematicae Japonicae, **21** (1976), 201-217.

[18] J. D. Son, *Customer selection problem with profit from a sideline*, European Journal of Operational Research, to appear.

[19] G.J. Stigler, *The economics of information*, The Journal of Political Economy, **65** (1961), 213-225.

[20] P. S. You, *Dynamic pricing policy in a selling problem*, Journal of Japan Society of Business Mathematics, (1997).

Mong Shan Ee: Doctoral Program in Policy and Planning Sciences, Graduate School of Systems and Information Engineering, University of Tsukuba, Tennodai 1-1-1,Tsukuba, 305-8573 Japan.
E-mail address: shan@sk.tsukuba.ac.jp

Seizo Ikuta: Retired from the University of Tsukuba in April 2005. Address: Fujizaki 1-1-13, 702 Tsuchiura, Ibaraki, Japan.
E-mail address: sikuta0716@yahoo.co.jp