C^*-ALGEBRAS ARISING FROM TWO HOMEOMORPHISMS WITH A CERTAIN RELATION

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Abstract. We introduce a notion of an associative trinary relation arising from two homeomorphisms that satisfy a certain relation and construct an associated C^*-algebra. We prove a sufficient condition for the C^*-algebra to have a maximal closed two-sided ideal. As an example, we study a trinary relation arising from two homeomorphisms of the ring of p-adic integers and show that the associated C^*-algebra has a maximal closed two-sided ideal.

1. Introduction

Pentagonal equations for operators play important roles in several aspects in operator algebras. For example, see [1], [3] and [11]. The author have studied pentagonal equations on Hilbert C^*-modules in [5], [6], [7] and [8]. In these studies, we are led to consider a tri-module instead of a bi-module, that is, we considered a module with three different actions of an algebra. One of our main examples comes from a groupoid. But a natural module arising from a groupoid is a bimodule because we have only the two natural maps, that is, the source map and the range map. This fact is based on that a groupoid is related to a binary relation. To consider a more general situation, it is desirable to have a notion of a trinary relation with three natural map. In this paper, we introduce a notion of a trinary relation arising from two homeomorphisms that satisfy a certain relation and study its associated C^*-algebra. As an example, we study a trinary relation arising from two homeomorphisms of the ring of p-adic integers.

In Section 2, we introduce a notion of an associative trinary relation arising from two homeomorphisms that satisfy a certain relation and construct its associated C^*-algebra A. In Section 3, we study basic properties of A. In Section 4, we study a certain property of a commutative subalgebra of A. In Section 5, we prove a sufficient condition for A to have a maximal closed two-sided ideal. In Section 6, we study a trinary relation arising from two homeomorphisms of the ring of p-adic integers and show that the associated C^*-algebra has a maximal closed two-sided ideal.

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2. Algebras associated with a trinary relation

Let X be a second countable compact Hausdorff space and let α and β be homeomorphisms of X onto itself. We suppose that there exists $\sigma \in \mathbb{Z} \setminus \{0, \pm 1\}$ such that $\beta\alpha = \alpha^\sigma \beta$.

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Set $\sigma(n) = \sigma^n$ for $n \in \mathbb{Z}$. Then we have $\beta^n \alpha^m = \alpha^{m\sigma(n)} \beta^n$ for $(n, m) \in \mathbb{N} \times \mathbb{Z}$, where $\mathbb{N} = \{0, 1, 2, \cdots\}$. Set $\mathcal{T} = \mathbb{N} \times \mathbb{Z} \times X$. Define maps q, r, $s : \mathcal{T} \longrightarrow X$ by $q(n, m, x) = \alpha^m(x)$, $r(n, m, x) = \beta^n(x)$ and $s(n, m, x) = \beta^n \alpha^m(x)$ respectively. We denote by $\mathcal{T} *_q \mathcal{T}$ the fibered product $\{(u, v) \in \mathcal{T}^2;\ s(u) = q(v)\}$. Define the fibered product $\mathcal{T} *_r \mathcal{T}$ similarly. Then we define a map $W : \mathcal{T} *_q \mathcal{T} \longrightarrow \mathcal{T} *_r \mathcal{T}$ by

$$\begin{aligned} &W((n_1, m_1, x(-n_1, m_2 - m_1\sigma(n_1))), (n_2, m_2, x)) \\ &\quad = ((n_2, m_2 - m_1\sigma(n_1), x), (n_1 + n_2, m_1, x(-n_1, m_2 - m_1\sigma(n_1)))), \end{aligned}$$

where $x(n, m) = \beta^n \alpha^m(x)$ for $x \in X$ and $(n, m) \in \mathbb{Z} \times \mathbb{Z}$. Note that W is continuous and injective with its inverse W^{-1};

$$\begin{aligned} &W^{-1}((n_1, m_1, x), (n_2, m_2, x(n_1 - n_2, m_1)) \\ &\quad = ((n_2 - n_1, m_2, x(n_1 - n_2, m_1)), (n_1, m_1 + m_2\sigma(n_2 - n_1), x)) \end{aligned}$$

but not surjective. If $W(u, v) = (u', v')$, then we have $q(u) = q(v')$, $r(u) = q(u')$, $r(v) = r(u')$ and $s(v) = s(v')$. We denote by $\mathcal{T} *_q \mathcal{T} *_q \mathcal{T}$ the fibered product $\{(u, v, w) \in \mathcal{T}^3;\ s(u) = q(v), s(v) = q(w)\}$. Define the fibered products $\mathcal{T} *_r \mathcal{T} *_q \mathcal{T}$, $\mathcal{T} *_q \mathcal{T} *_r \mathcal{T}$ and $\mathcal{T} *_r \mathcal{T} *_r \mathcal{T}$ similarly. We also denote by $(\mathcal{T} \times \mathcal{T}) * \mathcal{T}$ the fibered product $\{(u, v, w) \in \mathcal{T}^2;\ s(u) = q(w), s(v) = r(w)\}$. Then we can define a map $W *_q I : \mathcal{T} *_q \mathcal{T} *_q \mathcal{T} \longrightarrow \mathcal{T} *_r \mathcal{T} *_q \mathcal{T}$ by $(W *_q I)(u, v, w) = (W(u, v), w)$. Similarly we can define the following maps; $I *_r W : \mathcal{T} *_r \mathcal{T} *_q \mathcal{T} \longrightarrow \mathcal{T} *_q \mathcal{T} *_r \mathcal{T}$, $W *_r I : \mathcal{T} *_q \mathcal{T} *_r \mathcal{T} \longrightarrow \mathcal{T} *_r \mathcal{T} *_r \mathcal{T}$ and $I *_q W : \mathcal{T} *_q \mathcal{T} *_q \mathcal{T} \longrightarrow (\mathcal{T} \times \mathcal{T}) * \mathcal{T}$. We can also define a map $W_{(13)} : (\mathcal{T} \times \mathcal{T}) * \mathcal{T} \longrightarrow \mathcal{T} *_r \mathcal{T} *_r \mathcal{T}$ by $W_{(13)}(u, v, w) = (v, W(u, w))$. Then we have the following proposition. The proof is straightforward and we omit it.

Proposition 2.1. *The map W satisfies the following pentagonal equation;*

$$(W *_r I)(I *_r W)(W *_q I) = W_{(13)}(I *_q W).$$

Since the pentagonal equation means an associativity of an operation represented by W, we will call the pair $(\mathcal{T}, W)$ an associative trinary relation.

For $x \in X$, set $\mathcal{T}_x = s^{-1}(x)$. We denote by λ_x the counting measure on $\mathcal{T}_x$. Let $\mathcal{S}$ be the image of W and $\mathcal{S}(v)$ the set $\{u \in \mathcal{T}; (u, v) \in \mathcal{S}\}$ for $v \in \mathcal{T}$. We denote by $C_c(\mathcal{T})$ the set of complex valued continuous functions on $\mathcal{T}$ with compact supports. For ξ, $\eta \in C_c(\mathcal{T})$, define a product $\xi * \eta$ in $C_c(\mathcal{T})$ by

$$(\xi * \eta)(v) = \int_{\mathcal{S}(v)} (\xi \otimes \eta)(W^{-1}(u, v)) d\lambda_{r(v)}(u).$$

Theorem 2.2. *The linear space $C_c(\mathcal{T})$ is an associative algebra over $\mathbb{C}$ with respect to the product $\xi * \eta$ defined above.*

Proof. It is enough to prove the associativity of the product. Set $W^{-1}(u, v) = (\Psi'(u, v), \Psi(u, v))$ for $(u, v) \in \mathcal{S}$. For $\xi, \eta, \zeta \in C_c(\mathcal{T})$, we have

$$\begin{aligned} &((\xi * \eta) * \zeta)(v) \\ &= \iint_{A(v)} (\xi \otimes \eta \otimes \zeta)((W *_q I)^{-1}(I *_r W)^{-1}(w, u, v)) d\lambda_{q(u)}(w) d\lambda_{r(v)}(u), \end{aligned}$$

where $A(v)$ is the set $\{(w, u) \in \mathcal{T} *_q \mathcal{T}; w \in \mathcal{S}(\Psi'(u, v)), u \in \mathcal{S}(v)\}$. On the other hand, we have

$$\begin{aligned} &(\xi * (\eta * \zeta))(v) \\ &= \iint_{B(v)} (\xi \otimes \eta \otimes \zeta)((I *_q W)^{-1} W_{(13)}^{-1}(w, u, v)) d\lambda_{r(u)}(w) d\lambda_{r(v)}(u), \end{aligned}$$

where $B(v)$ is the set $\{(w,u) \in \mathcal{T} *_r \mathcal{T}; w \in \mathcal{S}(\Psi(u,v)), u \in \mathcal{S}(v)\}$. The set $A(v)$ coincides with the set $\{(w,u) \in \mathcal{T} *_q \mathcal{T}; (w,u,v) \in \mathrm{Im}(I *_r W)(W *_q I)\}$, and the set $B(v)$ coincides with the set $\{(w,u) \in \mathcal{T} *_r \mathcal{T}; (w,u,v) \in \mathrm{Im} W_{(13)}(I *_q W)\}$. Since W satisfies the pentagonal equation, we have $W(A(v)) = B(v)$. Then we have

$$\begin{aligned}
&\iint_{B(v)} f(W^{-1}(w,u))\, d\lambda_{r(u)}(w) d\lambda_{r(v)}(u) \\
&= \iint_{\mathcal{S}} f(W^{-1}(w,u)) \chi_{A(v)}(W^{-1}(w,u))\, d\lambda_{r(u)}(w) d\lambda_{r(v)}(u) \\
&= \iint_{\mathcal{T} *_q \mathcal{T}} f(w,u) \chi_{A(v)}(w,u)\, d\lambda_{q(u)}(w) d\lambda_{r(v)}(u) \\
&= \iint_{A(v)} f(w,u)\, d\lambda_{q(u)}(w) d\lambda_{r(v)}(u)
\end{aligned}$$

for every $f \in C_c(\mathcal{T} *_q \mathcal{T})$, where $\chi_{A(v)}$ is the characteristic function of $A(v)$. It follows from the pentagonal equation that we have $((\xi * \eta) * \zeta)(v) = (\xi * (\eta * \zeta))(v)$. □

We denote by $\tilde{\mathcal{A}}$ the opposite algebra of $C_c(\mathcal{T})$, that is, the product $\xi\eta$ in $\tilde{\mathcal{A}}$ is defined by $\xi\eta = \eta * \xi$. Then we have

$$(\xi\eta)(n,m,x) = \sum_{0 \le j \le n} \sum_{k \in \mathbb{Z}} \xi(j, k + m\sigma(n-j), \alpha^{-k}\beta^{n-j}(x)) \eta(n-j, m, x).$$

We denote by $\lambda_{\mathbb{N}}$ and $\lambda_{\mathbb{Z}}$ the counting measures on $\mathbb{N}$ and $\mathbb{Z}$ respectively. Let μ be a positive regular Radon measure on X whose support is X. Moreover we suppose that $\mu(X) = 1$. We define a measure λ on $\mathcal{T}$ by $\lambda = \lambda_{\mathbb{N}} \times \lambda_{\mathbb{Z}} \times \mu$. We denote by H the Hilbert space $L^2(\mathcal{T}, \lambda)$. For $\xi \in \tilde{\mathcal{A}}$, define a linear map $\tilde{\pi}(\xi) : C_c(\mathcal{T}) \longrightarrow C_c(\mathcal{T})$ by $\tilde{\pi}(\xi)\eta = \eta * \xi$.

Lemma 2.3. *There exists a positive number M such that $\|\tilde{\pi}(\xi)\eta\|_H \le M\|\eta\|_H$ for every $\eta \in C_c(\mathcal{T})$.*

Proof. Let K_1 and K_2 be finite sets of $\mathbb{N}$ and $\mathbb{Z}$ respectively such that the support of ξ is contained in $K_1 \times K_2 \times X$. Then we have

$$|(\xi\eta)(n,m,x)| \le \|\xi\|_\infty \#(K_1)^{1/2} \#(K_2) \left(\sum_{0 \le j \le n} \chi_{K_1}(j) |\eta(n-j,m,x)|^2 \right)^{1/2}.$$

This implies that $\|\tilde{\pi}(\xi)\eta\|_H \le \#(K_1)\#(K_2)\|\xi\|_\infty \|\eta\|_H$. □

It follows from the above lemma that one can extend $\tilde{\pi}(\xi)$ to a bounded operator on H, which we denote again by $\tilde{\pi}(\xi)$. Then the map $\tilde{\pi} : \tilde{\mathcal{A}} \longrightarrow \mathcal{B}(H)$ is a homomorphism of algebras. We denote by $\mathcal{A}$ the quotient algebra $\tilde{\mathcal{A}}/\mathrm{Ker}\,\tilde{\pi}$ and denote by π the representation of $\mathcal{A}$ on H corresponding to $\tilde{\pi}$. Define a unitary operator $U \in \mathcal{B}(H)$ by $(U\xi)(n,m,x) = \xi(n, m+1, x)$. We denote by A the C^*-subalebra of $\mathcal{B}(H)$ generated by $\pi(\mathcal{A})U^m$ $(m \in \mathbb{Z})$. Let $\tilde{\mathcal{A}}_0$ be the set of $\xi \in \tilde{\mathcal{A}}$ such that the support of ξ is contained in $\{0\} \times \mathbb{Z} \times X$. For $\xi \in \tilde{\mathcal{A}}_0$, the operator $\tilde{\pi}(\xi)$ satisfies

$$\begin{aligned}
(\tilde{\pi}(\xi)\eta)(n,m,x) &= \sum_{k \in \mathbb{Z}} \xi(0, k + m\sigma(n), \alpha^{-k}\beta^n(x)\eta(n,m,x) \\
&= \sum_{k \in \mathbb{Z}} \xi(0, k, \alpha^{-k}\beta^n\alpha^m(x))\eta(n,m,x).
\end{aligned}$$

We denote by A_0 the C^*-subalgebra of A generated by $\tilde{\pi}(\tilde{\mathcal{A}}_0)$.

Lemma 2.4. *The C^*-algebra A_0 is $*$-isomorphic to the C^*-algebra $C(X)$ of continuous functions on X.*

Proof. Note that Ker $\tilde{\pi}$ is the set of $\xi \in \tilde{\mathcal{A}}$ such that $\sum_{m\in\mathbb{Z}} \xi(n,m,\alpha^{-m}(x)) = 0$ for every $n \in \mathbb{N}$ and $x \in X$. We define a map $\tilde{h} : \tilde{\mathcal{A}}_0 \longrightarrow C(X)$ by $\tilde{h}(\xi)(x) = \sum_{m\in\mathbb{Z}} \xi(0,m,\alpha^{-m}(x))$. Let $\mathcal{A}_0$ be the subalgebra of $\mathcal{A}$ corresponding to $\tilde{\mathcal{A}}_0$. Then there exists an isomorphism h of $\mathcal{A}_0$ onto $C(X)$ such that $h([\xi]) = \tilde{h}(\xi)$, where $[\xi] \in \mathcal{A}$ is the class of $\xi \in \tilde{\mathcal{A}}$. Since we have $\|\pi(h^{-1}(f))\| = \|f\|$ and $\pi(h^{-1}(f))^* = \pi(h^{-1}(\bar{f}))$, there exists a $*$-isomorphism $\hat{h}$ of A_0 onto $C(X)$ such that $\hat{h}(\pi(a)) = h(a)$ for every $a \in \mathcal{A}_0$. □

3. Basic properties of the associated algebras

Define $V \in \mathcal{B}(H)$ by $(V\xi)(n,m,x) = \xi(n+1,m,x)$ for $\xi \in H$ and $(n,m,x) \in \mathcal{T}$. Note that V^* is an isometry, that is, $VV^* = I$. Let α_* and β_* be automorphisms of $C(X)$ defined by $\alpha_*(f)(x) = f(\alpha^{-1}(x))$ and $\beta_*(f)(x) = f(\beta^{-1}(x))$ respectively for $f \in C(X)$ and $x \in X$. Let δ_n and δ_m be Kronecker's δ on $\mathbb{N}$ and $\mathbb{Z}$ respectively. We have

$$(\tilde{\pi}(\delta_n \times \delta_m \times f)\eta)(n',m',x) = f(\alpha^{-m}\beta^{n'-n}\alpha^{m'}(x))\eta(n'-n,m',x)$$

if $n' \geq n$ and $(\tilde{\pi}(\delta_n \times \delta_m \times f)\eta)(n',m',x) = 0$ if $0 \leq n' \leq n-1$, and we have $\tilde{h}(\delta_0 \times \delta_m \times f) = \alpha_*^m(f)$. Therefore we have

$$\begin{aligned}\pi([\delta_n \times \delta_m \times f]) &= (V^*)^n \pi([\delta_0 \times \delta_m \times f]) \\ &= (V^*)^n \pi(h^{-1}(\alpha_*^m(f))).\end{aligned}$$

We also have

$$\pi(h^{-1}(f))V = V\pi(h^{-1}(\beta_*(f))).$$

For $k \in \mathbb{Z}$, define V_k by $V_k = V^k$ if $k \geq 0$ and by $V_k = (V^*)^{-k}$ if $k < 0$ and define a projection e_k of $\mathcal{B}(H)$ by $e_k = I$ if $k \geq 0$ and $e_k = (V^*)^{-k}V^{-k}$ if $k < 0$. Then we have $V_j e_k = e_{j+k} V_j$ and $V_j V_k = V_{j+k} e_{-k}$ for $j,\, k \in \mathbb{Z}$. For $k \in \mathbb{Z}$, define an injective $*$-homomorphism π_k of $C(X)$ to A by

$$(\pi_k(f)\xi)(n,m,x) = f(\beta^n \alpha^{m-k}(x))\xi(n,m,x)$$

for $f \in C(X)$, $\xi \in H$ and $(n,m,x) \in \mathcal{T}$. Then we have $\pi(h^{-1}(f)) = \pi_0(f)$ and $U^{-k}\pi_j(f)U^k = \pi_{j+k}(f)$. We denote by $\tilde{A}$ the dense $*$-subalgebra of A generated algebraically by $\pi(\mathcal{A})U^m$ ($m \in \mathbb{Z}$) and denote by B_0 the C^*-subalgebra of A generated by $\pi_k(C(X))$ ($k \in \mathbb{Z}$). Then every element of $\tilde{A}$ is a finite sum of elements of the form $e_k V_n U^m a$ with $k,n,m \in \mathbb{Z}$ and $a \in B_0$. We denote by B_I the commutative C^*-subalgebra of A generated by $B_0 e_n$ ($n \in \mathbb{Z}$). Then every element of $\tilde{A}$ is a finite sum of elements of the form $V_n U^m a$ with $n,m \in \mathbb{Z}$ and $a \in B_I$. For $(t,s) \in \mathbb{T}^2$, define a unitary operator $u(t,s)$ on H by $(u(t,s)\xi)(n,m,x) = \exp(-2\pi i(nt+ms))\xi(n,m,x)$ and define a homomorphism ϕ of $\mathbb{T}^2$ to $\mathrm{Aut}(A)$ by $\phi_{(t,s)} = \mathrm{Ad}\, u_{(t,s)}$. Then we have $\phi_{(t,s)}(V_n U^m a) = \exp(2\pi i(nt+ms))V_n U^m a$ for $n,m \in \mathbb{Z}$ and $a \in B_I$. Define a faithful conditional expectation E of A onto B_I by

$$E(x) = \int_{\mathbb{T}^2} \phi_{(t,s)}(x)\, dtds.$$

For $x = V_n U^m a$ with $a \in B_I$, we have $E(x) = 0$ if $(n,m) \neq (0,0)$ and $E(a) = a$.

4. A property of a commutative algebra

We denote by $C_b(\mathcal{T})$ the C^*-algebra of bounded continuous functions on $\mathcal{T}$. For $k \in \mathbb{Z}$, define an injective $*$-homomorohism $\kappa_k : C(X) \longrightarrow C_b(\mathcal{T})$ by

$$\kappa_k(f)(n, m, x) = f(\beta^n \alpha^{m-k}(x))$$

for $f \in C(X)$ and $(n, m, x) \in \mathcal{T}$. We denote by C the C^*-subalgebra of $C_b(\mathcal{T})$ generated by $\kappa_k(C(X))$ $(k \in \mathbb{Z})$. For $n \in \mathbb{N}$, define $\varepsilon_n \in C_b(\mathcal{T})$ by $\varepsilon_n(k, m, x) = 1$ if $k \geq n$ and $\varepsilon_n(k, m, x) = 0$ if $k < n$. We denote by C_I the C^*-subalgebra of $C_b(\mathcal{T})$ generated by $C\varepsilon_n$ $(n \in \mathbb{N})$. For $f \in C_I$, we have $f(n, m + 1, x) = f(n, m, \alpha(x))$. Define a $*$-isomorphism ρ_I of C_I onto B_I by $(\rho_I(f)\xi)(n, m, x) = f(n, m, x)\xi(n, m, x)$ for $f \in C_I$ and $\xi \in H$. Set $C_0 = \kappa_0(C(X))$. We have $\rho_I(C) = B_0$ and $\rho_I(C_0) = A_0$. Define an automorphism u of C_I by $u(f)(n, m, x) = f(n, m + 1, x)$. For $k \in \mathbb{N}$, define a $*$-endomorhphism v_k of C_I by $v_k(f)(n, m, x) = f(n+k, m, x)$ and define a $*$-endomorphism v_{-k} of C_I by $v_{-k}(f)(n, m, x) = f(n - k, m, x)$ if $n \geq k$ and $v_{-k}(f)(n, m, x) = 0$ if $0 \leq n < k$. We consider the following property, which is an analogue of the Rohlin property (cf. [2], Lemma VIII.3.7). We say that C_I has Property (R) if, for every $K \in \mathbb{N}$, there exist $f_j \in C_I$ with $|f_j| = 1$ $(j = 1, \cdots, N)$ such that

$$\sum_{j=1}^{N} f_j v_n u^m(\overline{f_j}) = 0$$

for n, $m \in \mathbb{Z}$ with $|n|$, $|m| \leq K$ and $(n, m) \neq (0, 0)$.

Proposition 4.1. *Suppose that C_I has Property* (R). *For every $a \in A$ and $\varepsilon > 0$, there exist $f_j \in C_I$ with $|f_j| = 1$ $(j = 1, \cdots, N)$ such that*

$$\left\| E(a) - \frac{1}{N}\sum_{j=1}^{N} \rho_I(f_j) a \rho_I(\overline{f_j}) \right\| < \varepsilon.$$

Proof. We have $V_n U^m \rho_I(f) = \rho_I(v_n u^m(f)) V_n U^m$ for $f \in C_I$ and $n, m \in \mathbb{Z}$. Let $a \in \tilde{A}$. Then we have $a = \sum_{n,m=-K}^{K} V_n U^m a_{(n,m)}$ with $a_{(n,m)} \in B_I$. There exist functions $f_j \in C_I$ $(j = 1, \cdots, N)$ which satisfy the equality in Property (R). Then we have

$$\frac{1}{N}\sum_{j=1}^{N} \rho_I(f_j) a \rho_I(\overline{f_j}) = a_{(0,0)} = E(a).$$

Since $\tilde{A}$ is dense in A, this completes the proof of the proposition. □

5. A maximal ideal of A

In this section, we assume that C_I has Property (R). For a subset S of C_I, we say that S is invariant if $u(S) = S$ and $v_k(S) \subset S$ for every $k \in \mathbb{Z}$. For a non-trivial closed invariant ideal S of C_I, we say that S is a maximal closed invariant ideal if there is no non-trivial closed invariant ideal which contains S.

Lemma 5.1. *Suppose that C_I has Property (R). Let J be a closed two-sided ideal of A. Then $E(J)$ is a subset of J. Moreover if S is a subset of C_I such that $\rho_I(S) = E(J)$, then S is a closed invariant ideal of C_I.*

Proof. Since J is a closed two-sided ideal, it follows from Proposition 4.1 that $E(x)$ is an element of J for every $x \in J$. Since E is a conditional expectation and ρ_I is an isomorphism, S is an ideal of C_I. Since $E(J) \subset J$ and J is closed, S is a closed subset. Since we have $\rho_I(u(a)) = U\rho_I(a)U^*$ for $a \in C_I$, we have $u(S) = S$. Since we have $\rho_I(v_k(a)) = V_k \rho_I(a) V_{-k}$ for $a \in C_I$ and $k \in \mathbb{Z}$, we have $v_k(S) \subset S$. □

Theorem 5.2. *Suppose that C_I has Property (R) and that there exists a maximal closed invariant ideal S of C_I. Then there exists a unique maximal closed two-sided ideal J of A such that $E(J) = \rho_I(S)$.*

Proof. Let $\tilde{J}$ be the closed two-sided ideal of A generated by $\rho_I(S)$. For $f \in S$ and $g, h \in C_I$, set $a = \rho_I(f)$, $x = V_n U^m \rho_I(g)$ and $y = V_k U^j \rho_I(h)$. Then we have $E(xay) = 0$ if $(n+k, m+j) \neq (0,0)$ and we have $E(xay) = \rho_I((v_n u^m)(gf)\varepsilon_{-k}h)$ if $(n+k, m+j) = (0,0)$. Since S is an invariant ideal, $E(xay)$ is an element of $\rho_I(S)$. Since S is closed, this implies that $E(\tilde{J}) \subset \rho_I(S)$. The reverse inclusion is clear. Therefore we have $E(\tilde{J}) = \rho_I(S)$. Let $\mathcal{I}$ be the set of closed two-sided ideals J' of A such that $E(J') = \rho_I(S)$. Note that $\tilde{J}$ belongs to $\mathcal{I}$. Since $\mathcal{I}$ is an inductive set, it has a maximal element J. Since S is a maximal closed invariant ideal, it follows from Lemma 5.1 that J is a maximal closed two-sided ideal of A. Let J' be another maximal closed two-side ideal of A such that $E(J') = \rho_I(S)$. Let J'' be the closure of $J + J'$. Then J'' is a closed two-sided ideal of A such that $E(J'') = \rho_I(S)$. Since J'' is non-trivial and J and J' are maximal, we have $J = J' = J''$. □

6. An example

In this section, we discuss the case when X is the ring of p-adic integers. Let p be a prime and $\mathbb{Z}_p$ the ring of p-adic integers. As for the notations and facts related to p-adic numbers, we refer the reader to [4, 9, 10]. For $\theta \in \mathbb{Z}_p^\times$ and $\sigma \in \mathbb{Z}_p^\times \cap \mathbb{Z}$ with $\sigma \neq \pm 1$, we define homeomorphisms α, $\beta : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ by $\alpha(x) = x + \theta$ and $\beta(x) = \sigma x$ respectively. Then we have $\beta\alpha = \alpha^\sigma \beta$. The action of α is minimal, that is, $\{\alpha^n(x); n \in \mathbb{Z}\}$ is dense in $\mathbb{Z}_p$ for every $x \in \mathbb{Z}_p$. For $k \in \mathbb{Z}$, let $\kappa_k : C(\mathbb{Z}_p) \longrightarrow C_b(\mathbb{N} \times \mathbb{Z}_p)$ be an injective $*$-homomorphism defined by $\kappa_k(f)(n,x) = f(\beta^n \alpha^{-k}(x))$ for $f \in C(\mathbb{Z}_p)$ and $(n,x) \in \mathbb{N} \times \mathbb{Z}_p$. We denote by C the C^*-subalgebra of $C_b(\mathbb{N} \times \mathbb{Z}_p)$ generated by $\kappa_k(C(\mathbb{Z}_p))$ $(k \in \mathbb{Z})$. We set $C_0 = \kappa_0(C(\mathbb{Z}_p))$. For $n \in \mathbb{N}$, define $\varepsilon_n \in C_b(\mathbb{N} \times \mathbb{Z}_p)$ by $\varepsilon_n(k,x) = 1$ if $k \geq n$ and $\varepsilon_n(k,x) = 0$ if $k < n$. We denote by C_I the C^*-subalgebra of $C_b(\mathbb{N} \times \mathbb{Z}_p)$ generated by C and $\{\varepsilon_n\}$. Note that the C^*-algebras C and C_I defined here are isomorphic to the algebras C and C_I defined in Section 4 respectively. Define an automorphism u of C_I by $u(f)(n,x) = f(n, \alpha^{-1}(x))$. For $k \in \mathbb{N}$, define a $*$-endomorphism v_k of C_I by $v_k(f)(n,x) = f(n+k, x)$ and define a $*$-endomorphism v_{-k} of C_I by $v_{-k}(f)(n,x) = f(n-k, x)$ if $n \geq k$ and by $v_{-k}(f)(n,x) = 0$ if $n \leq k-1$. Note that maps u, v_k and v_{-k} defined here correspond to the maps u^{-1}, v_k and v_{-k} defined in Section 4 respectively.

Let n_k be the least positive integer such that $\sigma^n \equiv 1 \pmod{p^{k+1}}$. Since n_{k-1} is a divisor of n_k, $(\mathbb{Z}/n_k\mathbb{Z})_{k \geq 0}$ is a projective system in a natural way. We denote by G the projective limit $\varprojlim \mathbb{Z}/n_k\mathbb{Z}$. Since $\lim_{k\to\infty} n_k = \infty$, G is a compact additive group which contains $\mathbb{Z}$ as a dense subgroup. For $k \in \mathbb{N}$, define an equivalence relation $\sim_k$ on $\mathbb{N}$ as follows; $n \sim_k m$ if and only if either $n = m$ if $n \leq n_k - 1$ or $m \geq n_k$ and $n \equiv m \pmod{n_k}$ if $n \geq n_k$. We set $Y_k = \mathbb{N}/\sim_k$ and denote by $[n]_k$ the equivalence class of n in Y_k. Define a map $F_k : Y_k \longrightarrow Y_{k-1}$ by $F_k([n]_k) = [n]_{k-1}$. Then $(Y_k, F_k)_{k\geq 0}$ is a projective system. We denote by Y the projective limit $\varprojlim Y_k$. Note that Y is a 2nd countable compact Hausdorff space. We denote by Ω and Ω_I the spectra of C and C_I respectively. For $n \in \mathbb{N}$ and $x \in \mathbb{Z}_p$, define $\omega_{(n,x)} \in \Omega$ by $\omega_{(n,x)}(f) = f(n,x)$ for $f \in C$. Define a map $j : \mathbb{N} \times \mathbb{Z}_p \longrightarrow \Omega$ by $j(n,x) = \omega_{(n,x)}$.

Proposition 6.1. *The map j is injective and continuous and the image of j is dense in Ω.*

Proof. It is clear that j is continuous. We can show that the image of j is dense in Ω as in [12] §6.5. We show that j is injective. Suppose that $\omega_{(n,x)} = \omega_{(m,y)}$ for (n,x), $(m,y) \in \mathbb{N} \times \mathbb{Z}_p$. Since we have $\omega_{(n,x)}(\kappa_k(f)) = f(\sigma^n(x - k\theta))$ for $f \in C(\mathbb{Z}_p)$ and $k \in \mathbb{Z}$, we have $\sigma^n(x - k\theta) = \sigma^m(y - k\theta)$ for all $k \in \mathbb{Z}$. We have $\sigma^n x = \sigma^m y$ when $k = 0$ and

$\sigma^n x - \sigma^m y = (\sigma^n - \sigma^m)\theta$ when $k = 1$. Then we have $\sigma^{n-m} = 1$. Since $\sigma \neq \pm 1$, we have $n = m$ and $x = y$. $\square$

For $k \in \mathbb{N}$ and $a \in \mathbb{Z}/p^{k+1}\mathbb{Z}$, we denote by $E_a^{(k)}$ the clopen set consisting of points $x \in \mathbb{Z}_p$ such that $x \equiv a \pmod{p^{k+1}}$ and by $\mathcal{E}^{(k)}$ the set of $E_a^{(k)}$ with $a \in \mathbb{Z}/p^{k+1}\mathbb{Z}$. Let $C(\mathcal{E}^{(k)})$ be the C^*-subalgebra of $C(\mathbb{Z}_p)$ generated by χ_E ($E \in \mathcal{E}^{(k)}$), where χ_E is the characteristic function of E and let C_k be the C^*-subalgebra of C generated by $\kappa_n(C(\mathcal{E}^{(k)}))$ ($n \in \mathbb{Z}$). Then $\{C_k\}$ is an increasing sequence whose union is dense in C. We denote by Ω_k the spectrum of C_k.

Lemma 6.2. *The spectrum Ω_k is identified with $\mathbb{Z}/n_k\mathbb{Z} \times \mathbb{Z}/p^{k+1}\mathbb{Z}$.*

Proof. For $n \in \mathbb{N}$ and $x \in \mathbb{Z}_p$, define $\omega_{(n,x)}^{(k)} \in \Omega_k$ by $\omega_{(n,x)}^{(k)}(f) = f(n,x)$ for $f \in C_k$ and define a map $j_k : \mathbb{N} \times \mathbb{Z}_p \longrightarrow \Omega_k$ by $j_k(n,x) = \omega_{(n,x)}^{(k)}$. One can show that the image of j_k is dense in Ω_k as in [12] §6.5. Since the equation $j_k(n,x) = j_k(m,y)$ is equivalent to the equations $n \equiv m \pmod{n_k}$ and $x \equiv y \pmod{p^{k+1}}$, the image of j_k is identified with $\mathbb{Z}/n_k\mathbb{Z} \times \mathbb{Z}/p^{k+1}\mathbb{Z}$. Since the image of j_k is finite and dense, j_k is surjective. $\square$

Theorem 6.3. *The spectrum Ω is homeomorphic to $G \times \mathbb{Z}_p$ and the spectrum Ω_I is homeomorphic to $Y \times \mathbb{Z}_p$.*

Proof. Define a map $\psi_k : \Omega_k \longrightarrow \Omega_{k-1}$ by $\psi_k(\omega) = \omega | C_{k-1}$, the restriction of ω to C_{k-1}. Then $(\Omega_k, \psi_k)_{k \geq 0}$ is a projective system. Set $E = \varprojlim \Omega_k$. Then E is the set of points $(\omega^{(k)})_{k \geq 0}$ of $\prod_{k \geq 0} \Omega_k$ such that $\psi(\omega^{(k)}) = \omega^{(k-1)}$. Define a map $\varphi_k : \Omega \longrightarrow \Omega_k$ by $\varphi_k(\omega) = \omega | C_k$ and define a map $\varphi : \Omega \longrightarrow E$ by $\varphi(\omega) = (\varphi_k(\omega))_{k \geq 0}$. It is clear that φ is injective and continuous. Since Ω is compact, $\varphi(\Omega)$ is closed. We can also show that $\varphi(\Omega)$ is dense in E. Therefore φ is surjective and E is homeomorphic to Ω. It follows form Lemma 6.2 that E is identified with $G \times \mathbb{Z}_p$.

Let $C_{I,k}$ be the C^*-subalgebra of C_I generated by C_k and $\varepsilon_0, \cdots, \varepsilon_{n_k}$. Then $\{C_{I,k}\}$ is an increasing sequence whose union is dense in C_I. We denote by $\Omega_{I,k}$ the spectrum of $C_{I,k}$. Then $\Omega_{I,k}$ is identified with $Y_k \times \mathbb{Z}/p^{k+1}\mathbb{Z}$. Since we have $\Omega_I = \varprojlim \Omega_{I,k}$, Ω_I is homeomorphic to $Y \times \mathbb{Z}_p$. $\square$

We denote by $\mu_k : Y \longrightarrow Y_k$ and $\nu_k : G \longrightarrow \mathbb{Z}/n_k\mathbb{Z}$ the canonical maps for the projective limits $Y = \varprojlim Y_k$ and $G = \varprojlim \mathbb{Z}/n_k\mathbb{Z}$ respectively. Define a map $\tilde{q}_k : Y_k \longrightarrow \mathbb{Z}/n_k\mathbb{Z}$ by $\tilde{q}_k([n]_k) = [n]'_k$, where $[n]'_k$ is the equivalence class of $n \in \mathbb{N}$ in $\mathbb{Z}/n_k\mathbb{Z}$. There exists a unique surjective continuous map $\tilde{q} : Y \longrightarrow G$ such that $\nu_k \tilde{q} = \tilde{q}_k \mu_k$. Define a map $i : \mathbb{N} \longrightarrow Y$ by $i(n) = ([n]_k)_{k \geq 0}$. Set $\widetilde{Y} = i(\mathbb{N})$ and $\partial Y = Y \setminus \widetilde{Y}$. Then $\widetilde{Y}$ is open and dense in Y. We denote by h the restriction of $\tilde{q}$ to ∂Y. Then the map $h : \partial Y \longrightarrow G$ is a homeomorphism of ∂Y onto G. Define a map $\gamma_k : Y_k \longrightarrow Y_k$ by $\gamma_k([n]_k) = [n+1]_k$ and a map $\gamma : Y \longrightarrow Y$ by $\gamma((x_k)_{k \geq 0}) = (\gamma_k(x_k))_{k \geq 0}$. Then γ is injective and continuous. Note that the restriction of γ to ∂Y is a homeomorphism of ∂Y onto itself and that we have $h(\gamma(x)) = h(x) + 1$ for $x \in \partial Y$. For a subset T of Y, we say that T is γ-invariant if $\gamma(T) \subset T$ and $\gamma^{-1}(T) \subset T$. The set ∂Y is the unique non-trivial closed γ-invariant subset of Y. Let $f \in C_I = C(Y \times \mathbb{Z}_p)$ and $(x,y) \in Y \times \mathbb{Z}_p$. Then we have $u(f)(x,y) = f(x, \alpha^{-1}(y))$. If $k \geq 0$, then $v_k(f)(x,y) = f(\gamma^k(x), y)$. If $k < 0$, then

$$v_k(f)(x,y) = \begin{cases} f(\gamma^k(x), y) & \text{if } x \in \operatorname{Im} \gamma^{-k}, \\ 0 & \text{if } x \notin \operatorname{Im} \gamma^{-k}. \end{cases}$$

We denote by S the subset of C_I consisting of elements f such that $f(x,y) = 0$ for all $x \in \partial Y$ and $y \in \mathbb{Z}_p$. Then the set S is the unique non-trivial closed invariant ideal of C_I.

Lemma 6.4. *The C^*-algebra C_I has Property (R).*

Proof. For every $k \in \mathbb{N}$, we show that there exist $\xi_s^{(k)} \in C_{I,k}$ with $|\xi_s^{(k)}| = 1$ $(s = 0, \cdots, n_k - 1)$ such that

$$\sum_{s=0}^{n_k-1} \xi_s^{(k)} v_n(\overline{\xi_s^{(k)}}) = 0$$

for $n \in \mathbb{Z}$ with $0 < |n| < n_k$. Let $\mu_k : Y \longrightarrow Y_k$ be the canonical map. For $j = 0, \cdots, 2n_k - 1$, set $D_j = \mu_k^{-1}([j]_k)$. Then $\{D_j\}$ is a partition of Y by clopen sets. We denote by a_j the characteristic function of $D_j \times \mathbb{Z}_p$. Then a_j is an element of $C_{I,k}$. For $j = 0, \cdots, n_k - 1$, define $b_j \in C_{I,k}$ by $b_j = a_j + a_{j+n_k}$. For $|n| \le n_k - 1$, we have (1) $v_n(b_j) = b_{j-n}$ if $j - n_k + 1 \le n \le j$, (2) $v_n(b_j) = b_{j-n+n_k}$ if $j + 1 \le n \le n_k - 1$ and (3) $v_n(b_j) = a_{j-n}$ if $-n_k + 1 \le n \le j - n_k$. Set $\lambda = \exp(2\pi\sqrt{-1}/n_k)$ and set

$$\xi_s^{(k)} = \sum_{j=0}^{n_k-1} \lambda^{sj} b_j$$

for $s = 0, \cdots, n_k - 1$. If $0 < n < n_k$, we have

$$v_n(\xi_s^{(k)}) = \sum_{j=0}^{n_k-1} \lambda^{s(j+n)} b_j$$

and we have

$$\sum_{s=0}^{n_k-1} \xi_s^{(k)} v_n(\overline{\xi_s^{(k)}}) = \sum_{s=0}^{n_k-1} \lambda^{-sn} = 0.$$

If $-n_k < n < 0$, we have

$$v_n(\xi_s^{(k)}) = \sum_{j=-n}^{n_k-1} \lambda^{s(j+n)} b_j + \sum_{j=n_k}^{n_k-n-1} \lambda^{s(j+n)} a_j$$

and we have

$$\sum_{s=0}^{n_k-1} \xi_s^{(k)} v_n(\overline{\xi_s^{(k)}}) = \sum_{s=0}^{n_k-1} \lambda^{-sn} \left(\sum_{j=-n}^{n_k-1} b_j + \sum_{j=n_k}^{n_k-n-1} a_j \right) = 0.$$

Similarly, we can show that there exist $\eta_t^{(k)} \in C_k$ with $|\eta_t^{(k)}| = 1$ $(t = 0, \cdots, p^{k+1} - 1)$ such that

$$\sum_{t=0}^{p^{k+1}-1} \eta_t^{(k)} u^m(\overline{\eta_t^{(k)}}) = 0$$

for $m \in \mathbb{Z}$ with $0 < |m| < p^{k+1}$. For $K \in \mathbb{N}$, we choose k to be $K < \min\{n_k, p^{k+1}\}$. Then we can take the family of functions f_j $(1 \le j \le n_k p^{k+1})$ to be $\xi_s^{(k)} \eta_t^{(k)}$ $(0 \le s \le n_k - 1, 0 \le t \le p^{k+1} - 1)$. $\square$

Let A be the C^*-algebra associated with the trinary relation arising from the homeomorphisms α and β introduced in this section. By virtue of Theorem 5.2, we have the following theorem.

Theorem 6.5. *Let A be the C^*-algebra as above and let S be the unique non-trivial closed invariant ideal of C_I. Then there exists a unique maximal closed two-sided ideal J of A such that $E(J) = \rho_I(S)$.*

Remark. We do not know whether J is the unique maximal closed two-sided ideal of A or not.

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