DISCRETE TIME PORTFOLIO MANAGING

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Abstract. In the paper we study a mathematical problem of optimally managing a portfolio of assets which are dependent on microeconomic and macroeconomic factors and where the investor's aim is to maximize the portfolio's risk sensitized growth rate. We consider an infinite horizon control problem of maximizing the portfolio's risk sensitized growth rate criterion. Using the so-called span semi-norm approach and imposing some technical assumptions we provide a complete solution to the above control problem.

1 Introduction

There have been several works where optimal investment models are reformulated as a risk sensitive stochastic control problems. Among them, Fleming [8], Fleming and Sheu [10], Bielecki and Pliska [1] explored the idea of risk sensitive control to the problems arising from portfolio management.

In particular, Bielecki and Pliska [1] applied risk sensitive control to a version of Merton's [17] intertemporal capital asset pricing model and introduced the approach of risk sensitive portfolio optimization based on the dependence of the assets on macroeconomic and microeconomic factors .The result was an optimal control problem with infinite horizon risk sensitive criterion. Using continuous time risk sensitive control theory (see [9], [18] and [23]) the authors showed that the optimal portfolio strategy is a simple function of the factor levels. Other studies of risk sensitive criterion include [2], [11], [12], [16], [19], [20]. In particular , in [2] Bielecki and Pliska considered the case with transaction costs having a fixed component.They used impulse control models to show that the solutions to the corresponding control problem can be obtained via so-called risk sensitive quasivariational inequalities.

Throughout all this works on risk sensitive portfolio management the security prices follow diffusion processes and underlying factor processes was taken to be Gaussian or, at least (see Nagai [19]) non-Gaussian processes.

Our interest is to reformulate the portfolio management problem, proposed by Bielecki and Pliska [1], in discrete time context and develop a procedure for computation of optimal strategies. Fortunately, the discrete time risk sensitive control theory offers a potential path to a solution to our problem.

Our financial model resembles the one described in [3], where dynamic programming approach was used to solve discrete time portfolio optimization control problem with restriction that the factors' state space is finite. In our paper we do not need restrictions on the factor space. However, our approach is different from [3] and is based on method

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applied by Di Masi and Stettner in [6].

Similar approach was also applied by Stettner (see [22]) to continuous time risk sensitive portfolio managing. Under assumption that investor can change his portfolio only at discrete times the cases with and without transaction costs were solved.

The paper is organized as follows. In Section 2 we give a detailed description of our financial model, state suitable control problem and introduce assumptions. In Section 3 we construct the corresponding Bellman equation and prove the verification theorem. Section 4 contains the main results of the paper which are a solution of the Bellman equation and a complete characterization of optimal trading strategy.

2 Financial Model and Assumptions

2.1 Financial Model We shall assume that the evolution of factors has a Markovian structure, namely it is described by discrete-time Markov process $X = (x_t), t = 0, 1, \dots,$ which is defined on a probability space $(\Omega, \mathcal{F}, P)$ and takes values on a compact separable metric space E endowed with a Borel σ- algebra $\mathcal{E}$. Moreover we assume that x_t has a transition operator $P(x_t, \cdot)$ at the generic period t.

We shall consider a financial market in which $m+1$ financial assets are traded in discrete time only.

One of them is a bank account with a constant interest rate r, so a deposit of one dollar gives a reward e^{rt} after t periods. We shall denote by B_t the part of wealth hold in the bank account at the time t.

There are also m risky assets with prices-per-share S_t^i at time t, which depend on the level of the factor process $X = (x_t)$.The assets are modelled in such way, that asset returns depend on the beginning and the end of the trading period. Namely, there is a Borel measurable function $\alpha : E \times E \times \mathbf{R}^m \mapsto R^+$ such that for

$$Z_{t+1} = \left(\frac{S_{t+1}^1}{S_t^1}, \cdots, \frac{S_{t+1}^m}{S_t^m} \right),$$

and given $X_t := \sigma((x_s),\, 0 \le s \le t)$ and $\mathcal{Y}_t := \sigma((x_s, S_s),\, 0 \le s \le t)$ with $S_t := (S_t^1, \cdots, S_t^m)$ we have

$$P\left(Z_{t+1} \in A \mid \mathcal{Y}_t, X_{t+1} \right) = \int_A \alpha(x_t, x_{t+1}, z)\mu(dz),$$

where $A \in \mathbf{R}^m$ and μ is a probability measure on $\mathbf{R}^m$.

Let us also consider an economic agent who starts with the initial capital $V_0 = v$ and invests it in the above $m+1$ assets. We assume that our investor can change his portfolio without transactions costs and is allowed to buy any nonnegative number (also parts) of the assets. We admit neither short selling nor short borrowing.

The trading strategy is modelled as follows.

Let V_t denote the agent's wealth at time t. Suppose that at time t our agent invests a proportion of his wealth, denoted by a_t^k, in the k-th risky asset and the rest of wealth is invested in the bank account, i.e. $B_t = (1 - a_t^1 - \cdots - a_t^m)V_t$.

Notice that the absence of short selling and short borrowing imposes some restrictions on the trading possibilities. Namely, we have

$$\forall\ k = 1, \cdots, m \quad a_t^k \ge 0 \quad \text{and} \qquad \sum_{k=1}^m a_t^k \le 1\,.$$

At time t our agent buys $h_t^k := \frac{a_t^k V_t}{S_t^k}$ shares of the asset k. Therefore at time $t+1$ the agent's wealth is

$$V_{t+1} = h_t^1 S_{t+1}^1 + \cdots + h_t^m S_{t+1}^m + B_t e^r .$$

Analogously, at time $t+1$ our investor can change portfolio by investing a proportion of his wealth V_{t+1} in the risky assets and so on.

Let $a_t = (a_t^1, \ldots, a_t^m)$ denote the vector of proportions invested at time t in the risky assets. Then the wealth of the investor evolves under trading strategy a_t and V_t is given accordingly by

$$V_{t+1} = V_t \left[e^r + a_t \circ (Z_{t+1} - e^r \mathbf{1}) \right] , \tag{1}$$

where by $\circ$ we denote an inner product of vectors and $\mathbf{1}$ represents the vector of $1'$s.

Indeed, we have

$$\begin{aligned} V_{t+1} &= \sum_{k=1}^m a_t^k V_t \frac{S_{t+1}^k}{S_t^k} + B_t e^r = \sum_{k=1}^m a_t^k V_t \frac{S_{t+1}^k}{S_t^k} + \left(1 - \sum_{k=1}^m a_t^k\right) V_t e^r \\ &= V_t \left[e^r + \sum_{k=1}^m a_t^k \left(\frac{S_{t+1}^k}{S_t^k} - e^r \right) \right] . \end{aligned}$$

To describe the concept of control formally we introduce the following notations and definition.

Let $\mathbf{A}$ be a compact subset of $\mathbf{R}^m$, whose elements $a = (a^1, \ldots a^m)$ represent vectors of proportions invested in the risky assets. Moreover, let $\hat{\mathbf{a}} = (a_t)$ denote an $\mathbf{R}^m$-valued investment process or trading strategy whose components are a_t^k, $k = 1, \cdots, m$.

Definition A *control strategy* $\hat{\mathbf{a}} = (a_t)$ is *admissible* if the following conditions are satisfied:

$$\begin{aligned} &(i) \quad \forall\, t \geq 0 \quad a_t \in \mathbf{A}, \quad a_t^k \geq 0\ , k = 1, \cdots, m \quad \text{and} \quad \sum_{k=1}^m a_t^k \leq 1\,, \\ &(ii) \quad a_t \text{ is } \quad \mathcal{Y}_t - \text{measurable.} \end{aligned}$$

With each admissible trading strategy $\hat{\mathbf{a}} = (a_t)$ we associate the following risk sensitive measure of performance (risk sensitized grown rate)

$$J^x(\hat{\mathbf{a}}) = \liminf_{T \to \infty} \left(-\frac{2}{\Theta} \right) \frac{1}{T} \ln E_x \left\{ \exp\left(-\frac{\Theta}{2} \ln V_T \right) \right\} . \tag{2}$$

where E_x denotes the conditional expectation under given $x_0 = x$.
The nonnegative parameter Θ captures the investor's attitudes about risk aversion.

Notice that by Taylor expansion for Θ close to 0 we have, that

$$\ln E \left\{ \exp\left(-\frac{\Theta}{2} \ln V_T \right) \right\} \approx -\frac{\Theta}{2} E (\ln V_T) + \frac{\Theta^2}{8} Var (\ln V_T) .$$

Consequently, for sufficiently small Θ the expression (2) gives an information about the long run growth rate of the wealth V_t together with the value of risk measured by the variance of $\ln V_t$.

The aim of our investor is to maximize the functional (2) .i.e. choose an admissible trading strategy $\hat{\mathbf{a}}^*$ such that

$$J^x(\hat{\mathbf{a}}^*) = \sup_{\hat{\mathbf{a}}} J^x(\hat{\mathbf{a}}) \,.$$

We can notice that the control problem of maximization of (4.2) is equivalent to minimization of the following functional

$$\mathcal{J}^x(\hat{\mathbf{a}}) = \limsup_{T\to\infty} \left(\frac{2}{\Theta}\right) \frac{1}{T} \ln E_x \left\{ \exp\left(-\frac{\Theta}{2} \ln V_T \right) \right\} \,. \tag{3}$$

2.2 Assumptions Let us first introduce assumptions on the factor process $X = (x_t)$. Namely, we shall need the following assumptions:

$(A1)$ $\forall f \in C(E)$ the mapping $E \ni x \mapsto Pf(x) = \int_E f(y) P(x, dy)$ is continuous.

$(A2)$ $\exists \delta < 1$ such that $\forall x, x' \in E$, $\forall B \in \mathcal{E}$

$$P(x', B) - P(x, B) \le \delta \,.$$

Let us introduce the following conditional expectation

$$\begin{aligned} \mu^\Theta(x_t, x_{t+1}, a_t) &= E\left\{ e^{\left(-\frac{\Theta}{2}\ln[e^r + a_t \circ (Z_{t+1} - e^r \mathbf{1})]\right)} | \mathcal{Y}_t, X_{t+1} \right\} \\ &= \int e^{\left(-\frac{\Theta}{2}\ln[e^r + a_t \circ (z - e^r \mathbf{1})]\right)} \alpha(x_t, x_{t+1}, dz) \,. \end{aligned} \tag{4}$$

The requirement for μ^Θ is that:

$(A3)$ There are constants $0 < b \le B$ such that $\forall x, y \in E$ and $\forall a \in \mathbf{A}$

$$e^{-\frac{\Theta}{2}\ln B} \le \mu^\Theta(x, y, a) \le e^{-\frac{\Theta}{2}\ln b} \,.$$

$(A4)$ $\forall y \in E$, $\forall a \in \mathbf{A}$ the mapping $E \ni x \to \mu^\theta(x, y, a)$ is continuous.

Remark 2.1 *In view of the construction of μ^Θ the assumption $(A3)$ introduces some restriction on the set $\mathbf{A}$. Indeed, it is required that the set $\mathbf{A}$ is such that, for each $a \in \mathbf{A}$, $z \in \mathbf{R}^m$*

$$b \le e^r + a \circ (z - e^r \mathbf{1}) \le B \quad a.s.\,.$$

The boundedness from below is needed to rule out the pathological cases while the boundedness from above is a stronger form of the natural assumption, that the conditional expectation (4) exists and is finite.

On the other hand, the assumption $(A3)$ holds under restrictions on the relative growth of prices of the risky assets. Namely, if there are constants $d \le D$ such that for t

$$d \le \frac{S^i_{t+1}}{S^i_t} \le D \quad for \quad i = 1, \cdots, m \,,$$

we have

$$e^{-\frac{\Theta}{2}\ln D} \le \mu^\Theta(x, y, a) \le e^{-\frac{\Theta}{2}\ln d} \,.$$

3 Optimality equation

Let us consider the following Bellman equation

$$e^{\omega(x)+\lambda} = \inf_{a\in\mathbf{A}} \left[e^{\frac{\Theta}{2}c^{\Theta}(x,a)} \int_E e^{\omega(x)} P^a_{\Theta}(x,dy) \right], \tag{5}$$

where $c^{\Theta}(x,a)$ and $P^a_{\Theta}(x,dy)$ are defined as follows:

$$P^a_{\Theta}(x,B) = \frac{\int_B \mu^{\Theta}(x,y,a)P(x,dy)}{\int_E \mu^{\Theta}(x,y,a)P(x,dy)}, \tag{6}$$

and

$$c^{\Theta}(x,a) = \frac{2}{\theta} \ln \left(\int_E \mu^{\Theta}(x,y,a)P(x,dy) \right). \tag{7}$$

The next verifications theorem states a relation between (5) and a given control problem.

Proposition 3.1 *Let assumptions* $(A1)$, $(A3)$ *and* $(A4)$ *be satisfied.*
If there exists a function $\omega \in C(E)$ *and a constants* λ *such that for* $x \in E$ *equation* (5) *is satisfied, then*

$$\lambda = \frac{\Theta}{2} \inf_{\hat{\mathbf{a}}} \mathcal{J}^x(\hat{a}) = \frac{\Theta}{2} \sup_{\hat{\mathbf{a}}} \mathcal{J}^x(u(X_t)),$$

where $u : E \to \mathbf{A}$ *is a Borel function for which the* inf *in* (5) *is attained.*

Before the proof of the above Proposition, we prove the following auxiliary lemma.

Lemma 3.1 *Let assumptions* $(A1)\,(A3)$ *and* $(A4)$ *be satisfied. Then for each* $f \in C(E)$ *the mapping*

$$E \times \mathbf{A} \ni (x,a) \mapsto P^a_{\Theta} f(x) = \int_E f(y) P^a_{\Theta}(x,dy)$$

is continuous.
Moreover, the function $c^{\Theta}(x,a)$ *is continuous with respect to both variables.*

Proof:
From the definition of $\mu^{\Theta}(x,y,a)$ and $(A3)$ immediately follows that $a \mapsto \mu^{\Theta}(x,y,a)$ is continuous. Therefore the mappings

$$a \mapsto P^a_{\Theta} f(x) \quad \text{and} \quad a \mapsto c^{\Theta}(x,a)$$

are continuous.
The continuity of μ^{Θ} and c^{Θ} with respect to x follows from assumption $(A4)$.
The boundedness of $c^{\Theta}(x,a)$ follows from $(A3)$. Indeed, we have

$$\ln \frac{1}{B} \le c^{\Theta}(x,a) \le \ln \frac{1}{b}.$$

This completes the proof.

□

Proof of Proposition 3.1

Let us first observe that by definition of c^{Θ} and $P^a_{\Theta}(x,\cdot)$ the equation (5) can be rewritten in the following form

$$e^{\omega(x)+\lambda} = \inf_{a\in\mathbf{A}} E_x\left[e^{\omega(x_1)}\mu^{\Theta}(x,x_1,a)\right] . \tag{8}$$

Indeed,

$$\begin{aligned} e^{\omega(x)+\lambda} &= \inf_{a\in\mathbf{A}}\left[e^{\frac{\Theta}{2}c^{\Theta}(x,a)}\int_E e^{\omega(x)}P^a_{\Theta}(x,dy)\right] \\ &= \inf_{a\in\mathbf{A}}\int_E e^{\omega(y)}\mu^{\Theta}(x,y,a)P(x,dy) \ . \end{aligned}$$

Without loss of generality we assume that $V_0 = 1$. Then

$$\begin{aligned} e^{\omega(x)+\lambda} &\le E_x\left[e^{\omega(x_1)}\mu^{\Theta}(x,x_1,a_0)\right] \\ &= E\left[e^{\omega(x_1)}\,E\left\{e^{-\frac{\Theta}{2}\ln[e^r+a_0(Z_1-e^r\mathbf{1})]}|\mathcal{Y}_0,X_1\right\}|X_0\right] \\ &= E\left[e^{\omega(x_1)}e^{-\frac{\Theta}{2}\ln V_1}|X_0\right] \\ &\le E\left[e^{-\frac{\Theta}{2}\ln V_1}\,E\left\{e^{-\lambda}e^{\omega(x_2)}\,E\left(e^{-\frac{\Theta}{2}\ln[e^r+a_1(Z_2-e^r\mathbf{1})]}|\mathcal{Y}_1,X_2\right)|\mathcal{Y}_1\right\}|X_0\right] \\ &= E\left[e^{-\frac{\Theta}{2}\ln V_2}e^{\omega(x_2)-\lambda}|X_0\right] . \end{aligned}$$

We may iterate T times to conclude that

$$e^{\omega(x)+\lambda T} \le E_x\left\{e^{-\frac{\Theta}{2}\ln V_T}e^{\omega(x_T)}\right\} .$$

Consequently, letting $T\to\infty$ we obtain

$$\lambda\frac{2}{\Theta} \le \limsup_{T\to\infty}\left(\frac{2}{\Theta}\right)\frac{1}{T}\ln E_x\left\{\exp\left(-\frac{\Theta}{2}\ln V_T\right)\right\} .$$

Notice that equality holds for the optimal control strategy. The proof is therefore completed.

□

4 Main result

We shall start from the proof of the following auxiliary lemma.

Lemma 4.1 *Let assumptions* $(A2)$, $(A3)$ *hold.*
For each $K < \infty$ *there is a constant* $\Delta < 1$ *such that for each* $x, x' \in E$, $a, a' \in \mathbf{A}$ *and* $B \in \mathcal{E}$

$$\sup_{\Theta \le K} \Big[P_{\Theta}^{a'}(x', B) - P_{\Theta}^{a}(x, B) \Big] \le \Delta \,. \tag{9}$$

Proof:
Suppose that the inequality (9) does not hold. That there are sequences (x_n), (x'_n), $(a_n), (a'_n)$, (Θ_n) and (B_n) such that

$$P_{\Theta_n}^{a'_n}(x'_n, B_n) - P_{\Theta_n}^{a_n}(x_n, B_n) \to 1 \quad \text{as} \quad n \to \infty \,.$$

Therefore

$$P_{\Theta_n}^{a'_n}(x'_n, B_n^c) \to 0 \quad \text{and} \quad P_{\Theta_n}^{a_n}(x_n, B_n) \to 0 \,.$$

Since

$$P_{\Theta}^{a}(x, B) = \frac{\int_B \mu^{\Theta}(x, y, a) P(x, dy)}{\int_E \mu^{\Theta}(x, y, a) P(x, dy)} \ge e^{-\frac{\Theta}{2} \ln \frac{B}{b}} \, P(x, B)$$

we have that

$$P(x'_n, B_n^c) \to 0 \ \text{ and } \ P(x_n, B_n) \to 0 \quad \text{as } n \to \infty \,.$$

Consequently

$$\lim_{n \to \infty} [P(x'_n, B_n^c) - P(x_n, B_n)] = 0$$

which contradicts the assumption $(A2)$.

□

Notice that by Lemma 3.3 in [8] we can transform (5) in an equivalent form

$$\omega(x) + \lambda = \inf_{a \in \mathbf{A}} \sup_{\nu \in \mathcal{P}(E)} \left[\frac{\Theta}{2} c^{\Theta}(x, a) + \int_E \omega(y) \nu(dy) - I(\nu, P_{\Theta}^{a}(x, \cdot)) \right] , \tag{10}$$

where $\mathcal{P}(E)$ is the space of probability measure on E and $I(\nu, \mu)$ is defined as follows

$$I(\nu, \mu) = \begin{cases} \int_E \log \frac{d\nu}{d\mu} \nu(dx) \,, & \text{when } \nu \ll \mu \,, \\ +\infty & \text{otherwise} \,. \end{cases}$$

Moreover, the sup in (10) is attained for

$$\hat{\nu}(B) = \frac{\int_B e^{\omega(z)} P_{\Theta}^{a}(x, dz)}{\int_E e^{\omega(z)} P_{\Theta}^{a}(x, dz)} \,.$$

For $g \in C(E)$ define the operator

$$T^{\Theta}g(x) = \inf_{a \in \mathbf{A}} \left[e^{\frac{\Theta}{2}c^{\Theta}(x,a)} \int_E e^{g(y)} P_{\Theta}^a(x, dy) \right], \tag{11}$$

or in equivalent form

$$T^{\Theta}g(x) = \inf_{a \in \mathbf{A}} \sup_{\nu \in \mathcal{P}(E)} \left[\frac{\Theta}{2} c^{\Theta}(x,a) + \int_E g(y)\nu(dy) - I(\nu, P_{\Theta}^a(x,\cdot)) \right]. \tag{12}$$

Notice that the sup in (12) is attained for

$$\hat{\nu}(B) = \frac{\int_B e^{g(z)} P_{\Theta}^a(x, dz)}{\int_E e^{g(z)} P_{\Theta}^a(x, dz)}.$$

The next Proposition was proved in [6] and we give only main ideas of the proof.

Proposition 4.1 *Let assumptions $(A1)\,(A2)\,(A3)$ and $(A4)$ be satisfied. Then the operator T^{Θ} is a local contraction in $C(E)$ endowed with the span norm $||g||_{sp} = \sup_{x \in E} g(x) - \inf_{y \in E} g(x)$, namely for each $M > 0$ there exists a constant $\rho(M) < 1$ such that for each $g_1, g_2 \in C(E)$ with $||g_1||_{sp} \le M$, $||g_2||_{sp} \le M$ we have*

$$||T^{\Theta}g_1 - T^{\Theta}g_2||_{sp} \le \rho(M)\, ||g_1 - g_2||_{sp}.$$

Sketch of proof:
Notice that under assumptions $(A1)$ and $(A4)$ the operator T^{Θ} transforms $C(E)$ into itself. Therefore the inf in (10) is attained.
For given $g_1, g_2 \in C(E)$ and $x_1, x_2 \in E$ choose a_1, a_2 such that

$$T^{\Theta}g_1(x_1) = \sup_{\nu \in \mathcal{P}(E)} \left[c^{\Theta}(x_1, a_1) + \int_E g_1(y)\nu(dy) - I(\nu, P_{\Theta}^{a_1}(x_1,\cdot)) \right], \tag{13}$$

and

$$T^{\Theta}g_2(x_2) = \sup_{\nu \in \mathcal{P}(E)} \left[c^{\Theta}(x_2, a_2) + \int_E g_2(y)\nu(dy) - I(\nu, P_{\Theta}^{a_2}(x_2,\cdot)) \right], \tag{14}$$

Moreover, let

$$\nu_1(B) = \frac{\int_B e^{g_2(z)} P_{\Theta}^{a_1}(x_1, dz)}{\int_E e^{g_2(z)} P_{\Theta}^{a_1}(x_1, dz)},$$

and

$$\nu_2(B) = \frac{\int_B e^{g_1(z)} P_{\Theta}^{a_2}(x_2, dz)}{\int_E e^{g_1(z)} P_{\Theta}^{a_2}(x_2, dz)}.$$

Then

$$
\begin{aligned}
&T^\Theta g_1(x_2) - T^\Theta g_2(x_2) - \left(T^\Theta g_1(x_1) - T^\Theta g_2(x_1)\right) \\
&\quad\leq \quad \tfrac{\Theta}{2} c^\Theta(x_2,a_2) + \int_E g_1(y)\nu_2(dy) - I(\nu_2, P_\Theta^{a_2}(x_2,\cdot)) \\
&\quad- \quad \tfrac{\Theta}{2} c^\Theta(x_2,a_2) + \int_E g_2(y)\nu_2(dy) - I(\nu_2, P_\Theta^{a_2}(x_2,\cdot)) \\
&\quad- \quad \tfrac{\Theta}{2} c^\Theta(x_1,a_1) + \int_E g_1(y)\nu_1(dy) - I(\nu_1, P_\Theta^{a_1}(x_1,\cdot)) \\
&\quad+ \quad \tfrac{\Theta}{2} c^\Theta(x_1,a_1) + \int_E g_2(y)\nu_1(dy) - I(\nu_2, P_\Theta^{a_1}(x_1,\cdot)) \\
&\quad= \quad \int_E \left(g_1(y) - g_2(y)\right)(\nu_2 - \nu_1)(dy) \leq ||g_1 - g_2||_{sp}(\nu_2 - \nu_1)(\Gamma)\,,
\end{aligned}
$$

where the set Γ we obtain from Hahn-Jordan decomposition of $\nu_2 - \nu_1$. Therefore

$$
||T^\Theta g_1 - T^\Theta g_2||_{sp} \leq ||g_1 - g_2||_{sp} \sup_{x,x' \in E} \sup_{a,a' \in A} \sup_{B \in \mathcal{E}} (\nu_{x,a,g_1} - \nu_{x',a',g_2}),
$$

where

$$
\nu_{x,a,g}(B) = \frac{\int_B e^{g(z)} P_\Theta^a(x,dz)}{\int_E e^{g(z)} P_\Theta^a(x,dz)}\,.
$$

Notice that

$$
\nu_{x,a,g}(B) \geq e^{-||g||_{sp}} P_\Theta^a(x,B).
$$

Therefore using the similar method as in Lemma 4.1 we obtain

$$
\sup_{g_1,g_2: ||g_1||_{sp}, ||g_2||_{sp} \leq M} \sup_{x,x' \in E} \sup_{a,a' \in A} \sup_{B \in \mathcal{E}} (\nu_{x,a,g_1} - \nu_{x',a',g_2}) = \rho(M) < 1\,,
$$

which completes the proof.

□

We shall make an additional assumption:

$$
(A5) \qquad \Delta e^{\frac{\Theta}{2} \ln \frac{B}{b}} < 1\,,
$$

where b, B come from $(A3)$ and Δ is specified as in Lemma 4.1.

We can now formulate our main result

Theorem 4.1 *Assume* $(A1)$, $(A2)$, $(A3)$, $(A4)$ *and* $(A5)$. *Then there exists at most one (up to an additive constant) function* $\omega \in C(E)$ *and a unique constant* λ *for which the Bellman equation* (5) *is satisfied.*

Proof:
Note first that under the assumption $(A5)$ we have

$$
\Delta e^{\frac{\Theta}{2} ||c^\Theta||_{sp}} < 1\,,
$$

where $||c^\Theta||_{sp} = \sup_{x,a} c^\Theta(x,a) - \inf_{x,a} c^\Theta(x,a)$. Indeed

$$
\Delta e^{\frac{\Theta}{2} ||c^\Theta||_{sp}} \leq \Delta \frac{\max_{x,a} \int_E \mu^\Theta(x,y,a) P(x,dy)}{\min_{x,a} \int_E \mu^\Theta(x,y,a) P(x,dy)} \leq \Delta e^{\frac{\Theta}{2} \ln \frac{B}{b}}\,.
$$

It is easy to check that

$$||T^{\Theta}\, 0||_{sp} \leq \frac{\Theta}{2}||c^{\Theta}||_{sp}\,.$$

where 0 is a null function defined on E. Moreover, under Proposition 2 [7] we have that

$$||\left(T^{\Theta}\right)^{n} 0||_{sp} \leq M\,,$$

where

$$M = \frac{\Theta}{2}||c^{\Theta}||_{sp} + \ln\left[\sum_{i=0}^{\infty}\left(\Delta e^{\frac{\Theta}{2}||c^{\Theta}||_{sp}}\right)^{i}\right]\,.$$

Then, by Proposition 4.1 we obtain

$$||\left(T^{\Theta}\right)^{n+1} 0 - \left(T^{\Theta}\right)^{n} 0||_{sp} \leq \rho(M)^{n}||\left(T^{\Theta}\right) 0||_{sp}\,.$$

Therefore the sequence $\left(T^{\Theta}\right)^{n} 0$ is convergent in the span semi-norm to a function ω satisfying $||T^{\Theta}\omega - \omega||_{sp} = 0$. Consequently, there is a constant λ such that

$$T^{\Theta}\omega = \omega + \lambda.$$

The uniqueness of λ and ω (up to an additive constant) follows from Proposition 4.1.

□

We can summarize the above results in the following form

Theorem 4.2 *Let assumptions* $(A1)$, $(A2)$, $(A3)$, $(A4)$ *and* $(A5)$ *be satisfied. Then there is an admissible trading strategy* **which is a function of current values of the factor levels** *and maximizes the functional* (2).
Moreover, the optimal trading strategy at time t depends only on values of the factor process $X = (x_t)$ at time t and can be computed by solving the equation (5).

The proof immediately follows from Proposition 3.1 and Theorem 4.1.

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