INNER INVARIANT MEANS ON DISCRETE SEMIGROUPS WITH IDENTITY

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Received January 16, 2006

Abstract. In the present paper, we initiate a study of a concept of amenability for discrete semigroups S with identity called strict inner amenability. We first study the relations between the concepts inner amenability, strict inner amenability, and amenability of S. We then introduce some suitable tools and properties to give several characterizations of strict inner amenability of S such as Reiter's condition and Følner's condition. We also offer a characterization of strict inner amenability in terms of a fixed point property. As applications of these results to discrete groups G, we obtain a number of equivalent statements describing strict inner amenability of G.

1 Introduction Throughout this paper, S denotes a discrete semigroup. We denote by $\ell^\infty(S)$ the set of all complex-valued bounded functions on S, and by $1_A \in \ell^\infty(S)$ the characteristic function of a subset A of S. Then $\ell^\infty(S)$ with the complex conjugation as involution, the pointwise operations and the sup-norm $\|.\|_\infty$ is a commutative C^*-algebra with identity $1 := 1_S$.

Also, for $1 \le p < \infty$, let $\ell^p(S)$ denote the Banach space of all complex-valued functions φ on S such that

$$\|\varphi\|_p := \left(\sum_{x\in S} |\varphi(x)|^p\right)^{1/p} < \infty.$$

Then $\ell^1(S)$ with the convolution multiplication defined by

$$(\varphi * \psi)(x) = \sum_{y\in S, x=yz} \varphi(y)\ \psi(z) \qquad (x \in S)$$

is a Banach algebra. Let us recall that every $\varphi \in \ell^p(S)$ is of the form

$$\varphi = \sum_{x\in S} \varphi(x)\ \delta_x,$$

where δ_x denotes the Dirac measure at $x \in S$.

A *mean* on $\ell^\infty(S)$ (or simply on S) is a positive functional m on $\ell^\infty(S)$ with norm one; m is called *invariant* if

$$m({}_xf) = m(f_x) = m(f)$$

for all $x \in S$ and $f \in \ell^\infty(S)$, where ${}_xf$ and f_x are defined on S by

$${}_xf(y) = f(xy) \quad \text{and} \quad f_x(y) = f(yx)$$

2000 *Mathematics Subject Classification.* 43A07, 43A15, 43A20, 46H05.

Key words and phrases. Discrete semigroup; Inner invariant mean; Fixed point property; Kazhdan's property; Reiter's condition and Følner's condition; Strict inner amenability.

for all $y \in S$. The semigroup S is called *amenable* if there is an invariant mean on $\ell^\infty(S)$. We refer the reader to Day [4] for an introductory exposition on amenable semigroup; see also Namioka [13].

A mean m on $\ell^\infty(S)$ is said to be *inner invariant* if

$$m({}_xf) = m(f_x)$$

for all $x \in S$ and $f \in \ell^\infty(S)$. Following Ling [12], S is called *inner amenable* if there is an inner invariant mean on $\ell^\infty(S)$.

In the case where S has an identity e, clearly δ_e as a functional on $\ell^\infty(S)$ is an inner invariant mean, and hence S is inner amenable; in particular, discrete groups are always inner amenable. So, all of the interesting statements equivalent to inner amenability of discrete semigroups obtained in [12] are automatically true for S and of course for discrete groups. This motivates us to consider means on $\ell^\infty(S)$ that are not equal to δ_e.

We say that a discrete semigroup S with identity e is *strictly inner amenable* if there is an inner invariant mean m on $\ell^\infty(S)$ such that $m \neq \delta_e$.

Note that if G is a discrete group, then a mean m is inner invariant if and only if

$$m({}_xf_{x^{-1}}) = m(f)$$

for all $x \in G$ and $f \in \ell^\infty(G)$, where ${}_xf_{x^{-1}}(y) = f(xyx^{-1})$ for all $y \in G$. The study of strict inner amenability for discrete groups was initiated by Effros [10] and pursued by Akemann [2], H. Choda and M. Choda [3], M. Choda [4, 5, 6], Choda and Watatani [7], Kaniuth and Markfort [11], Paschke [14], and Watatani [15]. However, strict inner amenability of discrete semigroups has not been touched so far.

Our purpose in this paper is to initiate a study of strict inner amenability of a discrete semigroup S with identity. In Section 2, we first have shown that the concept of strict inner amenability is stronger than inner amenability and weaker than amenability. We then have introduced a suitable subspace of $\ell^\infty(S)$ to recover some classical characterizations for strict inner amenability of S. Section 3 is devoted to a Reiter's condition describing strict inner amenability. In Section 4, we have characterized strict inner amenability in terms of a fixed point property. Finally, in Section 5, we have investigated some Følner's conditions for strict inner amenability under some cancellative properties on S.

2 The general results

We begin this section with the following result.

Proposition 2.1 *Let S and T be semigroups with identity. If S is strictly inner amenable, then $S \times T$ is strictly inner amenable.*

Proof. Let e and e' denote the identities of S and T respectively. Let n be an inner invariant mean on $\ell^\infty(S)$ with $n \neq \delta_e$. For each $f \in \ell^\infty(S \times T)$ and $t \in T$, define $p(t)f \in \ell^\infty(S)$ by

$$(p(t)f)(s) = f(s,t)$$

for all $s \in S$. It follows that

$$p(e')({}_{(x,y)}f\,) = {}_x(p(y)f) \quad \text{and} \quad p(e')(f_{(x,y)}) = (p(y)f)_x$$

for all $x \in S$ and $y \in T$. Now, define the mean m on $\ell^\infty(S \times T)$ by

$$m(f) = n(p(e')f) \qquad (f \in \ell^\infty(S \times T)).$$

Then for each $x \in S$ and $y \in T$,

$$\begin{aligned} m({}_{(x,y)}f) &= n(p(e')({}_{(x,y)}f)) = n({}_x(p(y)f)) \\ &= n((p(y)f)_x)) = n((p(e')(f_{(x,y)})) \\ &= m(f_{(x,y)}). \end{aligned}$$

That is m is inner invariant mean.

Furthermore, $m \neq \delta_{(e,e')}$. Indeed, we have $n(h) \neq h(e)$ for some $h \in \ell^\infty(S)$. Now, define the function $f \in \ell^\infty(S \times T)$ by $f(s,t) = h(s)$ for all $s \in S$ and $t \in T$. Therefore $m(f) = n(h) \neq h(e) = f(e,e')$ as required. $\square$

Remark 2.2 (a) It is easy to see that any amenable non-trivial semigroup with identity, is strictly inner amenable. But the converse is not true in general. In fact, there exists an extensive family of strictly inner amenable semigroups with identity which are not amenable. Indeed, let S be a strictly inner amenable semigroup with identity; for example a discrete semigroup with non-trivial center. Then for every non-amenable semigroup T with identity, $S \times T$ is not amenable; see for example [8], page 515. Now invoke Proposition 2.1 to conclude that $S \times T$ is strictly inner amenable.

(b) The strictly inner amenable semigroups with identity form a proper subclass of inner amenable semigroups. For example, let S be a set with at least three elements and choose $e \in S$. Then S with the multiplication defined by $xy = x$ for all $x, y \in S - \{e\}$ and $xe = ex = x$ for all $x \in S$ is an inner amenable semigroup with identity e which is not strictly inner amenable.

Proposition 2.3 *Let S be an inner amenable semigroup. Suppose that T is a semigroup with identity e which is not strictly inner amenable and σ is a homomorphism from S onto T. Then $m(f) = m(f1_{\sigma^{-1}(\{e\})})$ for all inner invariant means m on $\ell^\infty(S)$ and $f \in \ell^\infty(S)$.*

Proof. For each $s, a \in S$ and $g \in \ell^\infty(T)$ we have

$$\begin{aligned} [({}_{\sigma(s)}g - g_{\sigma(s)}) \circ \sigma](a) &= g(\sigma(s)\sigma(a)) - g(\sigma(a)\sigma(s)) \\ &= g(\sigma(sa) - g(\sigma(as)) \\ &= (g \circ \sigma)(sa) - (g \circ \sigma)(as) \\ &= {}_s(g \circ \sigma)(a)) - (g \circ \sigma)_s(a). \end{aligned}$$

That is

$$({}_{\sigma(s)}g - g_{\sigma(s)}) \circ \sigma ={}_s (g \circ \sigma) - (g \circ \sigma)_s.$$

Now, suppose m is an inner invariant mean on $\ell^\infty(S)$. Then the mean $n : g \mapsto m(g \circ \sigma)$ is inner invariant on $\ell^\infty(T)$; indeed, for each $s \in S$ we have

$$n({}_{\sigma(s)}g - g_{\sigma(s)}) = n({}_s(g \circ \sigma) - (g \circ \sigma)_s) = 0.$$

Since σ is onto, this implies that n is inner invariant. Thus $n = \delta_e$. This implies that

$$m(1_{\sigma^{-1}(\{e\})}) = m(1_{\{e\}} \circ \sigma) = n(1_{\{e\}}) = 1$$

Therefore for any $f \in \ell^\infty(S)$ we have

$$\begin{aligned} |m(f - f1_{\sigma^{-1}(\{e\})})| &\leq m(|f - f1_{\sigma^{-1}(\{e\})}|) \\ &\leq \|f\|_\infty \, m(1 - 1_{\sigma^{-1}(\{e\})}) = 0 \end{aligned}$$

That is $m(f) = m(f1_{\sigma^{-1}(\{e\})})$ for all $f \in \ell^\infty(S)$. $\square$

As an immediate consequence of Proposition 2.3 we have the following improvement of Lemma 1.2 of Kaniuth and Markfort [11].

Corollary 2.4 *Let G and H be groups, e be the identity of H and σ be a homomorphism from G onto H. If H is not strictly inner amenable, then*

$$m(f) = m(f1_{\sigma^{-1}(\{e\})})$$

] for all inner invariant means m on $\ell^\infty(G)$ and $f \in \ell^\infty(G)$.

Let R be a congruence relation on S; that is, an equivalence relation R such that $xt\ R\ yt$ and $tx\ R\ ty$ for all $x, y, t \in S$ with $x\ R\ y$. We denote by S/R the semigroup of all equivalence classes x/R $(x \in S)$ induced by R with the usual operation

$$(x/R)\ (y/R) = xy/R \qquad (x, y \in S).$$

Corollary 2.5 *Let S be a semigroup with identity e and R be a congruence relation such that S/R is not strictly inner amenable. Then*

$$m(f) = m(f1_{e/R})$$

for all inner invariant means m on $\ell^\infty(S)$ and $f \in \ell^\infty(S)$.

Proof. This follows from Proposition 2.3 together with the fact that the quotient map $\pi : S \longrightarrow S/R$ is an onto homomorphism. $\square$

Before we give some characterizations of strict inner amenability, let us give a lemma.

Lemma 2.6 *Let S be a semigroup with identity e. Then S is strictly inner amenable if and only if there is an inner invariant mean m on $\ell^\infty(S)$ with $m(1_{\{e\}}) = 0$.*

Proof. The "if" part is trivial. To prove the converse, suppose that S is inner amenable, and let n be an inner invariant mean on $\ell^\infty(S)$ not equal to δ_e. It follows that $n(1_{\{e\}}) < 1$. Now define the functional m on $\ell^\infty(S)$ by

$$m(f) = \frac{1}{1 - n(1_{\{e\}})}\, [\, n(f) - n(1_{\{e\}}) f(e)\,] \qquad (f \in \ell^\infty(S)).$$

Then m is an inner invariant mean on $\ell^\infty(S)$ with $m(1_{\{e\}}) = 0$. $\square$

Let S be a semigroup. Let $\mathcal{H}(S)$ denote the complex linear span of functions of the form ${}_xf - f_x$, where $x \in S$ and $f \in \ell^\infty(S)$. Also, let $\mathcal{H}_{\mathbb{R}}(S)$ denote the real linear space of all real-valued functions in $\mathcal{H}(S)$.

Theorem 2.7 *Let S be a semigroup with identity e. Then the following statements are equivalent.*

(a) *S is strictly inner amenable.*

(b) $\sup\{h(x) : x \in S\} \geq 0$ *for all* $h \in \mathcal{H}_{\mathbb{R}}(S) \oplus \mathbb{R}\, 1_{\{e\}}$.

(c) $\inf \left\{\| 1 - h \|_\infty : h \in \mathcal{H}(S) \oplus \mathbb{C}\, 1_{\{e\}} \right\} = 1$.

Proof. (a)⇒(b). Suppose that m is an inner invariant mean on $\ell^\infty(S)$ with $m(1_{\{e\}}) = 0$. Then

$$\sup\{h(x) : x \in S\} \geq m(h) = 0 \quad \text{for all} \quad h \in \mathcal{H}_{\mathbb{R}}(S) \oplus \mathbb{R}\,1_{\{e\}}.$$

(b)⇒(c). It follows from $0 \in \mathcal{H}(S) \oplus \mathbb{C}\,1_{\{e\}}$ that

$$\inf\left\{\|1-h\|_\infty : h \in \mathcal{H}(S) \oplus \mathbb{C}\,1_{\{e\}}\right\} \leq 1.$$

If the equality does not hold, then

$$\sup\{-\mathrm{Re}\, h(x) : x \in S\} < 0$$

for some $h \in \mathcal{H}(S) \oplus \mathbb{C}\,1_{\{e\}}$. Since

$$-\mathrm{Re}\, h \in \mathcal{H}_{\mathbb{R}}(S) \oplus \mathbb{R}\,1_{\{e\}},$$

this contradicts the assumption.

(c)⇒(a). By the Hahn-Banach theorem, there is a linear functional n on $\ell^\infty(S)$ with norm one such that $n(\mathcal{H}(S) \oplus \mathbb{C}\,1_{\{e\}}) = \{0\}$ and

$$n(1) = \inf\left\{\|1-h\|_\infty : h \in \mathcal{H}(S) \oplus \mathbb{C}\,1_{\{e\}}\right\}.$$

It follows that if (c) holds, then n is an inner invariant mean on $\ell^\infty(S)$ with $n(1_{\{e\}}) = 0$. □

Theorem 2.8 *Let S be a left or right cancellative semigroup with identity e. Then S is strictly inner amenable if and only if $\mathcal{H}(S) \oplus \mathbb{C}\,1_{\{e\}}$ is not norm dense in $\ell^\infty(S)$.*

Proof. The "only if" part is trivial. To prove the converse, choose a nonzero self-adjoint functional $n \in \ell^\infty(S)^*$ such that

$$n(\mathcal{H}(S) \oplus \mathbb{C}\,1_{\{e\}}) = \{0\}.$$

Then we may write $n = n^+ - n^-$, where

$$\begin{aligned} n^+(f) &= \sup\{n(g) : 0 \leq g \leq f\} \\ n^-(f) &= -\inf\{n(g) : 0 \leq g \leq f\} \end{aligned}$$

for all $f \in \ell^\infty(S)$ with $f \geq 0$. Now the argument as in the proof of Theorem 2 of [12] shows that n^+ and n^- are inner invariant positive functional on $\ell^\infty(S)$. The result will follow if we note that $n(1_{\{e\}}) = 0$ and therefore $n^+(1_{\{e\}}) = n^-(1_{\{e\}}) = 0$. □

Corollary 2.9 *Let G be a discrete group with identity e. Then the following statements are equivalent.*

(a) *G is strictly inner amenable.*

(b) $\sup\{h(x) : x \in G\} \geq 0$ *for all* $h \in \mathcal{H}_{\mathbb{R}}(G) \oplus \mathbb{R}\,1_{\{e\}}$.

(c) $\inf\left\{\| 1-h \|_\infty : h \in \mathcal{H}(G) \oplus \mathbb{C}\,1_{\{e\}}\right\} = 1$.

(d) $\mathcal{H}(G) \oplus \mathbb{C}\,1_{\{e\}}$ *is not norm dense in* $\ell^\infty(G)$.

3 Reiter's condition We commence with the following characterizations of strict inner amenability called Reiter's condition. First, let us recall that a mean in $\ell^1(S)$ is called a *finite mean* if it is a convex combination of the Dirac measures.

Theorem 3.1 *Let S be a semigroup with identity and $1 \leq p < \infty$. Then the following statements are equivalent.*

(a) *S is strictly inner amenable.*

(b) *There exists a net (φ_α) of finite means such that $\varphi_\alpha(e) = 0$ and*

$$\|\delta_x * \varphi_\alpha - \varphi_\alpha * \delta_x\|_1 \longrightarrow 0 \qquad (x \in S).$$

(c) *There exists a net (ϕ_α) in $\ell^p(S)$ with $\|\phi_\alpha\|_p = \sum_{x\in S} \phi_\alpha(x)^p = 1$ such that $\phi_\alpha(e) = 0$ and*

$$\|\delta_x * \phi_\alpha - \phi_\alpha * \delta_x\|_p \longrightarrow 0 \qquad (x \in S).$$

Proof. (a) $\Rightarrow$ (b). Suppose that there is an inner invariant mean m on $\ell^\infty(S)$ such that $m \neq \delta_e$. Since the finite means are weak* dense in the set of means, we can find a net (ξ_β) of finite means such that

$$\delta_x * \xi_\beta - \xi_\beta * \delta_x \longrightarrow 0 \qquad (x \in S)$$

in the weak* topology of $\ell^\infty(S)^*$, the dual of $\ell^\infty(S)$. Since $m(1_{\{e\}}) = 0$, we may assume that $\xi_\beta(e) = 0$ for all β. Now, by a standard argument, we can obtained a net (φ_α) such that every φ_α is a convex combination of the elements of (ξ_β) and

$$\|\delta_x * \varphi_\alpha - \varphi_\alpha * \delta_x\|_1 \longrightarrow 0 \qquad (x \in S).$$

Clearly $\varphi_\alpha(e) = 0$ for all α.

(b) $\Rightarrow$ (c). For every α, set $\phi_\alpha = \varphi_\alpha^{1/p}$ and note that

$$\|\phi_\alpha\|_p = \sum_{x\in S} \phi_\alpha(x)^p = 1 \quad \text{and} \quad \phi_\alpha(e) = 0.$$

Furthermore, for each $x \in S$ we have

$$\|\delta_x * \phi_\alpha - \phi_\alpha * \delta_x\|_p \leq \|\delta_x * \varphi_\alpha - \varphi_\alpha * \delta_x\|_1^{1/p} \longrightarrow 0.$$

(c) $\Rightarrow$ (a). Since for every $x \in S$,

$$\begin{aligned} \|\delta_x * \phi_\alpha^p - \phi_\alpha^p * \delta_x\| &= \|(\delta_x * \phi_\alpha - \phi_\alpha * \delta_x)^p\|_1 \\ &= \|\delta_x * \phi_\alpha - \phi_\alpha * \delta_x\|_p^p \longrightarrow 0. \end{aligned}$$

it follows that

$$\delta_x * \phi_\alpha^p - \phi_\alpha^p * \delta_x \longrightarrow 0$$

in the weak* topology of $\ell^\infty(S)^*$. On the other hand, $\|\phi_\alpha^p\|_1 = 1$, $\phi_\alpha^p \geq 0$ and $\phi_\alpha^p(e) = 0$ for each α. Therefore any weak* cluster point of (ϕ_α^p) is an inner invariant mean with $m \neq \delta_e$. $\square$

Let G be a discrete group and $f \in \ell^\infty(G)$. Then $f^* \in \ell^\infty(G)$ is defined by

$$f^*(x) = \overline{f(x^{-1})} \quad (x \in G).$$

Recall that a mean m on $\ell^\infty(G)$ is *symmetric* if

$$m(f^*) = m(f) \qquad (f \in \ell^\infty(G)).$$

Theorem 3.2 *Let G be a discrete group with identity e and $1 \le p < \infty$. Then the following statements are equivalent.*

(a) *G is strictly inner amenable.*

(b) *There exists a net (ψ_α) of symmetric finite means such that $\psi_\alpha(e) = 0$ and*

$$\|\delta_x * \psi_\alpha - \psi_\alpha * \delta_x\|_1 \longrightarrow 0 \qquad (x \in G).$$

(c) *There exists a net (ψ_α) in $\ell^p(G)$ with $\|\psi_\alpha\|_p = \sum_{x\in S} \psi_\alpha(x)^p = 1$ and $\psi_\alpha = \psi_\alpha^*$ such that $\psi_\alpha(e) = 0$ and*

$$\|\delta_x * \psi_\alpha - \psi_\alpha * \delta_x\|_p \longrightarrow 0 \qquad (x \in G).$$

(d) *There exists a symmetric inner invariant mean m on $\ell^\infty(G)$ such that $m(1_{\{e\}}) = 0$.*

Proof. Suppose that S is strictly inner amenable. By Theorem 3.1, there exists a net (φ_α) of finite means such that $\varphi_\alpha(e) = 0$ and

$$\|\delta_x * \varphi_\alpha - \varphi_\alpha * \delta_x\|_1 \longrightarrow 0 \qquad (x \in G).$$

So if we put $\psi_\alpha = 2^{-1}(\varphi_\alpha + \varphi_\alpha^*)$, then ψ_α is a symmetric finite mean, $\psi_\alpha(e) = 0$ and

$$\|\delta_x * \psi_\alpha - \psi_\alpha * \delta_x\|_1 \longrightarrow 0 \qquad (x \in G).$$

That is (b) holds. The other implications are trivial. □

4 A fixed point property Let S be a semigroup and let us point out that $\ell^\infty(S)$ can be considered as the dual space $\ell^1(S)^*$ of $\ell^1(S)$ under the duality given by

$$f(\varphi) = \sum_{x\in S} f(x)\,\varphi(x) \qquad (f \in \ell^\infty(S),\ \varphi \in \ell^1(S)\,).$$

Furthermore, $\ell^\infty(S)^*$ is a Banach algebra with the *first Arens product* $\odot$ defined by the equations

$$\begin{aligned} (m \odot n)(f) &= m(nf) \\ (nf)(\varphi) &= n(f\varphi), \\ (f\varphi)(\psi) &= f(\varphi * \psi) \end{aligned}$$

for all $m, n \in \ell^\infty(S)^*, f \in \ell^\infty(S)$, and $\varphi, \psi \in \ell^1(S)$; see Arens [1].

Let $\mathcal{B}(\ell^\infty(S)^*)$ denote the Banach space of bounded linear operators on $\ell^\infty(S)^*$. For every $\varphi \in \ell^1(S)$, define the operator $T_\varphi \in \mathcal{B}(\ell^\infty(S)^*)$ by

$$T_\varphi(m) = \varphi \odot m \qquad (m \in \ell^\infty(S)^*).$$

and set $T_x := T_{\delta_x}$ for all $x \in S$.

By *weak* operator topology* on $\mathcal{B}(\ell^\infty(S)^*)$, we shall mean the locally convex topology determined by the family

$$\{q(m, f) : m \in \ell^\infty(S)^*, f \in \ell^\infty(S)\}$$

of seminorms on $\mathcal{B}(\ell^\infty(S)^*)$, where

$$q(m, f)(T) = |T(m)(f)| \quad (T \in \mathcal{B}(\ell^\infty(S)^*)).$$

We denote by $\mathcal{P}(\ell^\infty(S)^*)$ the closure of the set

$$\left\{T_\varphi : \varphi \in \ell^1(S), \ \ \|\varphi\|_1 = \sum_{x\in S}\varphi(x) = 1\right\}$$

in the weak* operator topology of $\mathcal{B}(\ell^\infty(S)^*)$. Note that $T_e \in \mathcal{P}(\ell^\infty(S)^*)$ is the identity operator on $\ell^\infty(S)^*$, and that $\mathcal{P}(\ell^\infty(S)^*)$ is a subsemigroup of the semigroup $\mathcal{B}(\ell^\infty(S)^*)$ with the ordinary multiplication of linear operators.

Theorem 4.1 *Let S be a semigroup with identity e. Then the following assertions are equivalent.*

(a) *S is strictly inner amenable.*

(b) *There exists $T \in \mathcal{P}(\ell^\infty(S)^*)$ with $T \neq T_e$ such that $T_xT = TT_x$ for all $x \in S$.*

Proof. The operator algebra $\mathcal{B}(\ell^\infty(S)^*)$ can be identified with the dual space $(\ell^\infty(S)^*\widehat{\otimes}\ell^\infty(S))^*$ of the projective tensor product $\ell^\infty(S)^*\widehat{\otimes}\ell^\infty(S)$ in a natural way; see for example Corollary VIII.2.2 of [9]. In particular, the weak* operator topology of $\mathcal{B}(\ell^\infty(S)^*)$ coincides with the weak* topology of $(\ell^\infty(S)^* \otimes \ell^\infty(S))^*$ on bounded subsets of $\mathcal{B}(\ell^\infty(S)^*)$, and therefore $\mathcal{P}(\ell^\infty(S)^*)$ is compact in the weak* operator topology of $\mathcal{B}(\ell^\infty(S)^*)$ by the Banach-Alaoglu theorem.

Now, suppose that S is strictly inner amenable. Using Theorem 3.1, there exists a net (φ_α) of finite means such that $\varphi_\alpha(e) = 0$ and

$$\| \varphi_\alpha * \delta_x - \delta_x * \varphi_\alpha \|_1 \longrightarrow 0$$

for all $x \in S$. It follows that there is a subnet (φ_β) of (φ_α) such that $T_{\varphi_\beta} \longrightarrow T$ in the weak* operator topology for some $T \in \mathcal{P}(\ell^\infty(S)^*)$ with $\| T \| \leq 1$. Thus, on the one hand, for each $x \in S$ we have

$$\| T_{\varphi_\beta}T_x - T_xT_{\varphi_\beta} \| \leq \| \varphi_\beta * \delta_x - \delta_x * \varphi_\beta \|_1 \longrightarrow 0,$$

and on the other hand

$$T_{\varphi_\beta}T_x \longrightarrow TT_x$$

and

$$T_xT_{\varphi_\beta} \longrightarrow T_xT$$

in the weak* operator topology. This shows that $TT_x = T_xT$. Furthermore,

$$T_{\varphi_\beta}(\delta_e)(1_{\{e\}}) = \varphi_\beta(1_{\{e\}}) = \varphi_\beta(e) = 0.$$

for all β from which it follows that $T(\delta_e)(1_{\{e\}}) = 0$. In particular, $T \neq T_e$.

Conversely, suppose that (b) holds and choose an element T of $\mathcal{P}(\ell^\infty(S)^*)$ such that $T \neq T_e$ and $TT_x = T_xT$ for all $x \in S$. To prove (a), choose a net (φ_α) in $\ell^1(S)$ with

$$\|\varphi_\alpha\|_1 = \sum_{x\in S}\varphi_\alpha(x) = 1$$

such that $T_{\varphi_\alpha} \longrightarrow T$ in the weak* operator topology of $\mathcal{B}(\ell^\infty(S)^*)$. Let m be a weak* cluster point of (φ_α) in $\ell^\infty(S)^*$. We show that m is an inner invariant mean on $\ell^\infty(S)$ with $m \neq \delta_e$. It is clear that m is a mean. Also, for each $x \in S$ we have

$$\begin{aligned}(T_{\varphi_\alpha}\delta_x)(f) &= (\varphi_\alpha \odot \delta_x)(f)\\ &= \varphi_\alpha(\delta_x f)\\ &\to m(\delta_x f)\\ &= (m \odot \delta_x)(f)\end{aligned}$$

and hence $T(\delta_x) = m \odot \delta_x$. This together with that $T \neq T_e$ shows that $m \neq \delta_e$. Furthermore, for each $f \in \ell^\infty(S)$ and $x \in S$ we have

$$\begin{aligned} m \odot \delta_x &= T(\delta_x) \\ &= T(\delta_x \odot \delta_e) \\ &= T(T_x \delta_e) \\ &= T_x(T\delta_e) \\ &= T_x(m \odot \delta_e) \\ &= T_x m \\ &= \delta_x \odot m. \end{aligned}$$

It follows that m is also inner invariant. □

Recall that a discrete group G satisfies *Kazhdan's property T* if the trivial representation is isolated in the dual space $\widehat{G}$ of equivalence classes of irreducible unitary representations of G endowed with the Jacobson topology; this property, presented by Kazhdan in 1968, has important applications to discrete groups. G is said to be an *ICC* group, if $\{xyx^{-1} : x \in G\}$ is infinite for all $y \in G \setminus \{e\}$.

Corollary 4.2 *Let G be an ICC discrete group with identity e. If there exists $T \in \mathcal{P}(\ell^\infty(G)^*)$ with $T \neq T_e$ such that $T_x T = T T_x$ for all $x \in G$, then G does not satisfy Kazhdan's property T.*

Proof. If G has Kazhdan's property T, then δ_e is the only inner invariant mean on $\ell^\infty(G)$ by Theorem 1.1 of [11]. Now apply Theorem 4.1. □

5 Følner's condition Our results in this section give some characterizations of strict inner amenability known as Følner's condition. In the following, we denote the cardinal number of a set A by $|A|$.

Proposition 5.1 *Let S be a right cancellative semigroup with identity e. Then S is strictly inner amenable if for every finite set $F \subseteq S$ and $\varepsilon > 0$, there exists a finite nonempty set $A \subseteq S \setminus \{e\}$ such that $|Ax \setminus xA| < \varepsilon |A|$ for all $x \in F$.*

Proof. By the assumption, there exists a net of finite nonempty sets $A_\alpha \subseteq S$ with $e \notin A_\alpha$ such that

$$|A_\alpha x \setminus x A_\alpha| / |A_\alpha| \longrightarrow 0 \quad (x \in S).$$

By Lemma 9 of [12] we have

$$\|1_{A_\alpha} * \delta_x - \delta_x * 1_{A_\alpha}\|_1 = |A_\alpha x \setminus x A_\alpha|.$$

Set $\varphi_\alpha = |A_\alpha|^{-1} 1_{A_\alpha}$. Then $\varphi_\alpha(e) = 0$ for each α and

$$\|\delta_x * \varphi_\alpha - \varphi_\alpha * \delta_x\|_1 \longrightarrow 0 \qquad (x \in S).$$

Now the proof is complete by Theorem 3.1. □

Theorem 5.2 *Let S be a cancellative semigroup with identity e. Then S is strictly inner amenable if and only if for every finite set $F \subseteq S$ and $\varepsilon > 0$, there is a finite nonempty set $A \subseteq S \setminus \{e\}$ such that $|Ax \Delta xA| < \varepsilon |A|$ for all $x \in F$.*

Proof. The "if" part follows from Proposition 5.1 and its dual together with the fact that for any $A \subseteq S$,

$$|Ax\Delta xA| = |Ax \setminus xA| + |xA \setminus Ax|.$$

Conversely, suppose that S is strictly inner amenable. By Theorem 3.1, for every finite subset F of S and $\varepsilon > 0$, there exist a finite mean φ with $\varphi(e) = 0$ such that

$$|F| \, \|\delta_x * \varphi - \varphi * \delta_x\|_1 < \varepsilon$$

for all $x \in F$. We know from [8] that φ is of the form

$$\varphi = \sum_{i=1}^{n} r_i \, |A_i|^{-1} \, 1_{A_i}$$

for some nonempty finite subsets $A_1, ..., A_n$ with $A_1 \subsetneqq ... \subsetneqq A_n$ and positive numbers $r_1, ..., r_n$ with $r_1 + ... + r_n = 1$. By Lemma 13 of [12] we have

$$\|\delta_x * \varphi - \varphi * \delta_x\|_1 = \sum_{i=1}^{n} r_i \, |A_i|^{-1} \, |A_i x \Delta x A_i| \qquad (x \in S).$$

It follows that

$$\sum_{i=1}^{n} \sum_{x \in F} r_i \, |A_i|^{-1} \, |A_i x \Delta x A_i| < \varepsilon.$$

Therefore there is $1 \leq i_0 \leq n$ such that

$$|A_{i_0} x \Delta x A_{i_0}| < \varepsilon |A_{i_0}|$$

for all $x \in F$. Since $\varphi(e) = 0$ we have $e \notin A_{i_0}$ and the proof is complete. □

Before stating our last result, recall that a subset A of a group G is *symmetric* if $A = \{x^{-1} : x \in A\}$.

Theorem 5.3 *Let G be a discrete group with identity e. Then G is strictly inner amenable if and only if for every finite set $F \subseteq S$ and $\varepsilon > 0$, there is a finite nonempty symmetric set $A \subseteq S \setminus \{e\}$ such that $|Ax\Delta xA| < \varepsilon|A|$ for all $x \in F$.*

Proof. First recall from Lemma 18 of [12] that if

$$\varphi = \sum_{i=1}^{n} r_i \, |A_i|^{-1} \, 1_{A_i}$$

is a symmetric finite mean, then the sets A_i are symmetric. So an argument similar to the proof of Theorem 5.2 by the aid of Theorem 3.2 instead of Theorem 3.1 gives the result. □

Acknowledgment. This research was partially supported by the Center of Excellence for Mathematics at the Isfahan University of Technology, Iran. The second author gratefully acknowledges support by the Isfahan University of Technology through the research project 1MAD831.

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