ON IMPLICATIVE BCI-ALGEBRAS

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Abstract. In this paper, we give an axiom system of implicative BCI-algebras, investigate some properties of the branches of an implicative BCI-algebra, which are similar to those of implicative BCK-algebras, and show that for every initial section of an implicative BCI-algebra, it with respect to the BCI-ordering forms a Boolean algebra.

As is well known, commutative BCK-algebras, positive implicative BCK-algebras and implicative BCK-algebras are three classes of the most important BCK-algebras. In order to get the similar classes in BCI-algebras, J. Meng and X. L. Xin in [9], [11] and [10] introduced commutative BCI-algebras, positive implicative BCI-algebras and implicative BCI-algebras respectively, and investigated their fundamental properties similar to those of the corresponding algebras in BCK-algebras. And the author in [1], [2] and [3] gave some further properties of theirs.

The ideas of this paper are originated from [1]. Like [1], we will mainly use lattices and branches as well as initial sections to explore implicative BCI-algebras in this paper. And we will obtain a number of interesting results similar to those of implicative BCK-algebras.

0 Preliminaries For the notations and elementary properties of BCK and BCI-algebras, we refer the reader to [5], [4] and [8]. And we will use some familiar notions and properties of lattices without explanation.

Recall that according to the H. S. Li's axiom system (see [7]), a *BCI-algebra* $(X; *, 0)$ means that it is an algebra of type $(2, 0)$, satisfying the following conditions: for any x, y, $z \in X$,

BCI-1 $((x*y)*(x*z))*(z*y)=0$,
BCI-2 $x*0=x$,
BCI-3 $x*y=0$ and $y*x=0$ imply $x=y$.

It is known that given a BCI-algebra X, the following identities are valid:

$$(x*y)*z=(x*z)*y, \tag{0.1}$$

$$x*y=x*(x*(x*y)), \tag{0.2}$$

$$0*(x*y)=(0*x)*(0*y), \tag{0.3}$$

$$(x*y)*x=0*x. \tag{0.4}$$

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And X with respect to its *BCI-ordering* $\leqslant$ forms a partially ordered set $(X; \leqslant)$ satisfying the following quasi-identities:

$$(x * y) * (x * z) \leqslant z * y, \tag{0.5}$$

$$(x * z) * (y * z) \leqslant x * y, \tag{0.6}$$

$$(x * (x * y)) * (x * (x * z)) \leqslant y * z, \tag{0.7}$$

where the binary relation $\leqslant$ on X is defined as follows: $x \leqslant y$ if and only if $x * y = 0$. Moreover, the following assertions hold: for any x, y, $z \in X$,

$$x \leqslant y \quad \text{implies} \quad z * y \leqslant z * x, \tag{0.8}$$

$$x \leqslant y \quad \text{implies} \quad x * z \leqslant y * z. \tag{0.9}$$

A *minimal element* a of X means that a is an element in X such that $x \leqslant a$ (i.e., $x*a = 0$) implies $x = a$ for any $x \in X$. Given a minimal element a of X, the set $\{x \in X \mid x \geqslant a\}$ is called a *branch* of X, denoted by $V(a)$.

Given an element c in X, the set $\{x \in X \mid x \leqslant c\}$ is called an *initial section* of X, denoted by $A(c)$.

Theorem 0.1 ([8], §1.3). *Assume that P is the set of all minimal elements of a BCI-algebra X. Then the collection $\{V(a) \mid a \in P\}$ of branches of X forms a partition of X, that is, $X = \bigcup_{a\in P} V(a)$ and $V(a) \cap V(b) = \varnothing$ if $a \neq b$ for any a, $b \in P$. Moreover, the following fold: for any x, $y \in V(a)$,*

$$0 * (0 * x) = a, \tag{0.10}$$

$$0 * (x * y) = 0. \tag{0.11}$$

Definition ([9], [11] and [10]). A BCI-algebra X is called *commutative* if

$$x \leqslant y \;\; \text{implies} \;\; x = y * (y * x) \quad \text{for all } x,\, y \in X;$$

it is called *positive implicative* if

$$(x * (x * y)) * (y * x) = x * (x * (y * (y * x))) \;\; \text{for all } x,\, y \in X;$$

it is called *implicative* if

$$x * (x * y) = (y * (y * x)) * (x * y) \;\; \text{for all } x,\, y \in X. \tag{0.12}$$

Theorem 0.2 ([8], §2.4). *A BCI-algebra X is commutative if and only if for any branch $V(a)$ of X, $x \in V(a)$ and $y \in V(a)$ imply*

$$x * (x * y) = y * (y * x). \tag{0.13}$$

Moreover, $(V(a); \leqslant)$ forms a lower semilattice such that for any x, $y \in V(a)$,

$$x \wedge y = y * (y * x), \tag{0.14}$$

$$x * y = x * (x \wedge y). \tag{0.15}$$

Theorem 0.3 ([1], Theorem 3.2). *If $A(c)$ is an initial section of a commutative BCI-algebra X, then $(A(c); \leqslant)$ is a distributive lattice with*

$$x \wedge y = y * (y * x) \quad \textit{and} \quad x \vee y = c * ((c * x) \wedge (c * y)).$$

Theorem 0.4 ([3], Corollary 3). *A BCI-algebra X is positive implicative if and only if*

$$x * y = ((x * y) * y) * (0 * y) \quad \text{for any } x,\, y \in X. \tag{0.16}$$

Thus

$$x * y = (x * y) * y \quad \text{if } y \geqslant 0. \tag{0.17}$$

Theorem 0.5 ([11], Theorem 6). *A BCI-algebra X is implicative if and only if it is commutative and positive implicative.*

1 An axiom system of implicative BCI-algebras Let's begin our discussion with giving an axiom system of implicative BCI-algebras.

Theorem 1.1. *An algebra $(X; *, 0)$ of type $(2, 0)$ is an implicative BCI-algebra if and only if it satisfies the following identities:*

(1) $x * 0 = x$;
(2) $x * x = 0$;
(3) $(x * y) * z = (x * z) * y$;
(4) $(x * z) * (x * y) = ((y * z) * (y * x)) * (x * y)$.

Proof. Necessity. (1) is just BCI-2. Repeatedly applying BCI-2, we have

$$x * x = ((x * 0) * (x * 0)) * (0 * 0).$$

Then BCI-1 implies $x * x = 0$, (2) holding. By (0.1), (3) is true. By the definition of the implicativity of X, we have

$$x * (x * y) = (y * (y * x)) * (x * y).$$

Right $*$ multiplying both sides of the last identity by z, we derive

$$(x * (x * y)) * z = ((y * (y * x)) * (x * y)) * z.$$

Then (0.1) gives $(x * z) * (x * y) = ((y * z) * (y * x)) * (x * y)$, showing (4).

Sufficiency. BCI-2 is just (1). Putting $z = 0$ in (4) and using (1), we have

$$x * (x * y) = (y * (y * x)) * (x * y), \tag{1.1}$$

which is the implicativity of X. It is easily seen from (1.1) and (1) that BCI-3 is true. It remains to show BCI-1. In fact, by (4), we have

$$(x * y) * (x * z) = ((z * y) * (z * x)) * (x * z).$$

Right $*$ multiplying both sides of the last identity by $z * y$, we obtain

$$((x * y) * (x * z)) * (z * y) = (((z * y) * (z * x)) * (x * z)) * (z * y). \tag{1.2}$$

By (3), the right side of (1.2) coincides with

$$(((z * y) * (z * y)) * (z * x)) * (x * z). \tag{1.3}$$

By (2), $(z * y) * (z * y) = 0 = z * z$, then (1.3) is identical with

$$((z * z) * (z * x)) * (x * z). \tag{1.4}$$

Using (3) once again, (1.4) is the same as

$$((z * (z * x)) * (x * z)) * z. \tag{1.5}$$

By (1.1), (1.5) is identical with $(x * (x * z)) * z$, that is, $(x * z) * (x * z)$ by (3). Now, since $(x * z) * (x * z) = 0$ by (2), we see that (1.2) is equivalent to

$$((x * y) * (x * z)) * (z * y) = 0,$$

showing BCI-1. The proof is complete. □

2 On branches of implicative BCI-algebras We now consider the branches of an implicative BCI-algebra. It is known very well that the identity $x * (y * x) = x$ is just the implicativity of BCK-algebras. It is interesting that the same identity holds in a branch of an implicative BCI-algebra.

Proposition 2.1. *Let X be a BCI-algebra. If X is implicative, then for any branch $V(a)$ of X, $x \in V(a)$ and $y \in V(a)$ imply $x * (y * x) = x$.*

Proof. Since x, $y \in V(a)$, we have $0 * (x * y) = 0$ by (0.11). Then (0.4) gives

$$(x * (y * x)) * x = 0 * (y * x) = 0. \tag{2.1}$$

On the other hand, replacing y by $y * x$ in (0.12), we have

$$x * (x * (y * x)) = ((y * x) * ((y * x) * x)) * (x * (y * x)). \tag{2.2}$$

Also, since every implicative BCI-algebra is positive implicative, by (0.16), we derive

$$y * x = ((y * x) * x) * (0 * x). \tag{2.3}$$

Right $*$ multiplying both sides of (2.3) by $(y * x) * x$, it follows

$$(y * x) * ((y * x) * x) = (((y * x) * x) * (0 * x)) * ((y * x) * x). \tag{2.4}$$

By (0.4), the right side of (2.4) is equal to $0 * (0 * x)$. Then

$$(y * x) * ((y * x) * x) = 0 * (0 * x). \tag{2.5}$$

Right $*$ multiplying both sides of (2.5) by $x * (y * x)$, it yields

$$((y * x) * ((y * x) * x)) * (x * (y * x)) = (0 * (0 * x)) * (x * (y * x)).$$

Comparison with (2.2) gives

$$x * (x * (y * x)) = (0 * (0 * x)) * (x * (y * x)),$$

which means from (0.1) that

$$x * (x * (y * x)) = (0 * (x * (y * x))) * (0 * x). \tag{2.6}$$

Moreover, since x, $y \in V(a)$, by (0.3) and (0.11) as well as BCI-2, we obtain

$$0 * (x * (y * x)) = (0 * x) * (0 * (y * x)) = (0 * x) * 0 = 0 * x.$$

Now, substituting $0 * x$ for $0 * (x * (y * x))$ in (2.6), and noticing $(0 * x) * (0 * x) = 0$, the following holds:

$$x * (x * (y * x)) = (0 * x) * (0 * x) = 0. \tag{2.7}$$

Combining (2.1) with (2.7) and using BCI-3, it follows $x * (y * x) = x$. □

It is a pity that unlike Theorem 0.2, the converse of Proposition 2.1 is not true as shown in the following counter example.

Example 2.1. The set $X = \{0, 1, 2, 3\}$ together with the operation $*$ on X given by the Cayley table

$*$	0	1	2	3
0	0	0	2	2
1	1	0	2	2
2	2	2	0	0
3	3	2	1	0

forms a BCI-algebra (see [6], the author H. Jiang denotes it by I_{4-2-1}). It is not difficult to see that the whole minimal elements of X are 0 and 2, and the branches $V(0) = \{0, 1\}$ and $V(2) = \{2, 3\}$. Now, it is easy to verify that for any branch $V(a)$ of X, $x \in V(a)$ and $y \in V(a)$ imply $x * (y * x) = x$. However, X is not implicative. That is because

$$3 * (3 * 1) = 1 \neq 0 = (1 * (1 * 3)) * (3 * 1).$$

Nevertheless, we have still the next interesting fact.

Proposition 2.2. *Let X be a BCI-algebra. If for any branch $V(a)$ of X, $x \in V(a)$ and $y \in V(a)$ imply $x * (y * x) = x$, then X is commutative.*

Proof. Let x and y be any elements in X such that $x \leqslant y$ (i.e., $x * y = 0$). By Theorem 0.1, there exists a minimal element a of X such that $x \in V(a)$. Since $a \leqslant x$ and $x \leqslant y$, we obtain $a \leqslant y$, that is, $y \geqslant a$. Then $y \in V(a)$. So our hypothesis gives $x * (y * x) = x$. Hence (0.6) implies

$$x * (y * (y * x)) = (x * (y * x)) * (y * (y * x)) \leqslant x * y = 0.$$

In other words, $x \leqslant y * (y * x)$. The opposite inequality is naturally true. Therefore $x = y * (y * x)$, and X is commutative. □

As an implicative BCI-algebra X must be commutative, according to Theorem 0.2, every branch $V(a)$ of X forms a lower semilattice $(V(a); \leqslant)$, thus the greatest lower bound of any two elements in $V(a)$ exists. And we have the following analogy.

Proposition 2.3. *Let X be an implicative BCI-algebra and $V(a)$ be a branch of X. Then for any x, y, $z \in V(a)$,*

(1) $(x * y) \wedge (y * x) = 0$;
(2) $(x \wedge y) * z = (x * z) \wedge (y * z)$;
(3) *the least upper bound $(z * x) \vee (z * y)$ of $z * x$ and $z * y$ exists and*

$$z * (x \wedge y) = (z * x) \vee (z * y).$$

Proof. (1) Since x, $y \in V(a)$, by (0.1) and Proposition 2.1, we have

$$(y * x) * (x * y) = (y * (x * y)) * x = y * x.$$

Then (0.14) gives

$$(x * y) \wedge (y * x) = (y * x) * ((y * x) * (x * y)) = (y * x) * (y * x) = 0.$$

(2) Since $x \wedge y \leqslant x$ and $x \wedge y \leqslant y$, it is easy to see from (0.9) that $(x \wedge y) * z$ is a lower bound of $x * z$ and $y * z$. Let t be any lower bound of $x * z$ and $y * z$. Then $t \leqslant x * z$ and $t \leqslant y * z$. By $t \leqslant x * z$ and (0.9), we have

$$t * ((x \wedge y) * z) \leqslant (x * z) * ((x \wedge y) * z).$$

Also, by (0.6) and (0.15), we obtain

$$(x * z) * ((x \wedge y) * z) \leqslant x * (x \wedge y) = x * y.$$

So, $t * ((x \wedge y) * z) \leqslant x * y$. Similarly, $t * ((x \wedge y) * z) \leqslant y * x$. Therefore (1) implies

$$t * ((x \wedge y) * z) \leqslant (x * y) \wedge (y * x) = 0.$$

That is, $t \leqslant (x \wedge y) * z$. We have shown that $(x \wedge y) * z$ is the greatest lower bound of $x * z$ and $y * z$. Consequently, $(x \wedge y) * z = (x * z) \wedge (y * z)$.

(3) It is easy to verify from (0.8) that $z * (x \wedge y)$ is an upper bound of $z * x$ and $z * y$. Let t be any upper bound of $z * x$ and $z * y$. Then $z * x \leqslant t$ and $z * y \leqslant t$. By $z * y \leqslant t$ and (0.8), we have $z * t \leqslant z * (z * y)$, that is, $x * t \leqslant y * (y * z)$ by (0.13). Then (0.9) gives

$$(z * t) * (y * (y * x)) \leqslant (y * (y * z)) * (y * (y * x)). \tag{2.8}$$

By (0.1) and (0.14), the left side of (2.8) is equal to $(z * (x \wedge y)) * t$; by (0.7), the right side is less than or equal to $z * x$. So, $(z * (x \wedge y)) * t \leqslant z * x$. Thus (0.9) implies

$$((z * (x \wedge y)) * t) * t \leqslant (z * x) * t.$$

Since $z * x \leqslant t$ and 0 is a minimal element of X, we derive

$$((z * (x \wedge y)) * t) * t = 0. \tag{2.9}$$

Also, since $z, x \in V(a)$, by (0.11), we have $0 * (z * x) = 0$, namely, $0 \leqslant z * x$. Note that $z * x \leqslant t$, it follows $0 \leqslant t$, that is, $t \geqslant 0$. Hence (0.17) implies

$$(z * (x \wedge y)) * t = ((z * (x \wedge y)) * t) * t. \tag{2.10}$$

Now, comparing (2.10) with (2.9), we derive $(z * (x \wedge y)) * t = 0$, i.e., $z * (x \wedge y) \leqslant t$. We have shown that $z * (x \wedge y)$ is just the least upper bound of $z * x$ and $z * y$. Therefore $(z * x) \vee (z * y)$ exists and $z * (x \wedge y) = (z * x) \vee (z * y)$. □

It is not difficult to see that two elements in a branch of an implicative BCI-algebra have generally not their least upper bound. If the least upper bound exists, we also have the following analogy.

Proposition 2.4. *Let x and y be any elements in a branch $V(a)$ of a BCI-algebra X. If the least upper bound $x \vee y$ of x and y exists, then the following hold:*

(1) *$(x \vee y) * x = y * x$ and $(x \vee y) * y = x * y$;*
(2) *the least upper bound $(x * z) \vee (y * z)$ of $x * z$ and $y * z$ exists and*

$$(x \vee y) * z = (x * z) \vee (y * z) \quad \text{for any } z \in V(a);$$

(3) *$z * (x \vee y) = (z * x) \wedge (z * y)$ for any $z \in V(a)$.*

Proof. (1) If $x \vee y$ exists, then there is $c \in X$ such that c is an upper bound of x and y. So x and y are in the initial section $A(c)$. Now, by Theorem 0.3, we have

$$x \vee y = c * ((c * x) \wedge (c * y)). \tag{2.11}$$

Right $*$ multiplying both sides of (2.11) by x and using (0.1), we obtain

$$(x \vee y) * x = (c * ((c * x) \wedge (c * y))) * x = (c * x) * ((c * x) \wedge (c * y)).$$

By (0.15) and Theorem 1.1(4), it follows

$$(c * x) * ((c * x) \wedge (c * y)) = (c * x) * (c * y) = ((y * x) * (y * c)) * (c * y).$$

Since $y \leqslant c$, the right side of the last expression is the same as $((y * x) * 0) * (c * y)$, namely, $(y * (c * y)) * x$ by BCI-2 and (0.1). Hence

$$(x \vee y) * x = (y * (c * y)) * x.$$

Therefore $(x \vee y) * x = y * x$ by Proposition 2.1.

In a similar fashion, we can prove that $(x \vee y) * y = x * y$.

(2) It is obvious that $(x \vee y) * z$ is an upper bound of $x * z$ and $y * z$. Let t be any upper bound of $x * z$ and $y * z$. Then $x * z \leqslant t$ and $y * z \leqslant t$. Now, putting (0.8), (0.6) and (1) together, it follows

$$((x \vee y) * z) * t \leqslant ((x \vee y) * z) * (x * z) \leqslant (x \vee y) * x = y * x,$$
$$((x \vee y) * z) * t \leqslant ((x \vee y) * z) * (y * z) \leqslant (x \vee y) * y = x * y.$$

So Proposition 2.3(1) implies

$$((x \vee y) * z) * t \leqslant (y * x) \wedge (x * y) = 0.$$

Thus $(x \vee y) * z \leqslant t$. Hence $(x \vee y) * z$ is the least upper bound of $x * z$ and $y * z$. Therefore $(x * z) \vee (y * z)$ exists and $(x \vee y) * z = (x * z) \vee (y * z)$.

(3) From (0.5) and (1), we have

$$(z * x) * (z * (x \vee y)) \leqslant (x \vee y) * x = y * x,$$
$$(z * y) * (z * (x \vee y)) \leqslant (x \vee y) * y = x * y.$$

Then

$$((z * x) * (z * (x \vee y))) \wedge ((z * y) * (z * (x \vee y))) \leqslant (y * x) \wedge (x * y). \tag{2.12}$$

Applying Proposition 2.3(2) to the left side of (2.12) and Proposition 2.3(1) to the right side, and noticing that 0 is a minimal element of X, it follows

$$((z * x) \wedge (z * y)) * (z * (x \vee y)) = 0.$$

So, $(z * x) \wedge (z * y) \leqslant z * (x \vee y)$. The opposite inequality can be seen from (0.8). Therefore $z * (x \vee y) = (z * x) \wedge (z * y)$. □

The next corollary is an immediate result of Proposition 2.4(2) and (0.15).

Corollary 2.5. *Let x and y be any elements in a branch $V(a)$ of an implicative BCI-algebra X. If $x \vee y$ exists, then $(x * y) \vee (y * x)$ exists and*

$$(x \vee y) * (x \wedge y) = (x * y) \vee (y * x).$$

3 On initial sections of implicative BCI-algebras Finally let's consider the initial sections of an implicative BCI-algebra. It is known that if X is a BCK-algebra and $A(c)$ is an initial section of X, then $(A(c); \leqslant)$ forms a Boolean algebra (refer to [5], Theorem 12). It is interesting that the same conclusion is true if X is an implicative BCI-algebra.

Theorem 3.1. *Let $A(c)$ be an initial section of an implicative BCI-algebra X. Then $(A(c); \leqslant)$ is a Boolean algebra with $x \wedge y = y * (y * x)$, $x \vee y = c * ((c * x) \wedge (c * y))$ and $x' = (c * x) * (0 * x)$ for any $x, y \in A(c)$.*

Proof. As any implicative BCI-algebra is commutative, by Theorem 0.3, $(A(c); \leqslant)$ is a distributive lattice with $x \wedge y = y * (y * x)$ and $x \vee y = c * ((c * x) \wedge (c * y))$ for any $x, y \in A(c)$. Also, c is clearly the unit element of the lattice $A(c)$. Moreover, it is easy to verify from Theorem 0.1 that there exists some branch $V(a)$ of X such that $A(c) \subseteq V(a)$. Because a is the least element of the branch $V(a)$, it is the zero element of the lattice $(A(c); \leqslant)$. It remains to show that $A(c)$ is a complemented lattice with $(c * x) * (0 * x)$ as the complement x' of x for any $x \in A(c)$. Let u denote $(c * x) * (0 * x)$. Then what we need to show is just the following facts:

$$\text{(i) } u \in A(c); \quad \text{(ii) } x \wedge u = a; \quad \text{(iii) } x \vee u = c.$$

In fact, by (0.6) and BCI-2, we have $(c * x) * (0 * x) \leqslant c * 0 = c$, that is, $u \leqslant c$. Then $u \in A(c)$, (i) holding. To show (ii) and (iii), let's first assert that $u * x = c * x$. In fact, since X is positive implicative, by (0.1) and (0.16), the following holds:

$$((c * x) * (0 * x)) * x = ((c * x) * x) * (0 * x) = c * x.$$

That is, $u * x = c * x$, as asserted. Now, we have

$$x \wedge u = u * (u * x) = u * (c * x).$$

Because of $x \in V(a)$, by (0.4) and (0.10), we obtain

$$u * (c * x) = ((c * x) * (0 * x)) * (c * x) = 0 * (0 * x) = a.$$

Therefore $x \wedge u = a$, showing (ii). Because X is commutative and $u \leqslant c$, we derive $c * (c * u) = u$. Then $(c * (c * u)) * x = u * x$, that is, $(c * x) * (c * u) = u * x$ by (0.1). So, the fact that $u * x = c * x$ gives $(c * x) * (c * u) = c * x$. Left $*$ multiplying both sides of the last equality by $c * x$, it follows

$$(c * x) * ((c * x) * (c * u)) = (c * x) * (c * x).$$

That is, $(c * u) \wedge (c * x) = 0$, in other words, $(c * x) \wedge (c * u) = 0$. Therefore

$$c * ((c * x) \wedge (c * u)) = c * 0 = c.$$

Note that $x \vee u = c * ((c * x) \wedge (c * u))$, it yields $x \vee u = c$, proving (iii). □

A BCI-algebra X is called *locally bounded* if every branch $V(a)$ of X is bounded, i.e., there is $m_a \in V(a)$ such that $x \leqslant m_a$ for all $x \in V(a)$.

Corollary 3.2 ([10], Theorem 5). *Assume that X is a locally bounded implicative BCI-algebra. Then for every branch $V(a)$ of X, it with respect to the BCI-ordering $\leqslant$ forms a Boolean algebra $(V(a); \leqslant)$.*

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