ON J-ALGEBRAS

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Abstract. In this paper we introduce the notion of J-algebra, and show that the minimal sharp J-algebras form a variety of d-algebras. Moreover, we discuss Smarandache disjointness and the disjointness digraph in J-algebras.

1 Introduction. Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([7, 8]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [5, 6] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Recently, Y. B. Jun, E. H. Roh and H. S. Kim ([10]) introduced a new notion, called a BH-algebra, i.e., (I), (II) and (V) $x*y=0$ and $y*x=0$ imply $x=y$, which is a generalization of $BCH/BCI/BCK$-algebras. They also defined the notions of ideals and boundedness in BH-algebras, and showed that there is a maximal ideal in bounded BH-algebras. J. Neggers and H. S. Kim ([14]) introduced and investigated a class of algebras which is related to several classes of algebras of interest such as $BCH/BCI/BCK$-algebras and which seems to have rather nice properties without being excessively complicated otherwise. Furthermore, they demonstrated a rather interesting connection between B-algebras and groups. P. J. Allen et al. ([1]) included several new families of Smarandache-type P-algebras and studied some of their properties in relation to the properties of previously defined Smarandache-types. Recently, P. J. Allen et al. ([2]) introduced the notion of a super commutative d-algebra and showed that if $(X;*)$ is a commutative binary system, then by adjoining an element 0 and adjusting the multiplication to $x*x=0$, they obtained a super commutative d-algebra, thereby demonstrating that the class of such algebras is very large. They noted that the class of super commutative d-algebras is Smarandache disjoint from the class of BCK-algebras, once more indicating that the class of d-algebras is quite a bit larger than the class of BCK-algebras and leaving the problem of finding further classes of d-algebras of special types which are Smarandache disjoint from the classes of BCK-algebras and super commutative d-algebras as an open question. Lastly the idea of a super Smarandache class of algebras was also defined and investigated.

It has been proved that BCK-algebras do not form a variety. W. H. Cornish ([4]) introduced the condition (J), and proved that the BCK-algebras satisfying the condition

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(J), called J*-variety*, form a variety. It was shown that the variety of commutative (positive implicative, implicative) BCK-algebras is a subvariety of the J-variety, and that no non-trivial variety of BCK-algebras is 1-based or can be defined by identities involving only two variables.

In this paper we introduce the notion of J-algebra, and show that the minimal sharp J-algebras form a variety of d-algebras. Moreover, we discuss Smarandache disjointness and the disjointness digraph in J-algebras.

2 J-algebras.

An algebra $(X;*)$ is said to be a J*-algebra* if, for any $x, y \in X$,

$$x*(x*(y*(y*x))) = y*(y*(x*(x*y))) \tag{J}$$

Example 2.1. Let $X := \{0,1,2,3\}$ be a set with the following tables:

$*_1$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	3
3	2	3	0	0

$*_2$	0	1	2	3
0	0	0	0	1
1	1	0	0	0
2	2	1	0	0
3	3	1	3	0

Then $(X;*_i), i = 1,2$, are J-algebras.

An algebra $(X;*)$ is said to be *sharp* if there exists $0 \in X$ such that $x*0 = x, \forall x \in X$. A J-algebra $(X;*)$ is said to be a *sharp* J*-algebra* if it is sharp. An algebra $(X;*)$ is said to be *precise* if there exists $0 \in X$ such that $x*y = 0 = y*x$, $(x, y \in X)$ implies $x = y$. A J-algebra $(X;*)$ is called a *precise* J*-algebra* if it is precise.

Theorem 2.2. *If an algebra $(X;*,0)$ is a sharp J-algebra, then it is a precise J-algebra.*

Proof. If $x*y = 0 = y*x, x, y \in X$, then

$$\begin{aligned} x*(x*(y*(y*x))) &= x*(x*(y*0)) \\ &= x*(x*y) && [X \text{ is sharp}] \\ &= x*0 \\ &= x && [X \text{ is sharp}] \end{aligned}$$

Similarly, $y*(y*(x*(x*y))) = y$. Since X is a J-algebra, we obtain $x = y$. □

An algebra $(X;*)$ is said to be *minimal* if there exists $0 \in X$ such that $0*x = 0$ for any $x \in X$.

Example 2.3. The algebra $(X;*_1)$ is a precise minimal J-algebra, but not sharp; $(X;*_2)$ is a precise sharp J-algebra in Example 2.1. Let $X := \{0,1,2,3\}$ be a set with the following table:

$*_3$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	1	0	0

Then $(X;*_3)$ is a minimal sharp J-algebra, but not precise.

J. Neggers and H. S. Kim ([13]) introduced the notion of d-algebras which is a generalization of BCK-algebras, and investigated several relations between d-algebras and BCK-algebras. A *d-algebra* ([13]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the axioms: (I) $x*x=0$ (*nilsquare*); (II) $0*x=0$ (*minimal*); (III) $x*y=0$ and $y*x=0$ imply $x=y$ (*precise*) for all $x,y\in X$.

Theorem 2.4. *If $(X;*,0)$ is a minimal sharp J-algebra, then it is a d-algebra.*

Proof. By Theorem 2.2, X is precise. If we let $y:=0$ in (J), then

$$\begin{aligned} 0 &= 0*(0*(x*(x*0))) && [X \text{ is minimal}] \\ &= x*(x*(0*(0*x))) && [\text{condition } (J)] \\ &= x*(x*0) && [X \text{ is sharp}] \\ &= x*x, \end{aligned}$$

proving the theorem. □

Notice that the minimal sharp J-algebras are seen to be a variety while the d-algebras are not a variety. Thus, *the minimal sharp J-algebras form a variety of d-algebras.*

An algebra $(X;*)$ is said to have an *inclusion condition* if $(x*y)*x=0$ for any $x,y\in X$.

Example 2.5. Let $X:=\{0,1,2,3\}$ be a set with the following tables:

$*_4$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	1

$*_5$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	3
3	3	3	1	0

Then $(X;*_4)$ is a J-algebra with inclusion condition, while $(X;*_5)$ is a sharp J-algebra, but not having the inclusion condition, since $(2*_5 3)*_5 2 = 1 \neq 0$.

Theorem 2.6. *If $(X;*,0)$ is a precise minimal J-algebra with inclusion condition, then it is a sharp J-algebra.*

Proof. Since $(X;*,0)$ has an inclusion condition, we obtain $(x*0)*x=0, \forall x\in X$. If we let $y:=0$ in (J), then $x*(x*(0*(0*x)))=0*(0*(x*(x*0)))$. Since X is minimal, $x*(x*(0*(0*x)))=x*(x*0)$ and $0*(0*(x*(x*0)))=0$, and hence we obtain $x*(x*0)=0$. Since X is precise, we obtain $x*0=x$, proving the theorem. □

Theorem 2.6 serves as a partial converse to Theorem 2.2 above.

3 Smarandache disjointness. Let $(X,*)$ be a binary system/algebra. Then $(X,*)$ is a *Smarandache-type P-algebra* if it contains a subalgebra $(Y,*)$, where Y is non-trivial, i.e., $|Y|\geq 2$, or Y contains at least two distinct elements, and $(Y,*)$ is itself of type P. Thus, we have *Smarandache-type semigroups* (the type P-algebra is a semigroup), *Smarandache-type groups* (the type P-algebra is a group), *Smarandache-type abelian groups* (the type

P-algebra is an abelian group). A Smarandache semigroup in the sense of Kandasamy is in fact a Smarandache-type group (see [11]). Smarandache-type groups are of course a larger class than Kandasamy's Smarandache semigroups since they may include non-associative algebras as well.

Given algebra types $(X, *)$ (type-P_1) and $(X, \circ)$ (type-P_2), we shall consider them to be *Smarandache disjoint* if the following two conditions hold:

(A) If $(X, *)$ is a type-P_1-algebra with $|X| > 1$ then it cannot be a Smarandache-type-P_2-algebra $(X, \circ)$;

(B) If $(X, \circ)$ is a type-P_2-algebra with $|X| > 1$ then it cannot be a Smarandache-type-P_1-algebra $(X, *)$.

Given a set X, define $x * y = a$ for some fixed $a \in X$, $\forall x, y \in X$, then the condition (J) holds obviously, as does the associative law and the commutative law. Thus $(X; *)$ is an example of a J-algebra which is also a commutative semigroup. Setting $a := 0$, we obtain $0 * x = 0$ for all $x \in X$, and the minimal condition applies as well. Obviously, the precision condition fails and thus, if one wants to use Theorem 2.2, the sharpness condition fails as well. In fact, if we consider the algebra $(X; *, a)$ with $x * y = a$ for all $x, y \in X$, where $a \in X$ (fixed), to be a *collapsed algebra* (a *trivial algebra*). then we have the following:

Theorem 3.1. *The class of precise J-algebras and the class of collapsed J-algebras are Smarandache disjoint.*

Proof. If $x * y = 0 = y * x$, $x, y \in X$, then $a = 0$ in the collapsed algebra $(X; *, a)$. Since X is collapsed, $x * y = 0 = y * x$ for any $x, y \in X$. It follows from X is precise that $x = y$, i.e., $|X| = 1$, whence Smarandache disjointness follows. □

Theorem 3.2. *The class of J-algebras and the class of commutative cancellative semigroups are Smarandache disjoint.*

Proof. Let $(X; \cdot)$ be both a J-algebra and a commutative cancellative semigroup. The condition (J) becomes $x^3y^2 = y^3x^2, \forall x, y \in X$, whence cancelling x^2y^2 from both sides we find that $x = y$ and thus $|X| = 1$, proving the theorem. □

An algebra $(X; *,)$ is called a *left* (*right*, resp.) *semigroup* if $x * y = x$ ($x * y = y$, resp.) for all $x, y \in X$. If $(X; *)$ is both a J-algebra and left (or right) semigroup, then $x = y$ for all $x, y \in X$ and thus $|X| = 1$, i.e.,

Theorem 3.3. *The class of J-algebras and the class of left (or right) semigroups are Smarandache disjoint.*

If $(X; *, e)$ is a J-algebra and a group, then $y := e$ in (J) yields $x * (x * (e * (e * x))) = x * (x * x)$ and $e * (e * (x * (x * e))) = x * x$, $\forall x \in X$. Hence $x * (x * x) = x * x$ and $x = e$, i.e., $|X| = 1$. We summarize:

Theorem 3.4. *The class of J-algebras and the class of groups are Smarandache disjoint.*

4 Disjointness digraph. In a minimal J-algebra $(X;*,0)$ and $x, y \in X$, we consider x to be *disjoint* from y (denoted $x \to y$) if $x*(x*y) = 0$.

Proposition 4.1. *Let $(X;*,0)$ be a minimal sharp J-algebra and $x, y \in X$. If $x \to y$, then $x \to y*(y*x)$.*

Proof. If $x \to y$, then $x*(x*y) = 0$. It follows from Theorem 2.4 that

$$\begin{aligned} x*(x*(y*(y*x))) &= y*(y*(x*(x*y))) && [\text{condition } (J)] \\ &= y*(y*0))) && \\ &= y*y && [X \text{ is sharp}] \\ &= 0. && [\text{Theorem 2.4}] \end{aligned}$$

□

Notice that $0*(0*x) = 0$ means $0 \to x$ for all $x \in X$. On the other hand $x*(x*0) = 0$ as well, so that $x \to 0$ for all $x \in X$. This means that in the *disjointness graph* of a minimal sharp J-algebra $(X;*,0)$, denoted $\Gamma_D(X)$, the vertex 0 is both a universal source $(0 \to x)$ and a universal sink $(x \to 0)$ or an index object(vertex) and a terminal object(vertex).

Example 4.2. Consider a BCK-algebra $X := \{0,1,2,3\}$ with

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	0
3	3	3	3	0

Then it is a minimal sharp J-algebra. By routine calculations we obtain the disjointness digraph $\Gamma_D(X)$ as follows:

$$\begin{tikzcd} 2 \bullet \arrow[d, leftrightarrow] & \bullet 3 \arrow[l] \arrow[d] \arrow[dl, leftrightarrow] \\ 0 \bullet \arrow[r, leftrightarrow] & \bullet 1 \end{tikzcd}$$

$\Gamma_D(X)$

If we relax the minimality condition $0*x = 0$ for all $x \in X$ to the condition $0*(0*x) = 0$, $\forall x \in X$ (*subminimal condition*) then in a subminimal J-algebra $(X;*,0)$, letting $x := 0$ in (J), we obtain $0 = 0*(0*(y*(y*0))) = y*(y*(0*(0*y))) = y*(y*0)$ so that we still maintain the conditions $0 \to x$ for all x and $x \to 0$ for all x in the disjointness graph $\Gamma_D(X)$.

We shall consider a J-algebra $(X;*,p)$ *pointed* with special element p, and with $x \to y$ meaning $x*(x*y) = p$ to produce the digraph $\Gamma_D(X)$. The element p is called a *null point* if $x \to p$ and $p \to x$ for all elements $x \in X$, i.e., if $p = x*(x*p) = p*(p*x)$ for all $x \in X$. Notice that if a null point exists then it is unique. In fact, if p and q are possibly null points then we obtain $p = q*(q*p) = p*(p*q) = q$. We shall denote the unique null point of a J-algebra X by 0 and consider $\Gamma_0(X) = \Gamma_D(X)$ to represent the disjointness digraph of the J-algebra $(X;*,0)$.

If $(X;*,a)$ has $x*y = a$ for all $x, y \in X$, then $x*(x*y) = a$ for all $x, y \in X$ and $x \to y$ in $\Gamma_D(X)$, i.e., $\Gamma_a(X)$ is a complete digraph (whose adjacency matrix in the finite case has all entries equal to 1). Notice that if $p \neq a$, then $\Gamma_p(X)$ contains no arrows whatsoever.

The following questions arise:

(A) If $(X,*)$ and $(X;\circ)$ are J-algebras and if for all $p \in X$, $\Gamma_p(X,*) = \Gamma_p(X;\circ)$, are $(X,*)$ and $(X;\circ)$ isomorphic as J-algebras ?

(B) If $(X,*)$ and $(X;\circ)$ are J-algebras and there is a one-one correspondence $\varphi : X \to X$ such that for all $p \in X$, $\Gamma_p(X;*) = \Gamma_{\varphi(p)}(X;\circ)$, are $(X,*)$ and $(X;\circ)$ isomorphic as J-algebras ?

The answer to these questions are seen to be no. Consider the following example.

Example 4.3. Let $(X;*,a)$ be an algebra with $x*y = a, \forall x,y \in X$ and let $\Gamma_a(X;*) = \Gamma_a(X;\circ)$. Then $(X;*)$ and $(X;\circ)$ are J-algebras. In fact, if $x*y = a, \forall x,y \in X$, then $x*(x*y) = a$, i.e., $x \to y$ in $\Gamma_a(X;*)$ and hence $x \to y$ in $\Gamma_a(X;\circ)$. Similarly, $y \to x$ in $\Gamma_a(X;\circ)$. This means $x\circ(x\circ y) = a = y\circ(y\circ x)$ and hence $x\circ(x\circ(y\circ(y\circ x))) = x\circ(x\circ a) = a$, $y\circ(y\circ(x\circ(x\circ y))) = y\circ(y\circ a) = a$, proving $(X;\circ)$ is a J-algebra. Let $X := \{a,b,c\}$ be a set with the following table:

$\circ$	a	b	c
a	a	a	a
b	a	a	b
c	a	a	b

Assume $x\circ(x\circ y) \neq a$ for some $x,t \in X$. Then $x \neq a$, i.e., we have to deal with expressions $b\circ(b\circ y)$ or $c\circ(c\circ y)$, i.e., with expressions $b\circ b = a$ or $b\circ a = a$, $c\circ b = a$ or $c\circ a = a$, whence $x\circ(x\circ y) = a$ for all $x,y \in X$, i.e., $\Gamma_a(X;\circ) = \Gamma_a(X;*)$. But it is not true that $x\circ y = a$ for all $x,y \in X$, i.e., $(X;*)$ and $(X;\circ)$ are not isomorphic as J-algebras.

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