ON A DISCRETE ARBITRATION PROCEDURE

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Abstract. We derive the equilibrium in non-cooperative two-players zero-sum game intended to model of final-offer arbitration procedure in which the arbitrator's settlement is concentrated in odd number of points $\{-n,-(n-1),...,-1,0,1,...,n-1,n\}$.

1. Introduction

We consider a non-cooperative zero-sum game related with a model of final-offer arbitration procedure where two players L and M, interpreted here as Labour and Management, respectively, have a dispute on an improvement in the wage rate. Player L makes an offer x, and player M - an offer y. We shall assume that x and y are arbitrary real numbers from the interval $[A,B]$.

To solve the conflict we use the so-called final-offer arbitration scheme [1-3] developed by Farber (1980). If $x \le y$, there is no conflict, and the players agree on a payoff equal to $(x+y)/2$. If otherwise, $x > y$, the parties call in the arbitrator (A). Assume that the arbitrator has a settlement he would like to impose, denoted by α. Then, after observing the offers, x and y, the arbitrator simply chooses the offer that is closer to α. We suppose that α is a random variable. Assume, that the Manager wants to minimize the expected wage settlement imposed by the Arbitrator and the Labour wants to maximize it.

If $\alpha = a$ almost sure it is evident that the equilibrium is the pair of strategies (a,a). If α is a random variable with continuous distribution the equilibrium often consists of pure strategies [2-3]. If the distribution support of α is concentrated in two points or three points the solutions were derived in [4-6]. In this paper we analyse a case where the arbitrator's settlement is concentrated in odd number of points $\{-n,-(n-1),...,-1,0,1,...,n-1,n\}$.

Another approach to solve the conflict between Labour and Management like a multistage arbitration game with "random" arbitrator was developed in [7-8].

2. Problem statement

Suppose that α is a random variable that assumes the values $\{-n,-(n-1),...,-1,0,1,...(n-1),n\}$ with equal probabilities $p = 1/(2n+1)$. A non-cooperative game where the strategies of players L and M are arbitrary numbers $x, y \in [-a,a]$ is considered. The payoff in the game has form $H(x,y) = EH_\alpha(x,y)$, where

$$H_\alpha(x,y) = \begin{cases} \frac{x+y}{2}, & if \quad x \le y \\ x, & if \quad x > y, |x-\alpha| < |y-\alpha| \\ y, & if \quad x > y, |x-\alpha| > |y-\alpha| \\ \alpha, & if \quad x > y, |x-\alpha| = |y-\alpha| \end{cases} \tag{1}$$

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We find that the equilibrium in the game lies among the mixed strategies. Denote $f(x)$ and $g(y)$ as player L's and M's mixed strategies respectively. Suppose that the distribution $g(y)(f(x))$ support lies on the negative (positive) semiaxis. That is,

$$f(x) \geq 0, x \in [0, a], \int_0^a f(x)dx = 1, \quad g(y) \geq 0, y \in [-a, 0], \int_{-a}^0 g(y)dy = 1.$$

By symmetry, it follows that the value of the game is equal to zero, and the optimum strategies must be symmetric in respect to the axis of ordinate, i.e. $g(y) = f(-y)$. It therefore suffices to build an optimum strategy for one of the players, e.g. L.

3. Optimal strategies

Theorem. *For* $a \in \left(0, \frac{2(n+1)^2}{2n+1}\right]$ *the optimal strategy is*

$$f(x) = \begin{cases} 0, & 0 \leq x < (\frac{n}{n+1})^2 a, \\ \frac{n\sqrt{a}}{2\sqrt{x^3}}, & (\frac{n}{n+1})^2 a \leq x \leq a \end{cases} \tag{2}$$

and for $a \in \left(\frac{2(n+1)^2}{2n+1}, +\infty\right)$

$$f(x) = \begin{cases} 0, & 0 \leq x < \frac{2n^2}{2n+1}, \\ \frac{n(n+1)}{\sqrt{2(2n+1)}} \frac{1}{\sqrt{x^3}} & \frac{2n^2}{2n+1} \leq x \leq \frac{2(n+1)^2}{2n+1}, \\ 0, & \frac{2(n+1)^2}{2n+1} < x \leq a. \end{cases} \tag{3}$$

Proof. First, consider the case $a \in (0, 2]$.

3.1. Case $0 < a \leq 2$.

According to (1) the payoff of M for $y \in [-a, 0]$ is equal to

$$H(f, y) = \frac{1}{2n+1}\left[n \int_0^a y f(x)dx + \left(\int_0^{-y} x f(x)dx + \int_{-y}^{a} y f(x)dx\right) + n \int_0^a x f(x)dx\right].$$

We shall be looking the strategy f in the following form

$$f(x) = \begin{cases} 0, & 0 \leq x < \alpha, \\ \varphi(x), & \alpha \leq x \leq \beta, \\ 0, & \beta < x \leq a, \end{cases} \tag{4}$$

where $\varphi(x) > 0, \quad x \in [\alpha, \beta]$ and φ has a continuous derivative in (α, β).

The strategy (4) will be optimal if $H(f, y) = 0$ for $y \in [-\beta, -\alpha]$ and $H(f, y) \geq 0$ for $y \in [-a, -\beta) \cup (-\alpha, 0]$. Notice that $H(f, 0) = \frac{n}{2n+1} \int_0^a x f(x)dx > 0$.

By $H(f, -\alpha) = H(f, -\beta) = 0$ it yields

$$H(f, -\alpha) = \frac{1}{2n+1}\left[-(n+1)\alpha + n \int_\alpha^\beta x\varphi(x)dx\right] = 0,$$

$$H(f, -\beta) = \frac{1}{2n+1}\left[-n\beta + (n+1) \int_\alpha^\beta x\varphi(x)dx\right] = 0.$$

¿From the system it follows

$$\int_{\alpha}^{\beta} x\varphi(x)dx = \frac{n+1}{n}\,\alpha = \frac{n}{n+1}\,\beta$$

and $\beta = (\frac{n+1}{n})^2\alpha$ or $\alpha = (\frac{n}{n+1})^2\beta$.

For $y = -a$, $H(f,-a) = \frac{1}{2n+1}[-na + n\beta] = \frac{n}{2n+1}(\beta - a)$. Consequently, if $\beta < a$ then $H(f,-a) < 0$. Hence, $\beta = a$ and $\alpha = \left(\frac{n}{n+1}\right)^2 a$, and

$$\int_0^a xf(x)dx = \int_{\alpha}^{\beta} x\varphi(x)dx = \frac{n}{n+1}a. \tag{5}$$

Let us find the function $\varphi(x)$. The condition $H(f,y) = 0, \quad y \in [\beta, -\alpha]$ yields $H'(f,y) = H''(f,y) = 0$. So,

$$H'(f,y) = 1 + 2yf(-y) + \int_{-y}^{a} f(x)dx = 0, H''(y) = 3f(-y) - 2yf'(-y) = 0.$$

Letting $y = -x$ we obtain the differential equation

$$3f(x) + 2xf'(x) = 0. \tag{6}$$

The solution is

$$f(x) = \frac{c}{\sqrt{x^3}}. \tag{7}$$

Because,

$$1 = \int_0^a f(x)dx = \int\limits_{(\frac{n}{n+1})^2 a}^{a} \frac{c}{\sqrt{x^3}} = \frac{2c}{n\sqrt{a}},$$

we find c:

$$c = \frac{n\sqrt{a}}{2}.$$

Finally,

$$f(x) = \begin{cases} 0, & 0 \le x < \left(\frac{n}{n+1}\right)^2 a, \\ n\frac{\sqrt{a}}{2\sqrt{x^3}}, & \left(\frac{n}{n+1}\right)^2 a \le x \le a. \end{cases}$$

Let us check the optimality conditions. For $y \in [-a, -\left(\frac{n}{n+1}\right)^2 a]$ we have

$$(2n+1)H(f,y) = ny + \int\limits_{(\frac{n}{n+1})^2 a}^{-y} n\frac{\sqrt{a}}{2\sqrt{x}}dx + y\int_{-y}^{a} n\frac{\sqrt{a}}{2\sqrt{x^3}}dx + \frac{n^2}{n+1}a$$

$$= ny + n\sqrt{a}\left(\sqrt{-y} - \frac{n}{n+1}\sqrt{a}\right) - n\sqrt{a}y\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{-y}}\right) + \frac{n^2}{n+1}a = 0.$$

For $y \in (-\left(\frac{n}{n+1}\right)^2 a, 0]$

$$(2n+1)H(f,y) = ny + y + \frac{n^2}{n+1}a = (n+1)\left[y + \left(\frac{n}{n+1}\right)^2 a\right] > 0.$$

It proves the optimality of (2).

3.2. Case $2 < a \le \frac{2(n+1)^2}{2n+1}$.

Let $a \in \left(2, \frac{2(n+1)^2}{2n+1}\right]$. Consider $H(f,y)$ for $y \in \left[-a, -\left(\frac{n}{n+1}\right)^2 a\right]$ with f satisfying (2). Because the support of f is $\left[\left(\frac{n}{n+1}\right)^2 a, a\right]$ and $a \le \frac{2(n+1)^2}{2n+1}$ we have $a - \left(\frac{n}{n+1}\right)^2 a \le 2$.

Hence, for $y \in [-a, -\left(\frac{n}{n+1}\right)^2 a]$

$$(2n+1)H(f,y) = n \int\limits_{\left(\frac{n}{n+1}\right)^2 a}^{a} yf(x)dx + \left(\int\limits_{\left(\frac{n}{n+1}\right)^2 a}^{-y} xf(x)dx + \int\limits_{-y}^{a} yf(x)dx \right) + n \int\limits_{\left(\frac{n}{n+1}\right)^2 a}^{a} xf(x)dx.$$

Differentiating it we obtain again the differential equation (5). It's solution is $f(x)$ of the form (2). It yields $H(f,y) \equiv 0$ for $y \in \left[-a, -\left(\frac{n}{n+1}\right)^2 a\right]$.

Let us show that $H(f,y) > 0$ for $y \in \left(-\left(\frac{n}{n+1}\right)^2 a, 0\right]$. First, we determine the sign of $H(f,y)$ at the interval $\left[-\left(\frac{n}{n+1}\right)^2 a, -\left(\frac{n}{n+1}\right)^2 a + 2\right]$.

If $y \in \left[-\left(\frac{n}{n+1}\right)^2 a, -a+2\right]$, then

$$H(f,y) = \frac{n+1}{2n+1} y + \frac{n}{2n+1} \int\limits_{\left(\frac{n}{n+1}\right)^2 a}^{a} xf(x)\,dx =$$

$$= \frac{n+1}{2n+1}\left[y + \left(\frac{n}{n+1}\right)^2 a\right] > 0.$$

For $y \in \left[-a+2, -\left(\frac{n}{n+1}\right)^2 a + 2\right]$

$$H(f,y) = \frac{n+1}{2n+1} y + \frac{1}{2n+1}\left(\int\limits_{\left(\frac{n}{n+1}\right)^2 a}^{2-y} xf(x)\,dx + \int\limits_{2-y}^{a} yf(x)\,dx \right) +$$

$$+ \frac{n-1}{2n+1} \int\limits_{\left(\frac{n}{n+1}\right)^2 a}^{a} xf(x)\,dx.$$

Then

$$H'(f,y) = \frac{1}{2n+1}\left[n+1 + (2y-2)f(2-y) + \int\limits_{2-y}^{a} f(x)\,dx \right] =$$

$$= \frac{1}{2n+1}\left[n+1+\frac{(y-1)n\sqrt{a}}{\sqrt{(2-y)^3}} - n + \frac{n\sqrt{a}}{\sqrt{2-y}}\right] =$$

$$= \frac{1}{2n+1}\left(1+\frac{n\sqrt{a}}{\sqrt{(2-y)^3}}\right) > 0.$$

Hence, $H(f,y) > 0$ for $y \in \left(-\left(\frac{n}{n+1}\right)^2 a, -\left(\frac{n}{n+1}\right)^2 a + 2\right]$.

If $-\left(\frac{n}{n+1}\right)^2 a + 2 \geq 0$ it finishes the proof. Otherwise, we shift the interval to the right and show that $H(f,y) > 0$ for $y \in \left(-\left(\frac{n}{n+1}\right)^2 a + 2, -\left(\frac{n}{n+1}\right)^2 a + 4\right]$, etc. So, we prove that (2) is optimal also for $a \in (2, \frac{2(n+1)^2}{2n+1}]$.

3.3. Case $\frac{2(n+1)^2}{2n+1} < a \leq \infty$.

Suppose now that $a \in (\frac{2(n+1)^2}{2n+1}, \infty)$. In this case the form of $H(f,y)$ is more complicated. As an example we consider the case with infinite horyzon $a = \infty$.

Suppose that the player L uses the strategy (3) and find the payoff function $H(f,y)$. For simplicity denote $\alpha = \frac{2n^2}{2n+1}$ and $\beta = \alpha + 2 = \frac{2(n+1)^2}{2n+1}$. Then, for $y \in (-\infty, -2n-\beta]$

$$H(f,y) = \int_\alpha^\beta xf(x)dx = \frac{2n(n+1)}{2n+1} > 0.$$

Let $k = 3\left[\frac{n}{2}\right]+2$ if n is odd and $k = 3\frac{n}{2}$ if n is even. For $y \in [-2n+2r-\beta, -2n+2r-\alpha]$ where $r = 0, 1, \ldots, n, \ldots, k-1$, and for $y \in [-2n+2r-\beta, 0]$, where $r = k$, we find

$$H(f,y) = \frac{r}{2n+1}y + \frac{1}{2n+1}\left[\int_\alpha^{-2n+2r-y} xf(x)dx + \int_{-2n+2r-y}^\beta yf(x)dx\right] + \frac{2n-r}{2n+1}\int_\alpha^\beta xf(x)dx$$

$$= \int_\alpha^\beta xf(x)dx - \frac{r}{2n+1}\int_\alpha^\beta (x-y)f(x)dx - \frac{1}{2n+1}\int_{-2n+2r-y}^\beta (x-y)f(x)dx. \quad (8)$$

Differentiating (8) with f of the form (2) we obtain

$$H'(f,y) = \frac{r}{2n+1} + \frac{1}{2n+1}\int\limits_{-2n+2r-y}^{\beta} f(x)dx + \frac{1}{2n+1}(2y+2n-2r)f(-2n+2r-y)$$

$$= \frac{r-n}{2n+1}\left(1+\frac{2n(n+1)}{\sqrt{2(2n+1)(-2n+2r-y)^3}}\right). \quad (9)$$

It follows from (9) that in the interval $y \in [-\beta, -\alpha]$ where $r = n$ the expected payoff $H(f,y)$ is constant and because

$$H(f,\beta) = \int_\alpha^\beta xf(x)dx - \frac{n}{2n+1}\int_\alpha^\beta (x+\beta)f(x)dx$$

$$= \frac{n+1}{2n+1}\int_\alpha^\beta xf(x)dx - \frac{n}{2n+1}\beta = \frac{n+1}{2n+1}\frac{2n(n+1)}{2n+1} - \frac{n}{2n+1}\frac{2(n+1)^2}{2n+1} = 0$$

it yields $H(f,y) \equiv 0$ for $y \in [-\beta, -\alpha]$.

For $r < n$ (9) gives $H'(f,y) < 0$ and for $r > n$ $H'(f,y) > 0$ in the intervals $y \in [-2n+2r-\beta, -2n+2r-\alpha]$.

Consequently, $H(f,y) \geq 0$ for all y. That proves the optimality of the strategy (3).

The full proof of the theorem for $a \in (\frac{2(n+1)^2}{2n+1}, \infty)$ is derived by the same analysis like in the case $a = \infty$.

4. Conclusion

We see that the optimal strategies in this discrete arbitration game with uniform distribution are randomized. It is different from the solution in the continuous version of the final-offer arbitration procedure with uniform distribution considered in [2-3] where the optimal strategies of the players are concentrated at the extreme points of the interval $[-a, a]$. But if $a = n$ it follows from Theorem that optimal strategy (2) has non-zero measure only at the interval $\left[\left(\frac{n}{n+1}\right)^2 a, a\right]$ which size tends to zero for large n. So, for large n the solutions of discrete and continuous versions of the arbitration game are similar.

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