L^2-BOUNDEDNESS OF MARCINKIEWICZ INTEGRALS ALONG SURFACES WITH VARIABLE KERNELS

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Abstract. In this paper, we give the L^2 estimates for the Marcinkiewicz integral with rough variable kernels associated to surfaces. As corollaries of this result, we show that similar properties still hold for parametric Littlewood-Paley area integral and parametric g_λ^* functions with rough variable kernels. We also show some sharp differene betweeen properties of singular integrals and the Marcinkiewicz integral with rough variable kernels. Some of the results are extensions of some known results.

1 Introduction In order to study the elliptic partial differential equations of order two with variable coefficients, A. P. Calderón and A. Zygmund [2] defined and studied the L^2 boundedness of singular integrals T with variable kernels. In 1980, N. E. Aguilera and E. O. Harboure [1] studied the problem of pointwise convergence of singular integral and the L^2 bounds of Hardy-Littlewood maximal function with variable kernels. In 2002, L. Tang and D. C. Yang [16] gave the L^2 boundedness of the singular integrals with rough variable kernels associated to surfaces. In order to state more precisely, first we give some definitions.

Definition 1. Let $k(x, y) : \mathbb{R}^n \times \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$. Then $k(x, y)$ is said to be a variable C-Z kernel if

(a) $k(x, y)$ is positively homogeneous in y of degree $-n$, namely $k(x, \lambda y) = \lambda^{-n}k(x, y)$ for any $\lambda > 0$;

(b) $\int_{S^{n-1}} k(x, y')d\sigma(y') = 0$ for a.e. $x \in \mathbb{R}^n$.

Define the variable Calderón-Zygmund singular integral operator T_Φ associated to surfaces of the form $\{x = \Phi(|y|)y'\}$ by

$$T_\Phi(f)(x) = p.v. \int_{\mathbb{R}^n} k(x, y)f(x - \Phi(|y|)y')dy \tag{1.1}$$

for $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$. The truncated maximal operator T_Φ^* is defined by

$$T_\Phi^*(f)(x) = \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} k(x, y)f(x - \Phi(|y|)y')dy \right|$$

Obviously, if we take $\Phi(|y|) = |y|$ for $y \in \mathbb{R}^n\backslash\{0\}$, then $T_\Phi = T$ is just the singular integral operator studied by A. P. Calderón and A. Zygmund in [2].

L. Tang and D. C. Yang gave the following result:

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Theorem A ([16]). *Suppose $k(x,y)$ be a variable kernel as in Definition 1 and satisfies for some $p > 2(n-1)/n$*

$$\int_{S^{n-1}} |k(x,y')|^p d\sigma(y') \le C_1 \qquad \text{for a.e.} \quad x \in \mathbb{R}^n.$$

Let $\Phi(t)$ be a nonnegative (or non-positive) C^1 function on $(0,\infty)$ satisfying:

(c) *Φ is strictly increasing (or decreasing);*

(d) *$\frac{\Phi(t)}{t} = C_2\Phi'(t)\varphi(t)$ for all $t \in (0,\infty)$, φ is defined on $(0,\infty)$ which is a monotonic and uniformly bounded function;*

Then T_Φ^* is bounded on $L^2(\mathbb{R}^n)$ and T_Φ can be uniquely extended to be a bounded operator on $L^2(\mathbb{R}^n)$. Moreover, for all $f \in L^2(\mathbb{R}^n)$,

$$\|T_\Phi(f)\|_2 \le C\|f\|_2 \quad \text{and} \quad \|T_\Phi^*\|_2 \le C\|f\|_2,$$

where the constant C is independent of f.

In order to state other related results, let's first give some definitions.

Definition 2. Let S^{n-1} be the unit sphere of $\mathbb{R}^n (n \ge 2)$ equipped with Lebesgue measure $d\sigma = d\sigma(x')$. A function $\Omega(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ is said in $L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ $(q \ge 1)$ if

(e) $\Omega(x,\lambda y) = \Omega(x,y)$, for any $x, y \in \mathbb{R}^n$ and $\lambda > 0$;

(f) $\|\Omega\|_{L^\infty \times L^q(S^{n-1})} = \sup_{x\in\mathbb{R}^n} \left(\int_{S^{n-1}} |\Omega(x,y')|^q d\sigma(y') \right)^{1/q} < \infty$, where $y' = y/|y|$ for any $y \in \mathbb{R}^n\backslash\{0\}$.

Now, we define the Marcinkiewicz integral with rough variable kernels associated with surfaces of the form $\{x = \Phi(|y|)y'\}$ by

$$\mu_\Omega^\Phi(f)(x) = \left(\int_0^\infty |F_{\Omega,t}(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,t}(x) = \int_{|y|\le t} \frac{\Omega(x,y)}{|y|^{n-1}} f(x - \Phi(|y|)y')dy.$$

If $\Phi(|y|) = |y|$, we denote $\mu_\Omega^\Phi = \mu_\Omega$. Then μ_Ω is just the Marcinkiewicz integral of higher dimension which was first defined and studied by E. M. Stein [11] in 1958. Since then, many works have been done about this integral (See for example, [4], [5], [8]). In 2005, Y. Ding, C. Lin and S. Shao [6] studied the L^2 boundedness of the operator μ_Ω when Ω satisfies (e), (f) and

$$\int_{S^{n-1}} \Omega(x,y')d\sigma(y') = 0. \tag{1.2}$$

Theorem B ([6]). *Suppose that $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ $(q > 2(n-1)/n)$ and satisfies (1.2). Then there is a constant C such that $\|\mu_\Omega(f)\|_2 \le C\|f\|_2$, where the constant C is independent of f.*

Therefore, it is natural to ask if the results in Theorem A still hold or not for the Marcinkiewicz integral with rough variable kernels.

The main purpose of this paper is to give a positive answer to the above question. We get the following L^2 estimates for Marcinkiewicz integrals with rough variable kernels along surfaces :

Theorem 1. *Suppose that $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ $(q > 2(n-1)/n)$ and satisfies (1.2). Let Φ be a positive and strictly increasing (or negative and decreasing) function and satisfy $\frac{\Phi(t)}{t} = C_2\Phi'(t)\varphi(t)$ for all $t \in (0,\infty)$, where φ is a function defined on $(0,\infty)$ and there exists two constants δ, M such that $0 < \delta \le \varphi(t) \le M$. Suppose moreover φ satisfies one of the following conditions:*

(i) *$t\varphi'(t)$ is bounded;*

(ii) *φ is a monotonic function.*

Then there is a constant C such that $\|\mu_\Omega^\Phi(f)\|_2 \le C\|f\|_2$, where constant C is independent of f.

Remark 1. There is no including relationship between condition (i) and condition (ii), this can be seen from the examples given in section 2. We also must point out that C_2 is positive under the condition of Φ and φ in Theorem 1.

Remark 2. In 1997, D. Fan and Y. Pan [7] studied the L^p $(p > 1)$ boundedness of the singular integral along surface without the variable kernels. We note that in their paper, the conditions assumed on the function φ, are much stronger than the condition (i) in Theorem 1 (with $\varphi \in L^\infty$).

By Remark 1, it is natural to ask whether Theorem A still hold or not under the condition (i) in Theorem 1. The following theorem gives a positive answer to this question.

Theorem 2. *Let Φ, φ, $k(x,y)$ be the same as in Theorem A, except the monotonicity of φ. If the function φ satisfies the condition that $t\varphi'(t)$ is bounded, then Theorem A still holds.*

As another main purpose of this paper, we will show below some different properties between singular integrals and the Marcinkiewicz integral. Let $\{Y_{k,j}\}$ $(k \ge 1, j = 1, 2, ..., D_k)$ be the complete system of normalized surface spherical harmonics of degree k. Define

$$\mu^\Phi_{Y_{k,j}}(f)(x) = \left(\int_0^\infty \left|\int_{|y|\le t} \frac{Y_{k,j}(y')}{|y|^{n-1}} f(x - \Phi(|y|)y')dy\right|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Then we have

Theorem 3. *Let $\Phi(t) = t^{-\alpha}$ for $\alpha > 0$. Then L^2 boundedness of $\mu^\Phi_{Y_{k,j}}$ doesn't hold.*

Remark 3. Note that $\Phi(t) = t^\alpha$, $\alpha < 0$ satisfies the condition in Theorem A, however Theorem 3 shows certain different properties between singular integrals and the Marcinkiewicz integral with variable kernels.

Our results can be extended to the parametric Marcinkiewicz integrals, parametric area integral and parametric μ_λ^* function, which are defined by

$$\mu_\Omega^{\Phi,\sigma}(f)(x) = \left(\int_0^\infty \left|\int_{|y|\le t} \frac{\Omega(x,y)}{|y|^{n-\sigma}} f(x - \Phi(|y|)y')dy\right|^2 \frac{dt}{t^{1+2\sigma}}\right)^{1/2},$$

$$\mu_S^{\Phi,\sigma}(f)(x) = \Big(\iint_{\Gamma(x)} \Big|\frac{1}{t^\sigma}\int_{|z|<t} \frac{\Omega(y,z)}{|z|^{n-\sigma}} f(y-\Phi(|z|)z')dz\Big|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2},$$

$$\mu_{\lambda,\Phi}^{*,\sigma}(f)(x)$$
$$= \Big(\iint_{\mathbb{R}_+^{n+1}} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \Big|\frac{1}{t^\sigma}\int_{|z|<t} \frac{\Omega(y,z)}{|z|^{n-\sigma}} f(y-\Phi(|z|)z')dz\Big|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2},$$

$$\mu_{Y_{k,j}}^{\Phi,\sigma}(f)(x) = \Big(\int_0^\infty \Big|\int_{|y|\le t} \frac{Y_{k,j}(y')}{|y|^{n-\sigma}} f(x-\Phi(|y|)y')dy\Big|^2 \frac{dt}{t^{1+2\sigma}}\Big)^{1/2},$$

$$\mu_{S,Y_{k,j}}^{\Phi,\sigma}(f)(x) = \Big(\iint_{\Gamma(x)} \Big|\frac{1}{t^\sigma}\int_{|z|<t} \frac{Y_{k,j}(z')}{|z|^{n-\sigma}} f(y-\Phi(|z|)z')dz\Big|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2},$$

$$\mu_{\lambda,Y_{k,j}}^{*,\sigma}(f)(x)$$
$$= \Big(\iint_{\mathbb{R}_+^{n+1}} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \Big|\frac{1}{t^\sigma}\int_{|z|<t} \frac{Y_{k,j}(z')}{|z|^{n-\sigma}} f(y-\Phi(|z|)z')dz\Big|^2 \frac{dydt}{t^{n+1}}\Big)^{1/2},$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : \quad |x-y| < t\}$ and $\lambda > 1$.

We get the following results:

Theorem 4. *Let $\sigma > 0$. Then Theorem 1 still holds for the parametric operator $\mu_\Omega^{\Phi,\sigma}$, $\mu_S^{\Phi,\sigma}$ and $\mu_{\lambda,\Phi}^{*,\sigma}$.*

Theorem 5. *Suppose $\sigma \ge 0$ and $\Phi(t) = t^{-\alpha}$ for $\alpha > 0$, or $\sigma < 0$ and $\Phi(t) = t^\alpha$ for $\alpha > 0$. Then L^2 boundedness of $\mu_Y^{\Phi,\sigma}(f)$, $\mu_{S,Y}^{\Phi,\sigma}(f)$ and $\mu_{\lambda,Y}^{*,\sigma}(f)$ doesn't hold.*

2 Proof of Theorem 1 We begin with recalling a known lemma. This lemma can be obtained by (2.19) in [15, p. 152] and Theorem 3.10 in [15, p. 158] (see also [16]).

Lemma 2.1 ([15]). *Let $n \ge 2$, $k \ge 0$ and $P(y)$ be a spherical harmonics of degree k. Then*

$$\int_{S^{n-1}} P(y')e^{-ix\cdot y'}d\sigma(y') = (-i)^k(2\pi)^{\frac{n}{2}} \frac{J_{\frac{n}{2}+k-1}(|x|)}{|x|^{\frac{n}{2}-1}} P(\frac{x}{|x|}).$$

Lemma 2.2. *Let $g(t)$ be a nonnegative (positive) and non-decreasing (strictly increasing) function on $(0,\infty)$ such that there exist $C_0 > 0$ and a bounded function $\varphi(t)$ satisfying*

$$\frac{g(t)}{t} = C_0 g'(t)\varphi(t).$$

If there exists $\delta > 0$ such that $0 < \delta \le \varphi(t)$ on $(0,\infty)$, then $\frac{[g^{-1}(t)]^\sigma}{t^\varepsilon}$ is non-decreasing (strictly increasing) on $(0,\infty)$ for $0 < \varepsilon \le C_0\sigma\delta$ ($0 < \varepsilon < C_0\sigma\delta$). Conversely, if $\frac{[g^{-1}(t)]^\sigma}{t^\varepsilon}$ is non-decreasing (strictly increasing) for some $\varepsilon > 0$, then $\varphi(t) \ge \frac{\varepsilon}{C_0\sigma}$ ($\varphi(t) > \frac{\varepsilon}{C_0}\sigma$).

Proof. It is easily seen that we only give the proof for $\sigma = 1$. Set $f(t) = \frac{g^{-1}(t)}{t^\varepsilon}$. Then

$$f'(t) = -\varepsilon\frac{g^{-1}(t)}{t^{1+\varepsilon}} + \frac{1}{t^\varepsilon}\frac{1}{g'(g^{-1}(t))} = -\varepsilon\frac{g^{-1}(t)}{t^{1+\varepsilon}} + \frac{1}{t^\varepsilon}\frac{C_0 g^{-1}(t)\varphi(g^{-1}(t))}{t}$$
$$= \frac{g^{-1}(t)}{t^{1+\varepsilon}}\big(C_0\varphi(g^{-1}(t)) - \varepsilon\big).$$

Thus we have $\frac{g^{-1}(t)}{t^\varepsilon}$ is non-decreasing (strictly increasing) if and only if $C_0\varphi(t) \ge \varepsilon$ ($C_0\varphi(t) > \varepsilon$). This implies the desired conclusion. □

Below we give four examples.

Example 1. Let $g(t) = t^\alpha \log^\beta(1+t)$, $\alpha > 0$, $\beta \ge 0$. It is easy to see that

$$g'(t) = \left(\alpha + \frac{\beta t}{(1+t)\log(1+t)}\right)\frac{g(t)}{t}.$$

So

$$\frac{g(t)}{t} = g'(t)\frac{(1+t)\log(1+t)}{\alpha(1+t)\log(1+t)+\beta t} = g'(t)\varphi(t).$$

Hence, we have

$$\varphi'(t) = \frac{\beta[t-\log(1+t)]}{[\alpha(1+t)\log(1+t)+\beta t]^2}.$$

Thus, for $t > 0$, $\varphi(t)$ is bounded and increasing, $t\varphi'(t)$ is bounded, and $\varphi(t) \ge 1/\alpha$.

Example 2. Let $g(t) = 2t^3 - 2t^2 + t$. Then $g'(t) = \frac{g(t)}{t}\frac{6t^3-4t^2+t}{2t^3-2t^2+t}$ and $\varphi(t) = \frac{2t^2-2t+1}{6t^2-4t+1}$. It is easy to check that $g(t)$ is positive and increasing, and if $t < 1 - \frac{1}{\sqrt{2}}$ or $t > 1 + \frac{1}{\sqrt{2}}$, then φ is increasing, and if $1 - \frac{1}{\sqrt{2}} < t < 1 + \frac{1}{\sqrt{2}}$ then φ is decreasing. Hence $(2-\sqrt{2})/2 \le \varphi(t) \le (2+\sqrt{2})/2$. Moreover, $t\varphi'(t) = \frac{2(2t^2-4t+1)}{[6(t-1/3)^2+1/3]^2} \in L^\infty(0,\infty)$. This $g(t)$ satisfies the condition in Theorem 1 in the case (i), but does not satisfy the condition (ii).

Example 3. Take a nondecreasing function $\psi(t) \in C^\infty(\mathbb{R})$ satisfying $0 \le \psi(t) \le 1$, $\psi(t) = 0$ on $(-\infty, 0)$, $\psi(t) = 1$ on $[1,\infty)$. Set $\varphi(t) = 2 - \sum_{j=1}^\infty 2^{-j}\psi\big(2^{2j}(t-2^j)\big)$. Then, we have $1 \le \varphi(t) \le 2$ on $(0,\infty)$, $\varphi(t)$ is decreasing, and

$$\limsup_{t\to+\infty} |t\varphi'(t)| = +\infty.$$

Put $g(t) = \exp(\int_1^t \frac{ds}{s\varphi(s)})$. Then $g(t)$ is positive and increasing on $(0,\infty)$, and $g'(t) = g(t)/(t\varphi(t))$ i.e. $g(t)/t = g'(t)\varphi(t)$. This $g(t)$ satisfies the condition in Theorem 1 in the case (ii), but does not satisfy the condition (i).

Example 4. Let $g(t) = \sum_{j=1}^k a_j t^j$ ($a_j \ge 0$, and $a_k > 0$). Then $g'(t) = \frac{g(t)}{t}\frac{\sum_{j=1}^k j a_j t^j}{g(t)}$. So $\varphi(t) = \frac{\sum_{j=1}^k a_j t^j}{\sum_{j=1}^k j a_j t^j}$ and it is easy to see that $\varphi'(t) < 0$, hence $\varphi(t)$ is strictly decreasing, and $1/k < \varphi(t) \le \lim_{t\to 0}\varphi(t)$. In this case, $t\varphi'(t)$ is also bounded.

Next, we prepare two more lemmas. Denote by J_ν be the Bessel function of order ν of the first kind. The following Lemma is given by L. Lorch and P. Szego, the old version of this type inequality is due to A. P. Calderón and A. Zygmund.

Lemma 2.3 ([9]). *Suppose ν and λ satisfy $|\nu| > 1/2, \lambda \ge -1/2$ or $\nu > -1$, $\lambda \ge 0$. Then*

$$\left|\int_o^r \frac{J_\nu(t)}{t^\lambda}dt\right| \le \frac{C}{|\nu|^\lambda}, \qquad \text{for} \quad 0 < r < \infty. \tag{2.1}$$

Lemma 2.4 ([1]). *Suppose $m \ge 1$ and $\lambda > 0$. Then*

$$\left|\frac{1}{r}\int_0^r \frac{J_{m+\lambda}}{t^\lambda}dt\right| \le \frac{C}{m^{\lambda+1}}, \qquad \text{for} \quad 0 < r < \infty.$$

Now we turn to the proof of Theorem 1.

Let $\mathcal{H}_k$ be the space of surface spherical harmonics of degree k on S^{n-1} with dimension D_k . By the same argument as in [2], one can reduce the proof of Theorem 1 to the case as follows:

$$f \in C_0^\infty \text{ and } \Omega(x, y') = \sum_{k\geq 1}\sum_{j=1}^{D_k} a_{k,j}(x) Y_{k,j}(y') \text{ is a finite sum,}$$

where $\{Y_{k,j}\}$ $(k \geq 1, j = 1, 2, ..., D_k)$ denotes the complete system of normalized surface spherical harmonics. Set

$$a_k(x) = \left(\sum_{j=1}^{D_k} |a_{k,j}(x)|^2\right)^{1/2} \text{ and } b_{k,j}(x) = \frac{a_{k,j}(x)}{a_k(x)}.$$

Then we get

$$\sum_{j=1}^{D_k} b_{k,j}^2(x) = 1 \text{ and } \Omega(x, y') = \sum_{k\geq 1} a_k(x) \sum_{j=1}^{D_k} b_{k,j}(x) Y_{k,j}(y').$$

Note that if we take $0 < \varepsilon < 1$ sufficiently close to 1, then by [2, p. 230, (4.4)] we have

$$\left(\sum_{k\geq 1} k^{-\varepsilon} a_k^2(x)\right)^{1/2} \leq C\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})} = C\|\Omega\|.$$

By Hölder's inequality, the above estimates and Fourier transform, we get

$$\begin{aligned}
\|\mu_\Omega^\Phi(f)\|_2^2 &= \int_{\mathbb{R}^n}\int_0^\infty \left|\int_{|y|\leq t} \sum_{k\geq 1} a_k(x) \sum_{j=1}^{D_k} b_{k,j}(x) \frac{Y_{k,j}(y')}{|y|^{n-1}} f(x - \Phi(|y|)y') dy\right|^2 \frac{dt}{t^3} dx \\
&\leq \int_{\mathbb{R}^n} \left(\sum_{k\geq 1} k^{-\varepsilon} a_k^2(x)\right) \sum_{k\geq 1} k^\varepsilon \int_0^\infty (\sum_{j=1}^{D_k} b_{k,j}^2(x)) \sum_{j=1}^{D_k} \left|\int_{|y|\leq t} \frac{Y_{k,j}(y')}{|y|^{n-1}}\right. \\
&\quad \times \left. f(x - \Phi(|y|)y') dy\right|^2 \frac{dt}{t^3} dx \\
&\leq C\|\Omega\|^2 \sum_{k\geq 1} k^\varepsilon \sum_{j=1}^{D_k} \int_0^\infty \int_{\mathbb{R}^n} \left|\int_{|y|\leq t} \frac{Y_{k,j}(y')}{|y|^{n-1}} f(x - \Phi(|y|)y') dy\right|^2 dx \frac{dt}{t^3} \\
&\leq C\|\Omega\|^2 \sum_{k\geq 1} k^\varepsilon \sum_{j=1}^{D_k} \int_0^\infty \int_{\mathbb{R}^n} \left|\left(\int_{|y|\leq t} \frac{Y_{k,j}(y')}{|y|^{n-1}} f(\cdot - \Phi(|y|)y') dy\right)^\wedge(\xi)\right|^2 d\xi \frac{dt}{t^3} \\
&= C\|\Omega\|^2 \sum_{k\geq 1} k^\varepsilon \int_{\mathbb{R}^n} \sum_{j=1}^{D_k} [\mu_\Omega^\Phi(Y_{k,j})(\xi)]^2 |\hat{f}(\xi)|^2 d\xi,
\end{aligned} \tag{2.2}$$

where

$$\mu_\Omega^\Phi(Y_{k,j})(\xi) = \left(\int_0^\infty \left|\frac{1}{t}\int_0^t \int_{S^{n-1}} e^{-i\Phi(r)\xi\cdot y'} Y_{k,j}(y') d\sigma(y') dr\right|^2 \frac{dt}{t}\right)^{1/2}.$$

So by Lemma 2.1, we only need to show

$$\sum_{j=1}^{D_k} \int_0^\infty \left|\frac{1}{t}\int_0^t \frac{J_{\frac{n}{2}+k-1}(\Phi(r)|\xi|)}{(\Phi(r)|\xi|)^{\frac{n}{2}-1}} dr Y_{k,j}(\xi')\right|^2 \frac{dt}{t} \leq Ck^{-2}. \tag{2.3}$$

Denote $N_t(\xi) = \frac{1}{t}\int_0^t \frac{J_{\frac{n}{2}+k-1}(\Phi(r)|\xi|)}{(\Phi(r)|\xi|)^{\frac{n}{2}-1}} dr$. Note that $\frac{\Phi(t)}{t} = C_2\Phi'(t)\varphi(t)$. Then

$$\frac{d(\Phi^{-1}(\frac{\rho}{|\xi|}))}{d\rho} = \frac{1}{\Phi'(\Phi^{-1}(\frac{\rho}{|\xi|}))} = \frac{C_2}{|\xi|}\frac{\Phi^{-1}(\frac{\rho}{|\xi|})}{\frac{\rho}{|\xi|}}\varphi(\Phi^{-1}(\frac{\rho}{|\xi|})) = \frac{C_2\Phi^{-1}(\frac{\rho}{|\xi|})}{\rho}\varphi(\Phi^{-1}(\frac{\rho}{|\xi|})).$$

Hence,

$$\begin{aligned} N_t(\xi) &= \frac{1}{t}\int_0^{\Phi(t)|\xi|} \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}-1}}(\Phi^{-1})'(\frac{\rho}{|\xi|})\frac{d\rho}{|\xi|} \\ &= \frac{C_2}{t}\int_0^{\Phi(t)|\xi|} \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho. \end{aligned}$$

Let $s = \Phi(t)|\xi|$. Then $ds = \Phi'(t)|\xi|dt = \frac{\Phi(t)}{C_2 t\varphi(t)}|\xi|dt = \frac{s}{C_2 t\varphi(t)}dt$. Hence, $\frac{dt}{t} = \frac{C_2\varphi(t)ds}{s} = \frac{C_2\varphi(\Phi^{-1}(\frac{s}{|\xi|}))ds}{s}$.

So

$$\int_0^\infty |N_t(\xi)|^2\frac{dt}{t} = C_2^3\int_0^\infty \frac{\varphi(\Phi^{-1}(\frac{s}{|\xi|}))}{\Phi^{-1}(\frac{s}{|\xi|})^2}\left|\int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\right|^2\frac{ds}{s}. \quad (2.4)$$

We only need to treat two cases (i) and (ii) in Theorem 1, where $\Phi(t)$ is positive and increasing on $(0,\infty)$.

First we consider Case (i): Since $J_{\frac{n}{2}+k-1}(\rho) > 0$ for $0 < \rho < n/2+k-1$ and $\Phi(t)$ is positive and increasing on $(0,\infty)$, together with Lemma 2.4, we have for $0 < s < k/2$

$$\begin{aligned} \left|\int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\right| &\le \left(\frac{1}{s}\int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}d\rho\right)s\Phi^{-1}(\frac{s}{|\xi|})\|\varphi\|_\infty \\ &\le \frac{Cs\Phi^{-1}(\frac{s}{|\xi|})}{(k-1)^{n/2+1}}. \end{aligned} \quad (2.5)$$

For $s \ge k/2$, take $0 < \varepsilon < \varepsilon + \eta < C_2\delta$ and then integrating by parts, we obtain

$$\begin{aligned} I &= \int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}-\varepsilon}}\frac{\Phi^{-1}(\frac{\rho}{|\xi|})}{\rho^\varepsilon}\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho \\ &= \left(\int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}-\varepsilon}}d\rho\right)\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^\varepsilon}\varphi(\Phi^{-1}(\frac{s}{|\xi|})) - \int_0^s\left(\int_0^\rho \frac{J_{\frac{n}{2}+k-1}(u)}{u^{\frac{n}{2}-\varepsilon}}du\right) \\ &\times\left\{\frac{\Phi^{-1}(\frac{\rho}{|\xi|})}{\rho^\varepsilon}\varphi^2(\Phi^{-1}(\frac{\rho}{|\xi|}))\frac{1}{\rho} + \frac{1}{\rho^\varepsilon}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi'(\Phi^{-1}(\frac{\rho}{|\xi|}))\frac{\Phi^{-1}(\frac{\rho}{|\xi|})}{\rho}\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))\right. \\ &\left. -\varepsilon\frac{1}{\rho^{1+\varepsilon}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))\right\}d\rho. \end{aligned}$$

Hence, by Lemma 2.3, we get the following estimate for I,

$$\begin{aligned} |I| &\le C\left(\int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}-\varepsilon}}d\rho\right)\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^\varepsilon}\|\varphi\|_\infty + \int_0^s\left|\int_0^\rho \frac{J_{\frac{n}{2}+k-1}(u)}{u^{\frac{n}{2}-\varepsilon}}du\right|\frac{d\rho}{\rho^{1-\eta}}\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^{\eta+\varepsilon}} \\ &\times\left\{\|\varphi\|_\infty^2 + \|t\varphi'(t)\|_\infty\|\varphi\|_\infty + \varepsilon\|\varphi\|_\infty\right\} \\ &\le C\frac{1}{(\frac{n}{2}+k-1)^{\frac{n}{2}-\varepsilon}}\Phi^{-1}(\frac{s}{|\xi|})\frac{1}{s^\varepsilon}. \end{aligned} \quad (2.6)$$

Then by (2.5) and (2.6), we get

$$\int_0^\infty |N_t(\xi)|^2\frac{dt}{t} \le C\|\varphi\|_\infty^3\int_0^{k/2}\frac{s^2}{(k-1)^{n+2}}\frac{ds}{s} + C\int_{k/2}^\infty \frac{1}{s^{2\varepsilon}}\frac{ds}{s}\frac{1}{(\frac{n}{2}+k-1)^{n-2\varepsilon}} \le \frac{C}{k^n}. \tag{2.7}$$

Case (ii). We may assume φ is increasing since the proof is similar for the case φ is decreasing. Letting $\nu = \frac{n}{2}+k-1$, we will consider the following four cases :

(1) For $0 \le s \le \nu$, since $J_\nu(\rho) > 0$ for $0 \le \rho \le \nu$, and since by Lemma 2.2, $\frac{\Phi^{-1}(\frac{\rho}{|\xi|})}{\rho^\varepsilon}$ is increasing for $0 < \varepsilon < \min\{1/4, C_2\delta\}$, we have

$$\begin{aligned}\Big|\int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\Big| &= \Big|\int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}-\varepsilon}}\frac{\Phi^{-1}(\frac{\rho}{|\xi|})}{\rho^\varepsilon}\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\Big| \\ &\le \frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^\varepsilon}\|\varphi\|_\infty\int_0^s\frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}-\varepsilon}}d\rho \\ &\le C\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^\varepsilon}\|\varphi\|_\infty\frac{1}{(\frac{n}{2}+k-1)^{\frac{n}{2}-\varepsilon}}.\end{aligned}$$

(2) For $\nu < s \le 2\nu$,

$$\begin{aligned}\Big|\int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\Big| &\le \Big|\int_0^\nu \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\Big| \\ &\quad + \Big|\int_\nu^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\Big| \\ &= I_1 + I_2.\end{aligned}$$

By (1), we know that

$$I_1 \le C\frac{\Phi^{-1}(\frac{\nu}{|\xi|})}{\nu^\varepsilon}\|\varphi\|_\infty\frac{1}{(\frac{n}{2}+k-1)^{\frac{n}{2}-\varepsilon}} \le C\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^\varepsilon}\|\varphi\|_\infty\frac{1}{(\frac{n}{2}+k-1)^{\frac{n}{2}-\varepsilon}}.$$

As for I_2, by using the second mean value theorem, and Lemma 2.3, we get

$$\begin{aligned}I_2 &= \Big|\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^\varepsilon}\varphi(\Phi^{-1}(\frac{s}{|\xi|}))\int_{s'}^s\frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}-\varepsilon}}d\rho\Big| \\ &= \frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^\varepsilon}\varphi(\Phi^{-1}(\frac{s}{|\xi|}))\Big|\int_{s'}^s\frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}-\varepsilon}}d\rho\Big| \le C\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^\varepsilon}\|\varphi\|_\infty\frac{1}{(\frac{n}{2}+k-1)^{\frac{n}{2}-\varepsilon}}.\end{aligned}$$

(3) For $2\nu < s < \nu^3$,

$$\begin{aligned}\Big|\int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\Big| &\le \Big|\int_0^{2\nu} \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\Big| \\ &\quad + \Big|\int_{2\nu}^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\Big| \\ &= I_3 + I_4.\end{aligned}$$

By (2), we know that

$$I_3 \le C\frac{\Phi^{-1}(\frac{2\nu}{|\xi|})}{(2\nu)^\varepsilon}\|\varphi\|_\infty\frac{1}{(\frac{n}{2}+k-1)^{\frac{n}{2}-\varepsilon}} \le C\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^\varepsilon}\|\varphi\|_\infty\frac{1}{(\frac{n}{2}+k-1)^{\frac{n}{2}-\varepsilon}}.$$

As for I_4, since $|J_\nu(x)| \le 1$ (see [17, p. 406]), it is easy to see $|J'_\nu(\rho)| \le |J_{\nu-1}(\rho) - J_{\nu+1}(\rho)|/2 \le 1$ (see also [17, p. 45 and p. 406]). Hence,

$$\Big|\int_{2\nu}^{s} \frac{J'_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n-1}{2}-\varepsilon}(\rho^2-\nu^2)} \frac{\Phi^{-1}(\frac{\rho}{|\xi|})}{\rho^\varepsilon}\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\Big| \le \frac{\Phi^{-1}(\frac{s}{|\xi|})\|\varphi\|_\infty}{s^\varepsilon}\int_{2\nu}^{s}\frac{1}{\rho^{\frac{n-1}{2}-\varepsilon}(\rho^2-\nu^2)}d\rho$$
$$\le C\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^\varepsilon}\|\varphi\|_\infty\frac{1}{k^{\frac{n+1}{2}-\varepsilon}}. \tag{2.8}$$

On the other hand, since $\frac{r}{r^{\frac{n-1}{2}-\varepsilon}(r^2-\nu^2)}$ is decreasing on $[2\nu,\infty)$, by using the second mean value theorem twice, we have

$$\Big|\int_{2\nu}^{s} \frac{J''_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n-1}{2}-\varepsilon}(\rho^2-\nu^2)} \frac{\rho\Phi^{-1}(\frac{\rho}{|\xi|})}{\rho^\varepsilon}\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\Big|$$
$$\le \frac{2\nu}{(2\nu)^{\frac{n-1}{2}-\varepsilon}((2\nu)^2-\nu^2)}\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^\varepsilon}\|\varphi\|_\infty\Big|\int_{\eta'}^{s'} J''_{\frac{n}{2}+k-1}(\rho)d\rho\Big| \tag{2.9}$$
$$\le C\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^\varepsilon}\|\varphi\|_\infty\frac{1}{k^{\frac{n+1}{2}-\varepsilon}}.$$

Thus by (2.8), (2.9) and the fact that

$$\frac{J_\nu(\rho)}{\rho^{\frac{n}{2}-\varepsilon}} = -\frac{J'_\nu(\rho)}{\rho^{\frac{n-1}{2}-\varepsilon}(\rho^2-\nu^2)} - \frac{\rho J''_\nu(\rho)}{\rho^{\frac{n-1}{2}-\varepsilon}(\rho^2-\nu^2)},$$

we get

$$I_4 \le C\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^\varepsilon}\|\varphi\|_\infty\frac{1}{k^{\frac{n+1}{2}-\varepsilon}}.$$

(4) For $\nu^3 < s$,

$$\Big|\int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\Big| \le \Big|\int_0^{\nu^3} \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\Big|$$
$$+\Big|\int_{\nu^3}^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\Big|$$
$$= I_5 + I_6.$$

By (3), we know that

$$I_5 \le C\frac{\Phi^{-1}(\frac{\nu^3}{|\xi|})}{(\nu^3)^\varepsilon}\|\varphi\|_\infty\frac{1}{k^{\frac{n}{2}-\varepsilon}} \le C\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^\varepsilon}\|\varphi\|_\infty\frac{1}{k^{\frac{n}{2}-\varepsilon}}.$$

Using the following inequality (see [17, p. 447]),

$$|J_\nu(x)| \le \frac{\sqrt{2/\pi}}{(x^2-\nu^2)^{1/4}} \qquad \text{for} \quad x \ge \nu \ge 1/2,$$

we see that $|J_\nu(\rho)| \le \frac{C}{\sqrt{\rho}}$ for $\rho > 2\nu$. Hence

$$I_6 = \Big|\int_{\nu^3}^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}-\varepsilon}}\frac{\Phi^{-1}(\frac{\rho}{|\xi|})}{\rho^\varepsilon}\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\Big| \le C\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^\varepsilon}\|\varphi\|_\infty\int_{\nu^3}^s\frac{1}{\rho^{\frac{n}{2}-\varepsilon+\frac{1}{2}}}d\rho$$
$$\le C\frac{\Phi^{-1}(\frac{s}{|\xi|})}{s^\varepsilon}\|\varphi\|_\infty\frac{1}{k^{\frac{n}{2}-\varepsilon}}.$$

By (2.5) and (1)-(4) above, we have

$$\int_0^\infty |N_t(\xi)|^2 \frac{dt}{t} \le C\|\varphi\|_\infty^3 \int_0^{k/2} \frac{s^2}{(k-1)^{n+2}} \frac{ds}{s} + C\int_{k/2}^\infty (\frac{1}{s^{2\varepsilon}k^{n-2\varepsilon}} + \frac{1}{s^{2\varepsilon}k^{n+1-2\varepsilon}}) \frac{ds}{s}$$
$$\le C(\frac{1}{k^n} + \frac{1}{k^{n+1}}).$$

Thus, in both cases (i) and (ii), we have

$$\int_0^\infty |N_t(\xi)|^2 \frac{dt}{t} \le \frac{C}{k^n}.$$

Therefore, by the fact $\sum_{j=1}^{D_k} |Y_{k,j}(\xi')|^2 = w^{-1}D_m \sim k^{n-2}$ (see [3, p. 255, (2.6)]), where w denotes the area of S^{n-1}, we get

$$\sum_{j=1}^{D_k} \int_0^\infty |N_t(\xi)(Y_{k,j})(\xi')|^2 \frac{dt}{t} \le Ck^{-2}.$$

Thus, inequality (2.3) holds and the proof of Theorem 1 is finished. □

3 Proofs of Theorems 2 and 4 To prove Theorem 2, we only need to treat two cases: (1) Φ is positive and increasing, and (2) Φ is positive and decreasing.

(1) The case where Φ is positive and increasing. Let h_1 and h_2 be any numbers satisfying $\Phi(0)|\xi| \le h_1 < h_2 \le \Phi(\infty)|\xi|$. As in the proof of Lemma 2.2 in [16], we treat

$$L = \int_{h_1}^{h_2} \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} \varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho.$$

First we consider the case $h_1 \le k/2 \le h_2$. We have

$$L = \int_{h_1}^{k/2} \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} \varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho + \int_{k/2}^{h_2} \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} \varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho = L_1 + L_2.$$

For L_1, since $J_{\frac{n}{2}+k-1}(\rho) > 0$, by Lemma 2.3, we get

$$|L_1| \le \|\varphi\|_\infty \int_{h_1}^{\frac{k}{2}} \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} d\rho \le C\|\varphi\|_\infty \frac{1}{k^{\frac{n}{2}}}.$$

For L_2, we take $0 < \varepsilon < n/2$. Then, integration by parts, together with the facts $\varphi \in L^\infty$ and $t\varphi'(t)$ is bounded, yields

$$|L_2| = \left| \int_{k/2}^{h_2} \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}-\varepsilon}} \frac{\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))}{\rho^\varepsilon} d\rho \right|$$
$$= \left| \left(\int_{k/2}^{h_2} \frac{J_{\frac{n}{2}+k-1}(u)}{u^{\frac{n}{2}-\varepsilon}} du \right) \frac{\varphi(\Phi^{-1}(\frac{h_2}{|\xi|}))}{h_2{}^\varepsilon} d\rho - \int_{k/2}^{h_2} \left(\int_{k/2}^{\rho} \frac{J_{\frac{n}{2}+k-1}(u)}{u^{\frac{n}{2}-\varepsilon}} du \right) \right.$$
$$\left. \times \left(\varphi'(\Phi^{-1}(\frac{\rho}{|\xi|}))\Phi^{-1}(\frac{\rho}{|\xi|})\varphi(\Phi^{-1}(\frac{\rho}{|\xi|})) - \varepsilon\varphi(\Phi^{-1}(\frac{\rho}{|\xi|})) \right) \frac{d\rho}{\rho^{1+\varepsilon}} \right|$$
$$\le \frac{C\|\varphi\|_\infty}{(h_2)^\varepsilon} \frac{1}{k^{\frac{n}{2}-\varepsilon}} + \frac{C\|\varphi\|_\infty(1 + \|t\varphi'(t)\|_\infty)}{(k/2)^\varepsilon} \frac{1}{k^{\frac{n}{2}-\varepsilon}}$$
$$\le \frac{C\|\varphi\|_\infty(1 + \|t\varphi'(t)\|_\infty)}{k^{\frac{n}{2}}}.$$

As for the cases $k/2 < h_1$, replace $k/2$ by h_1 and repeat the step as we have dealt with L_2, we can obtain

$$|L| \le C\frac{C\|\varphi\|_\infty(1+\|t\varphi'(t)\|_\infty)}{k^{\frac{n}{2}}}.$$

Finally, for the case $k/2 > h_2$, similarly as we have dealt with L_1, we get

$$|L| \le \|\varphi\|_\infty \int_{h_1}^{\frac{k}{2}} \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} d\rho \le C\|\varphi\|_\infty \frac{1}{k^{\frac{n}{2}}}.$$

(2) The case Φ is positive and decreasing. In this case we need to treat h_1, h_2 satisfying $\Phi(\infty)|\xi| \le h_1 < h_2 \le \Phi(0)|\xi|$. So, we have the same estimate as in the case (1).

In any case we have

$$\left|\int_{h_1}^{h_2} \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}}\varphi(\Phi^{-1}(\frac{\rho}{|\xi|}))d\rho\right| \le \frac{C\|\varphi\|_\infty(1+\|t\varphi'(t)\|_\infty)}{k^{\frac{n}{2}}}.$$

The rest of the proof of Theorem 2 is the same as in [16], so we omit the detail. □

Next, we shall give the proof of Theorem 4.

First, we know that $\mu_S^{\Phi,\sigma}(f)(x) \le 2^{\lambda n}\mu_{\lambda,\Phi}^{*,\sigma}(f)(x)$. On the other hand,

$$\begin{aligned}
&\|\mu_{\lambda,\Phi}^{*,\sigma}(f)\|_2^2 \\
&= \int_{\mathbb{R}^n}\iint_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\frac{1}{t^\sigma}\int_{|z|<t}\frac{\Omega(y,z)}{|z|^{n-\sigma}}f(y-\Phi(|z|)z')dz\right|^2\frac{dydt}{t^{n+1}}dx \\
&= \int_0^\infty\int_{\mathbb{R}^n}\left(\frac{1}{t^n}\int_{\mathbb{R}^n}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}dx\right)\left|\frac{1}{t^\sigma}\int_{|z|<t}\frac{\Omega(y,z)}{|z|^{n-\sigma}}f(y-\Phi(|z|)z')dz\right|^2\frac{dydt}{t} \\
&\le C\|\mu_\Omega^{\Phi,\sigma}(f)\|_2^2.
\end{aligned} \tag{3.1}$$

Hence we only need to give the estimates for $\mu_\Omega^{\Phi,\sigma}(f)$. Similarly as (2.2), we get

$$\begin{aligned}
&\|\mu_\Omega^{\Phi,\sigma}(f)\|_2^2 \\
&= \int_{\mathbb{R}^n}\int_0^\infty\left|\int_{|y|\le t}\sum_{k\ge1}a_k(x)\sum_{j=1}^{D_k}b_{k,j}(x)\frac{Y_{k,j}(y')}{|y|^{n-\sigma}}f(x-\Phi(|y|)y')dy\right|^2\frac{dt}{t^{1+2\sigma}}dx \\
&\le \int_{\mathbb{R}^n}\left(\sum_{k\ge1}k^{-\varepsilon}a_k^2(x)\right)\sum_{k\ge1}k^\varepsilon\int_0^\infty(\sum_{j=1}^{D_k}b_{k,j}^2(x)) \\
&\quad\times \sum_{j=1}^{D_k}\left|\int_{|y|\le t}\frac{Y_{k,j}(y')}{|y|^{n-\sigma}}f(x-\Phi(|y|)y')dy\right|^2\frac{dt}{t^{1+2\sigma}}dx \\
&\le C\|\Omega\|^2\sum_{k\ge1}k^\varepsilon\sum_{j=1}^{D_k}\int_0^\infty\int_{\mathbb{R}^n}\left|\int_{|y|\le t}\frac{Y_{k,j}(y')}{|y|^{n-\sigma}}f(x-\Phi(|y|)y')dy\right|^2dx\frac{dt}{t^{1+2\sigma}} \\
&\le C\|\Omega\|^2\sum_{k\ge1}k^\varepsilon\sum_{j=1}^{D_k}\int_0^\infty\int_{\mathbb{R}^n}\left|\left(\int_{|y|\le t}\frac{Y_{k,j}(y')}{|y|^{n-\sigma}}f(\cdot-\Phi(|y|)y')dy\right)^\wedge(\xi)\right|^2d\xi\frac{dt}{t^{1+2\sigma}} \\
&= C\|\Omega\|^2\sum_{k\ge1}k^\varepsilon\sum_{j=1}^{D_k}\int_{\mathbb{R}^n}\sum_{j=1}^{D_k}[\mu_\Omega^{\Phi,\sigma}(Y_{k,j})(\xi)]^2|\hat{f}(\xi)|^2d\xi,
\end{aligned}$$

where

$$\mu_\Omega^{\Phi,\sigma}(Y_{k,j})(\xi) = \left(\int_0^\infty \left| \frac{1}{t^\sigma} \int_0^t \int_{S^{n-1}} r^{\sigma-1} e^{-i\Phi(r)\xi\cdot y'} Y_{k,j}(y') d\sigma(y') dr \right|^2 \frac{dt}{t} \right)^{1/2}.$$

By Lemma 2.1, we have

$$\begin{aligned}
&\frac{1}{t^\sigma} \int_0^t \int_{S^{n-1}} r^{\sigma-1} e^{-i\Phi(r)\xi\cdot y'} Y_{k,j}(y') d\sigma(y') dr \\
&= \frac{1}{t^\sigma} \int_0^t r^{\sigma-1} e^{-i\Phi(r)\xi\cdot y'} \frac{J_{\frac{n}{2}+k-1}(\Phi(r)|\xi|)}{(\Phi(r)|\xi|)^{\frac{n}{2}-1}} dr Y_{k,j}(\xi') \\
&= \frac{C}{t^\sigma} \int_0^{\Phi(t)|\xi|} [\Phi^{-1}(\frac{\rho}{|\xi|})]^{\sigma-1} \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} \Phi^{-1}(\frac{\rho}{|\xi|}) \varphi(\Phi^{-1}(\frac{\rho}{|\xi|})) dr Y_{k,j}(\xi') \\
&= \frac{C}{t^\sigma} \int_0^{\Phi(t)|\xi|} \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} [\Phi^{-1}(\frac{\rho}{|\xi|})]^\sigma \varphi(\Phi^{-1}(\frac{\rho}{|\xi|})) dr Y_{k,j}(\xi').
\end{aligned} \tag{3.2}$$

For any $\sigma > 0$, if we take $0 < \varepsilon < \min\{1/4, C_2\sigma\delta\}$, then we see by Lemma 2.2 that $\frac{[g^{-1}(t)]^\sigma}{t^\varepsilon}$ is strictly increasing on $(0, \infty)$. Thus, Theorem 4 follows from repeating the steps in the proof of Theorem 1. □

4 Proofs of Theorem 3 and Theorem 5 It is easy to see that if $\Phi(t) = t^{-\alpha}$ for $\alpha > 0$, then $\Phi'(t) = -\alpha\frac{\Phi(t)}{t}$ with $\varphi(t) = -\frac{1}{\alpha}$.

$$\begin{aligned}
N_t(\xi) &= \frac{1}{\Phi^{-1}(\frac{s}{|\xi|})} \int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} \Phi^{-1}(\frac{\rho}{|\xi|}) \varphi(\Phi^{-1}(\frac{\rho}{|\xi|})) d\rho \\
&= -\frac{1}{\alpha(\frac{s}{|\xi|})^{-\frac{1}{\alpha}}} \int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}}} (\frac{\rho}{|\xi|})^{-\frac{1}{\alpha}} d\rho \\
&= -\frac{s^{\frac{1}{\alpha}}}{\alpha} \int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}+\frac{1}{\alpha}}} d\rho.
\end{aligned}$$

If $0 < \alpha \le 1/k$, then from the fact $J_{\frac{n}{2}+k-1}(\rho) \sim \rho^{\frac{n}{2}+k-1} (\rho \to 0)$, it follows that $|N_t(\xi)| = \infty$. If $\alpha > 1/k$, by the Weber and Schafheitilin formula in [17, p. 391], we know that the integral $\int_0^\infty \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}+\frac{1}{\alpha}}} d\rho$ converges. Thus, we write

$$\int_0^s \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}+\frac{1}{\alpha}}} d\rho = \int_0^\infty \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}+\frac{1}{\alpha}}} d\rho - \int_s^\infty \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}+\frac{1}{\alpha}}} d\rho.$$

On the other hand,

$$\begin{aligned}
\int_s^\infty \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}+\frac{1}{\alpha}}} d\rho &= \int_s^\infty \left(\int_0^\rho \frac{J_{\frac{n}{2}+k-1}(u)}{u^{\frac{n}{2}+\frac{1}{\alpha}-\varepsilon}} du \right)' \frac{d\rho}{\rho^\varepsilon} \\
&= \int_0^s \frac{J_{\frac{n}{2}+k-1}(u)}{u^{\frac{n}{2}+\frac{1}{\alpha}-\varepsilon}} du \cdot \frac{1}{s^\varepsilon} - \varepsilon \int_s^\infty \left(\int_0^\rho \frac{J_{\frac{n}{2}+k-1}(u)}{u^{\frac{n}{2}+\frac{1}{\alpha}-\varepsilon}} du \right) \frac{d\rho}{\rho^{1+\varepsilon}}.
\end{aligned}$$

Therefore, by Lemma 2.3,

$$\left| \int_s^\infty \frac{J_{\frac{n}{2}+k-1}(\rho)}{\rho^{\frac{n}{2}+\frac{1}{\alpha}}} d\rho \right| \le \frac{C}{k^{\frac{n}{2}+\frac{1}{\alpha}-\varepsilon}} (\frac{1}{s^\varepsilon} + \int_s^\infty \frac{d\rho}{\rho^{1+\varepsilon}}) \le C \frac{s^{-\varepsilon}}{k^{\frac{n}{2}+\frac{1}{\alpha}-\varepsilon}}.$$

Taking $\varepsilon > \frac{1}{\alpha}$, we have $|N_t(\xi)| \approx Cs^{\frac{1}{\alpha}}$ if $s \to +\infty$, which means $\int_0^\infty |N_t(\xi)|^2 \frac{dt}{t} = +\infty$. Hence for $f \in L^2(\mathbb{R}^n)$ with $\|f\|_2 > 0$,

$$\begin{aligned}\|\mu^{\Phi}_{Y_{k,j}}(f)\|_2^2 &= \int_{\mathbb{R}^n}\int_0^\infty \left|\int_{|y|\le t} \frac{Y_{k,j}(y')}{|y|^{n-1}} f(x-\Phi(|y|)y')dy\right|^2 \frac{dt}{t^3}dx \\ &= \int_{\mathbb{R}^n} [\mu^{\Phi}_{\Omega}(Y_{k,j})(\xi)]^2 |\hat{f}(\xi)|^2 d\xi = C\int_{\mathbb{R}^n}\left(\int_0^\infty N_t(\xi)^2 \frac{dt}{t}|Y_{k,j}(\xi')|^2\right)|\hat{f}(\xi)|^2\, d\xi \\ &= +\infty,\end{aligned}$$

which proves Theorem 3. Similarly, by (3.1), (3.2) and the proof procedure of Theorem 3, we can prove Theorem 5, and so we omit the proof. □

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