ON LATTICES OF RADICALS OF INVOLUTION RINGS

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Abstract. We continue the study of lattice of radicals of involution rings which commenced in [3]. In particular, we consider atoms in the lattice of invariant hereditary radicals, and the lattice of special radicals. We also introduce a generalised ADS property and study the lattice of radicals whose semisimple classes have this property.

1. Introduction

Let R be an associative ring. The notation $A \lhd R$ means that A is an ideal of R. We recall that an *involution* on R is an anti-isomorphism $*$ of R onto itself (the image of r being denoted r^*) such that $(r^*)^* = r$ for all $r \in R$. An *involution ring* is a pair $(R,*)$ such that R is a ring and $*$ is an involution on R. If R is any ring and R^{op} is the anti-isomorphic image of R, then $(R \oplus R^{op}, e)$ is an involution ring where $(r,s)^e := (s,r)$ for all $(r,s) \in R \oplus R^{op}$. The involution e is called the *exchange involution*. The varieties of rings and involution rings will be denoted $\underline{Rng}$ and $\underline{IR}$, respectively. Recall that if $(R,*)$, $(S,*) \in \underline{IR}$, then f is a homomorphism in $\underline{IR}$ if $f : R \to S$ is a ring homomorphism, and $f(r^*) = (f(r))^*$ for all $r \in R$. The ideals of $(R,*)$ are the kernels of the homomorphisms. The notation $(A,*) \lhd (R,*)$ will mean that $(A,*)$ is an ideal of the involution ring $(R,*)$.

All subclasses of $\underline{Rng}$ and $\underline{IR}$ considered are *abstract*, i.e. closed under isomorphism. A class $\mathcal{C}$ is called *hereditary* if any ideal of an element of $\mathcal{C}$ is again an element of $\mathcal{C}$.

In this paper, "radical" will mean a radical in the sense of Kurosh and Amitsur. If $\mathcal{R}$ is a radical either in $\underline{Rng}$ or in $\underline{IR}$, its semisimple class will be denoted $\mathcal{SR}$. If $\mathcal{C}$ is a subclass of either $\underline{Rng}$ or $\underline{IR}$, the *lower radical determined by* $\mathcal{C}$ is the smallest radical in that variety which contains $\mathcal{C}$, and will be denoted by $\mathcal{LC}$. If $\mathcal{C}$ is a hereditary subclass of either $\underline{Rng}$ or $\underline{IR}$, the *upper radical determined by* $\mathcal{C}$ is the largest radical in that variety such that $\mathcal{C} \subseteq \mathcal{SR}$, and will be denoted by $\mathcal{UC}$. We remark that $\mathcal{UC}$ consists of those elements of the variety in question which have no nonzero homomorphic image in $\mathcal{C}$.

The radical theory in $\underline{IR}$ differs in certain ways from that in $\underline{Rng}$. Radicals in $\underline{Rng}$ have the *ADS property*, i.e. if $I \lhd R$, then $\mathcal{R}(I) \lhd R$ for any radical $\mathcal{R}$. This is not true in general for radicals in $\underline{IR}$. Radicals in $\underline{IR}$ which have this property are called *ADS-radicals*. Snider [11] showed that the class $\mathbb{L}^{\underline{Rng}}$ has a natural complete lattice structure (although it is not a set) with respect to inclusion, and this is in fact true for radicals in any universal class. The lattice $\mathbb{L}^{\underline{IR}}$ of radicals in $\underline{IR}$ was studied in [3]. For further details concerning general radical theory, we refer to any of the standard texts, for example [7].

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2. Atoms in sublattices of $\mathbb{L}^{\underline{IR}}$

In [3] it was shown that the classes $\mathbb{L}_h^{\underline{IR}}$ and $\mathbb{L}_i^{\underline{IR}}$ of hereditary and invariant radicals are complete sublattices of $\mathbb{L}^{\underline{IR}}$. Hence the class $\mathbb{L}_{ih}^{\underline{IR}}$ of hereditary invariant radicals is also a complete sublattice of $\mathbb{L}^{\underline{IR}}$. We will characterise the atoms of $\mathbb{L}_{ih}^{\underline{IR}}$.

Let $\mathcal{C}$ be a hereditary, homomorphically closed class of rings. It is well known that $\mathcal{LC} = \{R \in \underline{Rng}$ |every nonzero homomorphic image of R has a nonzero accessible subring which is in $\mathcal{C}\}$. A similar characterization may be given for involution rings, that is, if $\mathcal{C}$ is a hereditary, homomorphically closed subclass of $\underline{IR}$, then $\mathcal{LC} = \{(R,*) \in \underline{IR}$ |every nonzero homomorphic image of $(R,*)$ has a nonzero accessible sub-involution ring which is in $\mathcal{C}\}$.

Lemma 2.1. *Let S be a simple ring. Then $\lambda\mathcal{L}\{S, S^{op}\} = \mathcal{L}\{(S,*) \mid *$ is an involution on $S\} \vee \mathcal{L}\{(S \oplus S^{op}, e)\}$.*

Proof. If $*$ is an involution on S, then by the definition of the operator λ, $(S,*) \in \lambda\mathcal{L}\{S, S^{op}\}$. Hence $\mathcal{L}\{(S,*) \mid *$ is an involution on $S\} \leq \lambda\mathcal{L}\{S, S^{op}\}$. Also, $S \oplus S^{op} \in \mathcal{L}\{S, S^{op}\}$ and so $(S \oplus S^{op}, e) \in \lambda\mathcal{L}\{S, S^{op}\}$, whence $\mathcal{L}\{(S \oplus S^{op}, e)\} \leq \lambda\mathcal{L}\{S, S^{op}\}$ and so $\mathcal{L}\{(S,*) \mid *$ is an involution on $S\} \vee \mathcal{L}\{(S \oplus S^{op}, e)\} \leq \lambda\mathcal{L}\{S, S^{op}\}$.

To prove the reverse inclusion, note that $\mathcal{L}\{(S,*) \mid *$ is an involution on $S\} \vee \mathcal{L}\{(S \oplus S^{op}, e)\} = \mathcal{LC}$, where $\mathcal{C} := \{(S,*) \mid *$ is an involution on $S\} \cup \{(S \oplus S^{op}, e)\}$. Since $\mathcal{C}$ consists of simple involution rings, it is homomorphically closed and hereditary. Suppose that $(R,*) \in \lambda\mathcal{L}\{S, S^{op}\}$. Then $R \in \mathcal{L}\{S, S^{op}\}$, whence every nonzero homomorphic image of R has a nonzero accessible subring which is isomorphic either to S or to S^{op}. In particular, if $(T,*)$ is a nonzero homomorphic image of $(R,*)$, then T contains a nonzero accessible subring K which is isomorphic either to S or to S^{op}. It is easily seen that $(K+K^*,*)$ is an accessible sub-involution ring of $(T,*)$. If $K = K^*$, then $(K+K^*,*) = (K,*) \in \{(S,*) \mid *$ is an involution on $S\} \subseteq \mathcal{C}$. If $K \neq K^*$ then $(K+K^*,*) \cong (S \oplus S^{op}, e) \in \mathcal{C}$ (cf. [1, Theorem 3.12], noting that the proof of this result does not make use of the existence of a unity in S). Thus in either case, $(T,*)$ has a nonzero accesssible sub-involution ring which is in $\mathcal{C}$, so $(R,*) \in \mathcal{LC}$, and the proof is complete. ∎

Proposition 2.2. *The lattice $\mathbb{L}_{ih}^{\underline{IR}}$ is atomic, and the atoms of $\mathbb{L}_{ih}^{\underline{IR}}$ are those radicals $\mathcal{A}$ of the form $\mathcal{A} = \mathcal{L}\{(S,*) \mid *$ is an involution on $S\} \vee \mathcal{L}\{(S \oplus S^{op}, e)\}$, where S is a simple ring.*

Proof. In [3, Propositions 2.4 and 3.11] it is shown that the lattice $\mathbb{L}_{sh}^{\underline{Rng}}$ of symmetric hereditary radicals of rings is atomic, and the mapping $\mathcal{R} \to \lambda\mathcal{R}$ is a lattice isomorphism of $\mathbb{L}_{sh}^{\underline{Rng}}$ onto $\mathbb{L}_{ih}^{\underline{IR}}$. Moreover the atoms of $\mathbb{L}_{sh}^{\underline{Rng}}$ are the radicals $\mathcal{L}\{S, S^{op}\}$ where S is a simple ring. Hence the atoms of $\mathbb{L}_{ih}^{\underline{IR}}$ are the radicals $\lambda\mathcal{L}\{S, S^{op}\}$ where S is a simple ring. The result now follows from Lemma 2.1. ∎

Proposition 2.3. *Let S be a simple ring with unity. Then $\mathcal{A} = \mathcal{L}\{(S,*) \mid *$ is an involution on $S\} \vee \mathcal{L}\{(S \oplus S^{op}, e)\}$ is an atom of $\mathbb{L}_i^{\underline{IR}}$.*

Proof. Let $\mathcal{R}$ be a symmetric radical in $\underline{Rng}$ such that $0 < \mathcal{R} \leq \mathcal{L}\{S, S^{op}\}$ and let $0 \neq R \in \mathcal{R}$. Then R has a a nonzero accessible subring I such that either $I \cong S$ or $I \cong S^{op}$. Assume the former. Since S has a unity, $I \lhd R$ and is a direct summand of R. Hence S is a homomorphic image of R and so is in $\mathcal{R}$. Since $\mathcal{R}$ is symmetric, $S^{op} \in \mathcal{R}$, and so $\{S, S^{op}\} \subseteq \mathcal{R}$. Similarly, $I \cong S^{op}$ implies $\{S, S^{op}\} \subseteq \mathcal{R}$. Hence $\mathcal{L}\{S, S^{op}\} = \mathcal{R}$, and so $\mathcal{L}\{S, S^{op}\}$ is an atom of $\mathbb{L}_s^{\underline{Rng}}$. The desired result follows from the fact that the mapping

$\mathcal{R} \to \lambda\mathcal{R}$ defines an isomorphism of $\mathbb{L}_s^{\underline{Rng}}$ onto $\mathbb{L}_i^{\underline{IR}}$ [3, Propositions 2.4 and 3.11] and Lemma 2.1. ∎

3. Special Radicals

It is well known [11] that the class $\mathbb{L}_{sp}^{\underline{Rng}}$ of special radicals in $\underline{Rng}$ is a lattice which is not a sublattice of $\mathbb{L}^{\underline{Rng}}$. Special radicals for involution rings were defined by Salavová [10]. Recall that an involution ring $(R,*)$ is called *prime* if $(A,*),(B,*) \lhd (R,*)$, $AB=0$ implies $A=0$ or $B=0$. A class $\mathcal{M}$ of involution rings is called a *special class* if (i) $\mathcal{M}$ consists of prime involution rings, (ii) $\mathcal{M}$ is hereditary and (iii) if $(A,*) \lhd (R,*)$, $(R,*)$ prime and $(A,*) \in \mathcal{M}$ implies that $(R,*) \in \mathcal{M}$. If $\mathcal{R}$ is a radical in $\underline{IR}$ such that $\mathcal{R} = \mathcal{UM}$ for some special class $\mathcal{M}$, then $\mathcal{R}$ is called a *special radical*. It was shown in [3] that the class $\mathbb{L}_{sp}^{\underline{IR}}$ of all special radicals in $\underline{IR}$ is a lattice, but not a sublattice of $\mathbb{L}^{\underline{IR}}$

The prime (Baer) radical in $\underline{Rng}$ will be denoted β and the prime radical in $\underline{IR}$ will be denoted β_*. Nore that $\beta_* = \lambda\beta$. If $\mathcal{C}$ is a class of rings or involution rings, the smallest special radical containing $\mathcal{C}$ will be denoted $\mathcal{L}_s\mathcal{C}$. Whether $\mathcal{L}_s\mathcal{C}$ is a radical of rings or involution rings will be clear from the context.

A ring R is called a $*$-*ring* ([8], [5]) if R is a prime ring and all non-isomorphic homomorphic images of R are β-radical rings. In view of the association of the symbol $*$ with the involution operation, we will refer to such rings as *s-rings* in the sequel. Similarly an involution ring $(R,*)$ which is prime, and whose non-isomorphic homomorphic images are β_*-radical rings will be called an *s-involution ring*. The smallest special radical containing a class $\mathcal{C}$ of rings or involution rings will be denoted $\mathcal{LC}$. H. France-Jackson (formerly H. Korulczuk) [8] has given a partial characterisation of atoms in $\mathbb{L}_{sp}^{\underline{Rng}}$.

Proposition 3.1. *Let R be an s-ring. Then $\mathcal{L}_s\{R\}$ is an atom of $\mathbb{L}_{sp}^{\underline{Rng}}$.*

Let π denote the class of all prime rings, and if A is any prime ring let π_A be the smallest special class containing A. Then

Proposition 3.2. [5, Theorem 3] *If R is an s-ring, then $\mathcal{L}_s\{R\} = \mathcal{U}(\pi \backslash \pi_R)$.*

Let Π denote the class of all prime involution rings and if $(A,*)$ is a prime involution ring, let $\Pi_{(A,*)}$ be the smallest special class containing $(A,*)$. We can prove analogies for involution rings of the preceding two results using proofs vitrtually identical to those in [8] and [5], that is:

Proposition 3.3. *Let R be an s-$*$-ring. Then $\mathcal{L}_s\{(R,*)\}$ is an atom of $\mathbb{L}_{sp}^{\underline{IR}}$.*

Proposition 3.4. *If $(R,*)$ is an s-involution ring, then $\mathcal{L}_s\{(R,*)\} = \mathcal{U}(\Pi \backslash \Pi_A)$.*

Proposition 3.5. *Let $(R,*)$ be an s-involution ring. Then $\mathcal{L}_s\{(R,*)\} = \mathcal{L}_s\{(S,*)\}$ for some simple idempotent involution ring $(S,*)$ if and only if $(R,*)$ contains a minimal ideal.*

Proof. Suppose that $(R,*)$ contains a minimal ideal $(K,*)$. Then $(K,*)$ is a simple involution ring ring. For suppose that $0 \neq (J,*) \lhd (K,*)$. Then $J \lhd K \lhd R$, whence, by the Andrunakievich Lemma, $(\overline{J})^3 \subseteq J \subseteq K$, where $\overline{J}$ denotes the ideal of R generated by J. Now $(\overline{J},*)$ is an ideal of $(R,*)$, so $(\overline{J}^3,*) = (\overline{J},*)^3$ is an ideal of $(R,*)$. Since $(R,*)$ is a prime involution ring, $(\overline{J},*)^3 \neq 0$. By the minimality of $(K,*)$, this implies that $(\overline{J},*)^3 = (K,*)$. But $(\overline{J},*)^3 \subseteq (J,*) \subseteq (K,*)$, so $(J,*) = (K,*)$. Hence $(K,*)$ is a simple involution ring. It follows from the hereditariness of $\mathcal{L}_s\{(R,*)\}$ that $(K,*) \in \mathcal{L}_s\{(R,*)\}$, whence $\mathcal{L}_s\{(K,*)\} \subseteq \mathcal{L}_s\{(R,*)\}$. Since $\beta_* \subset \mathcal{L}_s\{(K,*)\}$ and $\mathcal{L}_s\{(R,*)\}$ is an atom of $\mathbb{L}_{sp}^{\underline{IR}}$, we have that $\mathcal{L}_s\{(R,*)\} = \mathcal{L}_s\{(K,*)\}$.

Conversely, suppose that $\mathcal{L}_s\{(R,*)\} = \mathcal{L}_s\{(S,*)\}$, where $(S,*)$ is a simple involution ring. Then $\mathcal{L}_s\{(S,*)\} = \mathcal{U}(\Pi\backslash\Pi_{(S,*)})$. Since $(R,*) \in \mathcal{L}_s\{(S,*)\}$, $(R,*)$ has no nonzero homomorphic image in $\Pi\backslash\Pi_{(S,*)}$. But $(R,*)$ is prime, and hence $(R,*) \in \Pi_{(S,*)}$. It follows from that $(R,*)$ has an accessible subring which is isomorphic to an accessible subring of $(S,*)$. Since $(S,*)$ is a simple idempotent involution ring, this implies that $(R,*)$ has an ideal $(K,*)$ which is isomorphic to $(S,*)$. Again since $(S,*)$ is simple, $(K,*)$ is a minimal ideal of $(R,*)$. ∎

Proposition 3.6. *Let R be an s-ring. Then $(R \oplus R^{op}, e)$ is an s-involution ring.*

Proof. Clearly $(R \oplus R^{op}, e)$ is a prime involution ring. Let $f : (R \oplus R^{op}, e) \to (S,*)$ be a surjective homomorphism with nonzero kernel. Suppose that $\ker f \cap R = 0$. Let $(x,y) \in \ker f$. If $r \in R$, then $(x,y)(r,0)(x,y) = (xrx,0) \in \ker f \cap R$. Hence $xRx = 0$. Since R is a prime ring, $x = 0$. Since $\ker f \cap R = 0$ and f is an involution ring homomorphism, it is easily verified that $\ker f \cap R^{op} = 0$. By similar reasoning to that employed above, we may deduce that $y = 0$. This contradicts our assumption that $\ker f$ is nonzero. Hence $\ker f \cap R \neq 0$, from which it is easily deduced that $\ker f \cap R^{op} \neq 0$. Since R is an s-ring, so is R^{op}, and so $f(R)$ and $f(R^{op})$ are β-radical rings. Hence S is a β-radical ring and so $(S,*)$ is a β_*-radical ring. Thus $(R \oplus R^{op}, e)$ is an s-involution ring. ∎

Proposition 3.7. *Let R be an s-ring which does not have minimal ideals. Then $(R \oplus R^{op}, e)$ is an s-involution ring which does not have minimal ideals.*

Proof. It follows from Proposition 3.6 that $(R \oplus R^{op}, e)$ is an s-involution ring. Suppose that (I,e) is a minimal ideal of $(R \oplus R^{op}, e)$. Using arguments similar to those employed in the proof of Proposition 3.6 it may be shown that $I \cap R \neq 0$ and $I \cap R^{op} \neq 0$. Since R does not have minimal ideals, there exists an ideal J of R with $0 \subset J \subset I$. But then $(J \oplus J^{op}, e)$ is a nonzero ideal of $(R \oplus R^{op}, e)$ which is properly contained in $(I \oplus I^{op}, e)$, which contradicts our assumption that $(I \oplus I^{op}, e)$ is a minimal ideal of $(R \oplus R^{op}, e)$. Hence $(R \oplus R^{op}, e)$ has no minimal ideals. ∎

France-Jackson [6] has given an example of an s-ring R which has no minimal ideals. Consequently, the radical $\mathcal{L}_s\{R\}$ is an atom of $\mathbb{L}_{sp}^{\underline{Rng}}$ which cannot be generated by a simple ring. It follows from Propositions 3.6 and 3.7 that $\mathcal{L}_s\{(R \oplus R^{op}, e)\}$ is an atom in $\mathbb{L}_{sp}^{\underline{IR}}$ which cannot be generated by a simple involution ring.

4. g-Radicals

It may be shown that for a Plotkin radical $\mathcal{R}$ in any universal class $\mathcal{C}$ the ADS property is equivalent to the statement:

G1 $I \lhd J \lhd R \in \mathcal{C}, I \in \mathcal{R} \Longrightarrow \exists X \lhd R$ such that $I \subseteq X \subseteq J$ and $J \in \mathcal{R}$.

Clearly, this condition may be satisfied by a class which is not a radical class. In the case that $\mathcal{C}$ is hereditary, condition G1 is equivalent to:

G2 $I \lhd J \lhd R, I \in \mathcal{R} \Longrightarrow \langle I \rangle \in \mathcal{R}$, where.$\langle I \rangle$ denotes the ideal of R which is generated by I.

In particular, because the semisimple classes of radicals in $\underline{Rng}$ are always hereditary, conditions G1 and G2 are equivalent in such classes. In [4] a complete characterisation was given for radical classes in $\underline{Rng}$ whose semisimple classes satisfy the equivalent conditions G1 and G2. It turns out [4] that these are those radical classes $\mathcal{R}$ such that either $\mathcal{R}$ or $\mathcal{SR}$ consists entirely of idempotent rings. Such radicals are called *g-radicals*. As the ADS property does not hold in general in $\underline{IR}$, semisimple classes are not necessarily hereditary in this variety. As this appears to cause some difficulty in the development of the theory of g-radicals, we will define as follows:

A radical $\mathcal{A}$ in $\underline{IR}$ is called a *g-radical* if $\mathcal{A}$ satisfies ADS and $(I,*)\lhd(J,*)\lhd(R,*)$, $(I,*)\in\mathcal{SA}\Longrightarrow\langle(I,*)\rangle\in\mathcal{SA}$, where.$\langle(I,*)\rangle$ denotes the ideal of $(R,*)$ which is generated by $(I,*)$.

It is not known whether the requirement that $\mathcal{A}$ satisfies ADS is redundant in this definition.

Proposition 4.1. *Let $\mathcal{A}$ be a radical in $\underline{IR}$ such that $\mathcal{A}\leq\mathcal{I}_*$, where $\mathcal{I}_*$ denotes the radical class of idempotent involution rings. Then $\mathcal{A}$ is a g-radical.*

The proof is similar to that of the corresponding result for rings [4, Proposition 2.1], and will therefore be omitted.

Proposition 4.2. *Let $\mathcal{A}$ be a radical in $\underline{IR}$ such that $\mathcal{SA}\subseteq\mathcal{I}_*$. Then $\mathcal{A}$ is a g-radical.*

Proof. Let $(I,*)\lhd(J,*\}\lhd(R,*)$, $(I,*)\in\mathcal{SA}$. Then I is idempotent and $I\lhd J\lhd R$. Hence $I\lhd R$, so $\langle I\rangle=I\in\mathcal{SA}$. Hence $\mathcal{A}$ is a g-radical. ∎

Lemma 4.3. *Let $\mathcal{R}$ be a symmetric radical. The $R\in\mathcal{SR}$ if and only if $(R\oplus R^{op},e)\in\mathcal{S}(\lambda\mathcal{R})$.*

Proof. Since $\mathcal{R}$ is symmetric,

$$\begin{aligned}\lambda\mathcal{R}(R\oplus R^{op},e) &= (\mathcal{R}(R\oplus R^{op}),e)\\ &= (\mathcal{R}(R)\oplus\mathcal{R}(R^{op}),e)\\ &= (\mathcal{R}(R)\oplus\mathcal{R}(R)^{op},e)\text{ (again since }\mathcal{R}\text{ is symmetric).}\end{aligned}$$

The result now follows easily. ∎

Proposition 4.4. *Let $\mathcal{R}$ be a symmetric in $\underline{Rng}$. Then $\lambda\mathcal{R}$ is a g-radical in $\underline{IR}$ if and only if $\mathcal{R}$ is a g-radical in $\underline{Rng}$.*

Proof. Suppose that $\mathcal{R}$ is a symmetric g-radical in $\underline{Rng}$. Suppose that $(I,*)\lhd(J,*)\lhd(R,*)$, and that $(I,*)\in\mathcal{S}\lambda\mathcal{R}$. Then $I\lhd J\lhd R$. Since $\mathcal{R}$ is symmetric, $\mathcal{R}(I)=\lambda\mathcal{R}(I,*)=0$, i.e. $I\in\mathcal{SR}$. Then $\langle I\rangle\in\mathcal{SR}$, since $\mathcal{R}$ is a g-radical. But $\langle(I,*)\rangle=(\langle I\rangle,*)$ and hence $\langle(I,*)\rangle\in\mathcal{SR}$. Hence $\lambda\mathcal{R}$ is a g-radical in $\underline{Rng}$.

Conversely, suppose that $\mathcal{R}$ is a symmetric radical in $\underline{Rng}$ which is not a g-radical. Then there exists a ring R with $I\lhd J\lhd R$, $I\in\mathcal{SR}$ and $\langle I\rangle\notin\mathcal{SR}$. Then $(I\oplus I^{op},e)\lhd(J\oplus J^{op},e)\lhd(R\oplus R^{op},e)$ and $(I\oplus I^{op},e)\in\mathcal{S}(\lambda\mathcal{R})$ by Lemma 4.3. Morover $\langle(I\oplus I^{op},e)\rangle=(\langle I\rangle\oplus\langle I\rangle^{op},e)\notin\mathcal{S}(\lambda\mathcal{R})$ by Lemma 4.3. Hence $\lambda\mathcal{R}$ is not a g-radical in $\underline{IR}$. ∎

In [4, Theorems 3.7 and 3.9] it was shown that $\mathbb{L}_g^{\underline{Rng}}$ and $\mathbb{L}_{gh}^{\underline{Rng}}$ are sublattices of $\mathbb{L}^{\underline{Rng}}$. As an immediate consequence of [3, Proposition 3.12] and Proposition 4.4 we have:

Proposition 4.5. *The mapping $\mathcal{R}\to\lambda\mathcal{R}$ is a lattice isomorphism of the lattice $\mathbb{L}_g^{\underline{Rng}}$ ($\mathbb{L}_{gh}^{\underline{Rng}}$) of (hereditary) g-radicals in $\underline{Rng}$ onto the lattice $\mathbb{L}_g^{\underline{IR}}$ ($\mathbb{L}_{gh}^{\underline{IR}}$) of (hereditary) invariant g-radicals in $\underline{IR}$.*

Proposition 4.5 enables us to give a full characterization of invariant g-radicals in $\underline{IR}$.

Proposition 4.6. *Let $\mathcal{A}$ be an invariant radical in $\underline{IR}$. Then $\mathcal{A}$ is a g-radical if and only if either $\mathcal{A}\leq\mathcal{I}_*$ or $\mathcal{SA}\subseteq\mathcal{I}_*$.*

Proof. Let A be a symmetric g-radical. Then by Proposition 4.5 $\mathcal{A}=\lambda\mathcal{R}$ for some invariant g-radical $\mathcal{R}$ in $\underline{Rng}$. Then from [4], either $\mathcal{R}\leq\mathcal{I}$ or $\mathcal{SR}\subseteq\mathcal{I}$. In the former case if $(R,*)\in\mathcal{A}$, then $R\in\mathcal{R}$, whence R is idempotent. Hence $\mathcal{A}\leq\mathcal{I}_*$ in this case.

Suppose that $\mathcal{SR} \subseteq \mathcal{I}$, and let $(R,*) \in \mathcal{SA}$. Then $\mathcal{A}(R,*) = \lambda\mathcal{R}(R,*) = \mathcal{R}(R)$ (since $\mathcal{R}$ is symmetric) $= 0$. Hence R is idempotent, and so $(R,*) \in \mathcal{I}_*$. Thus $\mathcal{SA} \subseteq \mathcal{I}_*$.

Conversely, suppose that $\mathcal{A} \leq \mathcal{I}_*$ or $\mathcal{SA} \subseteq \mathcal{I}_*$. If $\mathcal{A} \leq \mathcal{I}_*$, then $\mathcal{A}$ is a g-radical by Proposition 4.1. If $\mathcal{SA} \subseteq \mathcal{I}_*$, it follows from Proposition 4.2 that A is a g-radical. ∎

In [4, Theorem 2.3] it is shown that a radical $\mathcal{R}$ is a g-radical in $\underline{Rng}$ such that $\mathcal{SR} \subseteq \mathcal{I}$ if and only if $\mathcal{SR}$ is contained in the subdirect closure of some set $\overline{\mathcal{K}} := \{F_1, ..., F_n\}$ where each of the F_i is a finite field. It follows that the elements of $\mathcal{SR}$ are all commutative rings, and hence that $\mathcal{R}$ is symmetric in this case. Note that the class $\mathcal{K}$ is special, and since $\mathcal{SR} \subseteq \overline{\mathcal{K}} := \mathcal{SUK}$, we have that $\mathcal{UK} \leq \mathcal{R}$. Hence $\lambda(\mathcal{UK}) \leq \lambda\mathcal{R}$. Since $\mathcal{UK}$ is the upper radical determined by the special class $\mathcal{K}$ it follows from [2, Proposition 3.2] that $\lambda\mathcal{UK} = \mathcal{UK}^*$, where $\mathcal{K}^* := \{(R,*) \in \underline{IR} \mid \exists I \lhd R$ such that $I \cap I^* = 0$ and $R/I \in \mathcal{K}\}$. Clearly $\mathcal{K}^* = \bigcup_{i=1}^{n} \mathcal{K}_i$, where $\mathcal{K}_i := \{(R,*) \in \underline{IR} \mid \exists I \lhd R$ such that $I \cap I^* = 0$ and $R/I \cong F_i\}$. Since F_i is a field, and hence a simple ring with unity, it may easily be verified that $\mathcal{K}_i := \{(F_i,*) \mid *$ is an involution on $F_i\} \cup \{(F_i \oplus F_i, e)\}$. We can now prove:

Proposition 4.7. *Let $\mathcal{A}$ be an invariant radical in $\underline{IR}$. Then $\mathcal{A}$ is a g-radical such that $\mathcal{SA} \subseteq \mathcal{I}_*$ if and only if $\mathcal{SA}$ is contained in the subdirect closure of a class $\mathcal{K} := \bigcup_{i=1}^{n} \{(S_i,*) \mid$ either S_i is a finite field or $(S_i,*) \cong (F_i \oplus F_i, e)$ for some finite field $F_i\}$.*

Proof. Suppose that $\mathcal{A}$ is an invariant g-radical in $\underline{IR}$ such that $\mathcal{SA} \subseteq \mathcal{I}_*$. Then $\mathcal{A} = \lambda\mathcal{R}$ for some g-radical $\mathcal{R}$ in $\underline{Rng}$ by Proposition 4.5. Clearly, $\mathcal{SR} \subseteq \mathcal{I}$. It follows from the preceding discussion that $\mathcal{SA}$ is contained in a class of the form in the statement of this proposition.

Conversely, suppose that $\mathcal{A}$ is an invariant radical in $\underline{IR}$ such that $\mathcal{SA}$ is contained in the subdirect closure of some class $\mathcal{K}$ of the form $\mathcal{K} := \bigcup_{i=1}^{n} \{(S_i,*) \mid$ either S_i is a finite field or $(S_i,*) \cong (F_i \oplus F_i, e)$ for some finite field $F_i\}$. Let $\mathcal{K}_1 := \{F \mid F$ is a finite field and $(F,*) \in \mathcal{K}\}$ and let $\mathcal{K}_2 := \{F \mid F$ is a finite field and $(F \oplus F, e) \in \mathcal{K}\}$. Suppose that $(R,*) \in \mathcal{SA}$. Then there exist elements $(S_i,*)$ of $\mathcal{K}$ and surjective homomorphisms θ_i of $(R,*)$ onto $(S_i,*)$, $i \in I$, such that $\bigcap_{i \in I} \ker\theta_i = 0$. For each $i \in I$, either $S_i \in \mathcal{K}_1$ or $S_i = F_i \oplus F_i$, where $F_i \in \mathcal{K}_2$. In the first instance, θ_i is a ring homomorphism of R onto S_i. In the second instance, it is easily verified that $\varphi_{i1} := \pi_1 \circ \theta_i$ and $\varphi_{i2} := \pi_2 \circ \theta_i$ are ring homomorphisms of R onto F_i, where π_k denotes projection of $S_i = F_i \oplus F_i$ onto the k-th component. Moreover, $\ker\theta_i = \ker\varphi_{i1} \cap \ker\varphi_{i2}$. Furthermore, $0 = \bigcap_{i \in I} \ker\theta_i = \bigcap_{S_i \in \mathcal{K}_1} \ker\theta_i \cap \bigcap_{S_i \in \mathcal{K}_2} \ker\theta_i = \bigcap_{S_i \in \mathcal{K}_1} \ker\theta_i \cap \bigcap_{S_i \in \mathcal{K}_2} (\ker\varphi_{i1} \cap \ker\varphi_{i2})$. It follows that R is a subdirect product of elements of $\mathcal{K}_1 \cup \mathcal{K}_2$. But the subdirect closure of $\mathcal{K}_1 \cup \mathcal{K}_2$ is the semisimple class of a g-radical $\mathcal{R}$ in $\underline{Rng}$, and consists of idempotent rings [4, Theorem 2.3]. It follows that R is idempotent, and hence $\mathcal{SA} \subseteq I_*$. It follows from Proposition 4.1 that $\mathcal{A}$ is a g-radical. ∎

In [4, Theorem 3.9] it is shown that the lattice $\mathbb{L}_{gh}^{Rng}$ of hereditary g-radicals in $\underline{Rng}$ is atomic and that its atoms are precisely the radicals $\mathcal{L}\{S\}$, where S is a simple idempotent ring. It is easily seen that the lattice $\mathbb{L}_{sgh}^{Rng}$ is also atomic and that its atoms are the radicals $\mathcal{L}\{S, S^{op}\}$, where S is a simple ring. It follows from Proposition 4.5 that the lattice $\mathbb{L}_{ihg}^{IR}$ of invariant hereditary g-radicals in $\underline{IR}$ is atomic, and that its atoms are precisely the radicals $\lambda\mathcal{L}\{S, S^{op}\}$, where S is a simple idempotent ring. Hence from Lemma 2.1 we have:

Proposition 4.8. *The lattice* $\mathbb{L}_{ihg}^{IR}$ *is atomic, and its atoms are the radicals* $\mathcal{L}\{(S,*) \mid *$ *is an involution on* $S\} \vee \mathcal{L}\{(S \oplus S^{op}, e)\}$*,where* S *is a simple idempotent ring.*

The question arises: are all g-radicals in $\underline{IR}$ invariant? The following two examples give a negative answer.

Example 4.9. *Let* $\mathbb{C}$ *be the field of complex numbers, and let* $\mathcal{A} := \mathcal{L}\{(\mathbb{C}, id)\}$*. It follows from Proposition 4.1 that* $\mathcal{A}$ *is a g-radical. Let* c *be the involution defined by* $z^c := \overline{z}$ *for all* $z \in \mathbb{C}$*. Then clearly* $(\mathbb{C}, id) \in A$*. Since* $(\mathbb{C}, c)$ *is a simple involution ring,* $(\mathbb{C}, id) \in A$ *would imply* $(\mathbb{C}, c) \cong (\mathbb{C}, id)$ *which is false. Hence* $(\mathbb{C}, c) \notin \mathcal{A}$*, so* $\mathcal{A}$ *is not invariant.*

Lemma 4.10. *Let* $(R,*)$ *be a subdirect product of the involution rings* $(R_i, id), i \in I$*, where* R_i *is commutative for all* i*. Then* $(R,*) = (R, id)$*.*

Proof. Let $\theta_i : (R,*) \to (R_i, id)$ be involution ring homomorphisms such that $\bigcap \ker \theta_i = 0$. If $r \in R$, then $\theta_i(r^*) = (\theta_i(r))^{id} = \theta_i(r)$ for all $i \in I$. It follows that $r^* = r = r^{id}$ and so $(R,*) = (R, id)$. ∎

Example 4.11. *Let* $\mathbb{Z}_p$ *be a prime field and let* $\mathcal{R} := \mathcal{U}\{\mathbb{Z}_p\}$*. Then* $\mathcal{R}$ *is a g-radical in* $\underline{Rng}$ *by* [4, Proposition 2.3.]*. Let* $\mathcal{A} := \mathcal{U}\{(\mathbb{Z}_p, id)\}$*. Suppose that* $(I,*) \lhd (J,*) \lhd (R,*)$*, and that* $(I,*) \in \mathcal{SA}$*. Then* $(I,*)$ *is a subdirect product of copies of.*$(\mathbb{Z}_p, id)$*. Hence* I *is a subdirect products of copies of* $\mathbb{Z}_p$*, so* $I \in \mathcal{SR}$*. . Moreover,* $I \lhd J \lhd R$*, and since* $\mathcal{R}$ *is a g-radical,* $\langle I \rangle \in \mathcal{SR}$*. It follows that* $\langle I \rangle$ *is a subdirect product of copies of* $\mathbb{Z}_p$*. Now* $\langle (I,*) \rangle = (\langle I \rangle, *)$*, whence by Lemma 4.10,* $\langle (I,*) \rangle = (\langle I \rangle, *) = (\langle I \rangle, id)$*. It follows that* $\langle (I,*) \rangle$ *is a subdirect product of copies of* $(\mathbb{Z}_p, id)$ *and so* $\langle (I,*) \rangle \in \mathcal{SA}$*. Hence* $\mathcal{A}$ *is a g-radical.*

It is clear that $(\mathbb{Z}_p \oplus \mathbb{Z}_p, id) \in \mathcal{SA}$ *so* $(\mathbb{Z}_p \oplus \mathbb{Z}_p, id) \notin \mathcal{A}$*. But* $(\mathbb{Z}_p \oplus \mathbb{Z}_p, e)$ *is a simple involution ring which is not a subdirect product of copies of* $(\mathbb{Z}_p, id)$*, and so is in* $\mathcal{A}$*. Hence* $\mathcal{A}$ *is not an invariant radical.*

Proposition 4.12. *The class* $\mathbb{L}_g^{IR}$ *of all g-radicals in* $\underline{IR}$ *is a complete lattice.*

Proof. Let $\{\mathcal{A}_i \mid i \in I\}$ be a class of g-radicals in $\underline{IR}$. Let $(I,*) \lhd (J,*) \lhd (R,*)$ such that $(I,*) \in \mathcal{S}\left(\bigvee_{i \in I} \mathcal{A}_i\right)$. Since the $\mathcal{A}_i$ satisfy ADS, the semisimple classes $\mathcal{SA}_i$ are hereditary, and so $\mathcal{S}\left(\bigvee_{i \in I} \mathcal{A}_i\right) = \bigcap_{i \in I} \mathcal{SA}_i$. Hence $(I,*) \in \mathcal{SA}_i$ for each $i \in I$. Since each $\mathcal{A}_i$ is a g-radical, $\langle (I,*) \rangle \in \mathcal{SA}_i$ for each $i \in I$, so $\langle (I,*) \rangle \in \bigcap_{i \in I} \mathcal{SA}_i = \mathcal{S}\left(\bigvee_{i \in I} \mathcal{A}_i\right)$. Hence $\bigvee_{i \in I} \mathcal{A}_i$ is a g-radical. It follows from a well-known lattice-theoretic result that $\mathbb{L}_g^{IR}$ is a lattice with the join defined as in $\mathbb{L}_g^{IR}$ and the meet $\bigwedge_s$ defined by $\bigwedge_s \mathcal{A}_i := \bigvee\{\mathcal{B}_i \in \mathbb{L}_g^{IR} \mid \mathcal{B}_i \leq \mathcal{A}_i$ for all $i \in I\}$. ∎

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