# N-PERSON SILENT GAME ON SALE

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Abstract. We consider a class of timing games which are suggested from a timing problem for putting some kind of farm products on the market. $N$ players, Player $1, 2, \ldots, n$, take possession of the right to put some kind of farm products at any time in $[0, 1]$. The price of them increases with time so long as none of the $n$ players sell them, however, if one of the $n$ players puts his farm products on the market the price falls discontinuously and then increases with time. Such discontinuous falls in the price of farm product arise successively until $n$ players complete to sell. All players have to put their farm product within the unit interval $[0, 1]$. In such a situation, each player wishes to delay action as late as possible, but does not wish to delay so late that his opponents can put earlier. We assume that all players learn neither when nor whether their opponents have put their farm products on the market. Each of the $n$ players has to decide his action time. This model yields us a certain class of $n$-person non-zero sum infinite games.

**1 Introduction** We consider a class of games which are suggested from correlative phenomena between the price and supply in a market on farm products. $N$ players, Player $1, \ldots, n$, take possession of the right to put some kind of farm products on the market with same ratio. We call such kind of products product F in this paper. We can harvest product F at a specific season every year periodically. Each of the $n$ players wants to decide the optimal time to put his product F on the market until the next harvest season. We consider one time period where the harvest time in each year is the beginning and the next harvest time is the end. The price of product F increases with time so long as none of the players put on the market and keep their own products. But, when one of the players puts his product F on the market, the price of product F possessed by the other $n-1$ players falls discontinuously and then increases with time continuously. Such discontinuous falls in the price on product F possessed by the other players arise successively whenever one of the rest players puts his product F, until all of the $n$ players complete to sell. In such a situation, each player wishes to delay action as late as possible to increase the price of his product F, but does not wish to delay so late that the price falls discontinuously several times without knowing the current price. Each player has to decide his optimal action time considering the current price and his opponents' action times, each other.

As is usual with the games of timing [1, 2], there are two kind of information available to the players. If a player is informed of his opponents' action times as soon as one of his opponents puts his product F on the market, we say they are in a noisy version. If neither player learns when nor whether each of the $n-1$ players has put his product F on the market, we say they are in a silent version. We consider the case where $n$-players are in a silent version in this paper.

This model yields us a certain class of $n$-person non-zero sum infinite games on the hyper unit-cube. The equilibrium strategies express the proper balance between the desire

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for the delay and the danger of delaying. Related to our model, we can point out three works [3, 4, 5].

**2 Notations and Assumptions** Since we consider one period game, we express the period as the unit interval $[0,1]$. Throughout of this paper, we use the following notations:

$v(t)$ is the price of product F at time $t \in [0,1]$, when any of the $n$-players doesn't put his product F. We assume that $v(t)$ is differentiable and $v'(t) > 0$ for $t \in (0,1)$. It is natural to assume $0 < v(0) < \infty$.

$r$ is the discount factor after one of the $n$-players has already put his product F on the market and is assumed $0 < r < 1$. That is, if one of the $n$-players sells his product F at time $t \in [0,1]$, the price of product F possessed by other $n-1$ players falls down from $v(t)$ to $rv(t)$ immediately. Hence if $k$ players have already put their product F in the market before time $t \in [0,1]$, the price of it possessed by other $n-k$ players decreases to $r^k v(t)$ at time $t \in [0,1]$.

If $k$ players put their product F at a same time $t \in [0,1]$, each of the $k$ players can get the current price $v(t)$, and then the price of product F decreases to $r^k v(t)$.

We now denote the expectations for the real valued function $M_i(x_1,\ldots,x_n)$ over $[0,1] \times \cdots \times [0,1]$ when player $i$ uses *cdf* $F_i(x_i)$ over $[0,1]$ as his mixed strategy as follows:

$$M_i(F_1,F_2,\ldots,F_n) = \int_0^1 \cdots \int_0^1 M_i(x_1,x_2,\ldots,x_n)\ dF_1(x_1)dF_2(x_2)\cdots dF_n(x_n)$$

and

$$M_i(x_1,F_2,\ldots,F_n) = \int_0^1 \cdots \int_0^1 M_i(x_1,x_2,\ldots,x_n)\ dF_2(x_2)\cdots dF_n(x_n).$$

**3 Formulations and Analysis** Since each of the $n$-players can't observe the other $n-1$ players' action time, he decides his putting time in advance and then he can find the value of his product F after putting on the market. Hence we establish the pure strategy for Player $i$ as $x_i \in [0,1],\ i = 1,2,\ldots,n$. Then the expected payoff kernel to Player 1 $M_1(x_1,x_2,\ldots,x_n)$ is given by

$$M_1(x_1,x_2,\ldots,x_n) = \begin{cases} v(x_1), & 0 \le x_1 \le y_{(1)} \\ rv(x_1), & y_{(1)} < x_1 \le y_{(2)} \\ r^2 v(x_1), & y_{(2)} < x_1 \le y_{(3)} \\ \vdots & \vdots \\ r^k v(x_1), & y_{(k)} < x_1 \le y_{(k+1)} \\ \vdots & \vdots \\ r^{n-1} v(x_1), & y_{(n-1)} < x_1 \le 1, \end{cases} \tag{1}$$

where $y_{(i)}$ denotes the $i$th smallest of $x_2,\ldots,x_n,\ i = 1,\ldots,n-1$. Hence, we suppose that $y_{(1)} \le y_{(2)} \le \cdots \le y_{(n-1)}$.

Observing the above payoff kernel, we can suppose that all of the $n$ players use the same mixed strategy (*cdf* on $[0,1]$) $F(x)$ and also suppose that $F(x)$ consists of its *pdf* $f(x) > 0$ over an interval $(a,1) \subset [0,1]$ by considering the result of two person game [5]. Since the expected payoff to Player 1 for any $x \in (a,1]$ is given by

$$M_1(x,F,F,\ldots,F) = v(x)\left[\sum_{k=0}^{n-1} \binom{n-1}{k} (rF(x))^k \{1-F(x)\}^{n-k-1}\right], \tag{2}$$

we have

$$M_1(x, F, F, \ldots, F) = \begin{cases} v(x), & 0 \le x < a \\ v(x)[1-(1-r)F(x)]^{n-1}, & a \le x \le 1. \end{cases} \tag{3}$$

Putting

$$M_1(x, F, F, \ldots, F) = \text{ const} \quad \text{ for } x \in (a, 1),$$

we get

$$v'(x)[1-(1-r)F(x)] = (n-1)(1-r)f(x)v(x), \quad a < x < 1,$$

which yields

$$F(x) = \{1/(1-r)\}[1-\{c/v(x)\}^{1/(n-1)}], \quad a < x < 1, \tag{4}$$

where $c$ is an integration constant. The boundary conditions

$$F(a) = 0 \quad \text{and} \quad F(1) = 1$$

give us

$$c = r^{n-1}v(1) \;\; ; \;\; v(a) = r^{n-1}v(1).$$

However, the above conditions are satisfied even when $v(0) \le r^{n-1}v(1)$.

So we consider the case where $v(0) \le r^{n-1}v(1)$ first. For this case, there exists the unique root $a^0$ which satisfies the equation $v(a) = r^{n-1}v(1)$ in the interval $[0, 1]$. And then we have the following relations:

$$M_1(x, F, F, \ldots, F) = \begin{cases} v(x) < v(a^0) = r^{n-1}v(1), & 0 \le x < a^0 \\ v(a^0) = r^{n-1}v(1), & a^0 \le x \le 1 \end{cases} \tag{5}$$

Hence we get Theorem 1.

**Theorem 1.** Assume that $v(0) \le r^{n-1}v(1)$. Let $a^0$ be the unique root of the equation of $v(a) = r^{n-1}v(1)$ in the interval $[0, 1]$. And consider the following mixed strategy:

$$F^0(x) = \begin{cases} 0, & 0 \le x < a^0 \\ \{1/(1-r)\}[1-\{v(a^0)/v(x)\}^{1/(n-1)}], & a^0 \le x \le 1. \end{cases}$$

Then $n$-tuple of mixed strategies $(F^0, F^0, \ldots, F^0)$ is a Nash equilibrium of our $n$-person non-zero sum silent game. And the corresponding equilibrium value $v_i$ to Player $i$ is given as

$$v_i = M_i(F^0, F^0, \ldots, F^0) = r^{n-1}v(1), \quad i = 1, \ldots, n.$$

Now we consider the case where $v(0) > r^{n-1}v(1)$DSince we assumed that each of $k$ players can get the current price $v(t)$ when $k$ players put their product F at a same time $t \in [0, 1]$, we have

$$M_1(x, 0, \ldots, 0) = r^{n-1}v(x) \le r^{n-1}v(1) < v(0), \quad 0 < x \le 1 \tag{6}$$

and also have

(7) $$M_1(0, x_2, \ldots, x_n) = v(0) > r^{n-1}v(1), \quad \text{for any } x_2, \ldots, x_n \text{ in } [0, 1].$$

Thus we get Theorem 2.

**Theorem 2.** Assume $v(0) > r^{n-1}v(1)$. Then $n$-tuple $(0, \ldots, 0)$ is a Nash equilibrium point of this non-zero sum silent game. The corresponding equilibrium value $v_i$ to Player $i$ is given as

$$v_i = M_i(0, \ldots, 0) = v(0), \quad i = 1, \ldots, n.$$

The inequality $v(0) > r^{n-1}v(1)$ holds for large $n$. It means that when the number of participants of such a game exceeds some level, each of them wants to sell his product F at the first time of the game.

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