DIRECT PRODUCTS OF ORDERED ABELIAN GROUPS

HIROSHI TANAKA

Received August 3, 2005; revised January 16, 2006

Abstract. In this paper we study some theories of direct products of ordered abelian groups.

1. Introduction

In [1], Komori showed that the direct product of $\mathbb{Z}$ and $\mathbb{Q}$ admits elimination of quantifiers in a language, where $\mathbb{Z}$ ($\mathbb{Q}$) is the ordered abelian group of integers (of rational numbers). Extending this, Weispfenning [7] showed that the direct product of finitely many copies of $\mathbb{Z}$ admits elimination of quantifiers in a language L, and the direct product of finitely many copies of $\mathbb{Z}$ and one $\mathbb{Q}$ admits elimination of quantifiers in the same language L. In this paper, we show the converse of them. We also show that the algebraic closure over the above structures satisfies the Exchange Principle.

2. Preliminaries

Let $\mathbb{N}$ be the set of natural numbers. Let $\mathbb{Z}$ be the ordered abelian group of integers. Let $\mathbb{Q}$ be the ordered abelian group of rational numbers. Let $k \in \mathbb{N} \setminus \{0\}$.

Suppose that $L = \{0, 1^{(1)}, 1^{(2)}, \ldots, 1^{(k)}, +, -, 0 < *, n \mid *\}_{n>0}$, where $0 < *$ and $n \mid *$ are unary relation symbols. For each i with $1 \leq i \leq k$, the terms $t + \cdots + t$ and $1^{(i)} + \cdots + 1^{(i)}$ (t and $1^{(i)}$ repeated n times) are written as nt and $n^{(i)}$, respectively. The term $t + (-s)$ is written as $t - s$. The formula $0 < t - s$ is written as $s < t$. The formulas $s < t \wedge t < u$ and $s < t \vee s = t$ are written as $s < t < u$ and $s \leq t$, respectively.

We now give some axioms for ordered abelian groups.

(1) The axioms for abelian groups:
$\forall x \forall y \forall z((x + y) + z = x + (y + z))$;
$\forall x(x + 0 = x)$;
$\forall x(x - x = 0)$;
$\forall x \forall y(x + y = y + x)$.

(2) The axioms for a linear ordering compatible with group structures:
$\forall x(x = 0 \vee 0 < x \vee 0 < -x)$;
$\forall x(\neg(0 < x \wedge 0 < -x))$;
$\forall x \forall y(0 < x \wedge 0 < y \rightarrow 0 < x + y)$.

(3) The axioms for a semi-discrete ordering:
$0 < 2^{(i+1)} < 1^{(i)}$ for each i with $1 \leq i \leq k - 1$;
$\forall x(2x < 1^{(i)} \vee 1^{(i)} < 2x)$ for each i with $1 \leq i \leq k$.

(4) The axioms for infinitesimals:
$\forall x(2x < 1^{(i)} \rightarrow nx < 1^{(i)})$ for each i with $1 \leq i \leq k$ and $n \geq 2$.

(5) $\forall x(n \mid x \leftrightarrow \exists y \exists z(-1^{(k)} < 2z < 1^{(k)} \wedge x = ny + z))$ for each $n > 0$.

2000 *Mathematics Subject Classification.* 03C10, 03C64, 06F20.
Key words and phrases. ordered abelian groups, quantifier elimination.

(6) $\forall x\big(\bigvee_{0\leq q_1,\dots,q_k\leq n-1}(n \mid x+q_1^{(1)}+\cdots+q_k^{(k)})\big)$ for each $n>1$.
(7) The axioms for divisible infinitesimals:
$\forall x(-1^{(k)}<2x<1^{(k)}\rightarrow \exists y(x=ny))$ for each $n>1$.
(8) The axiom for discrete ordering:
$\forall x(\neg(0<x<1^{(k)}))$.
(9) The axiom for existence of infinitesimals:
$\exists x(0<x<1^{(k)})$.

Let $SS_k := (1)\cup(2)\cup(3)\cup(4)\cup(5)\cup(6)$. Let $DC_k := SS_k\cup(7)\cup(8)$ and $SC_k := SS_k\cup(7)\cup(9)$. We consider the lexicographic order from left to right on the ordered abelian group $\mathbb{Z}^k=\mathbb{Z}\times\cdots\times\mathbb{Z}$ ($\mathbb{Z}$ repeated k times). We similarly consider the lexicographic order on the ordered abelian group $\mathbb{Z}^k\times\mathbb{Q}$. In the ordered abelian group $\mathbb{Z}^k$, we interpret $1^{(1)}$, $1^{(2)}$, …, $1^{(k)}$ as $(1,0,\dots,0)$, $(0,1,0,\dots,0)$, …, $(0,\dots,0,1)$. In the ordered abelian group $\mathbb{Z}^k\times\mathbb{Q}$, we interpret $1^{(1)}$, $1^{(2)}$, …, $1^{(k)}$ as $(1,0,\dots,0)$, $(0,1,0,\dots,0)$, …, $(0,\dots,0,1,0)$. Then $\mathbb{Z}^k$ is a model of DC_k, and $\mathbb{Z}^k\times\mathbb{Q}$ is a model of SC_k. Weispfenning showed that both $\mathrm{Th}_L(\mathbb{Z}^k)$ and $\mathrm{Th}_L(\mathbb{Z}^k\times\mathbb{Q})$ admit elimination of quantifiers.

In section three, we show that both DC_k and SC_k admit elimination of quantifiers and that they are complete. We show the converse of Weispfenning's results. Namely, we show that if M is a model of SS_k and $\mathrm{Th}(M)$ admits elimination of quantifiers, then M is a model of either DC_k or SC_k.

In section four, we show that for each model M of either DC_k or SC_k, the algebraic closure over M satisfies the Exchange Principle.

I would like to thank Associate Professor Katsumi Tanaka for useful discussions.

3. Quantifier eliminable ordered abelian groups

To show that both DC_k and SC_k admit elimination of quantifiers, we first prove some lemmas needed later.

Lemma 1. *Let $1\leq i\leq k$. Then, the formula $\forall x\forall y(-1^{(i)}<2x<1^{(i)}\wedge -1^{(i)}<2y<1^{(i)}\rightarrow -1^{(i)}<2(x+y)<1^{(i)})$ holds in SS_k.*

Proof. Without loss of generality, we may assume that $0<2x<1^{(i)}$ and $0<2y<1^{(i)}$ hold. Suppose for a contradiction that $1^{(i)}<2(x+y)$ holds. Then we have $2\{1^{(i)}-(x+y)\}<1^{(i)}$. Thus, by Axiom (4), we have $5\{1^{(i)}-(x+y)\}<1^{(i)}$. Therefore $4^{(i)}<5x+5y$ holds. Now $0<2x<1^{(i)}$ and $0<2y<1^{(i)}$ hold. Thus, by Axiom (4), we have $0<5x<1^{(i)}$ and $0<5y<1^{(i)}$. Therefore it follows $5x+5y<2^{(i)}$, a contradiction. □

Lemma 2. *Let $1\leq i\leq k-1$ and $n>0$. Then, the formula $\forall x\big(-1^{(i)}<2x<1^{(i)}\rightarrow \bigvee_{0\leq q_{i+1},\dots,q_k\leq n-1}(n\mid x+q_{i+1}^{(i+1)}+\cdots+q_k^{(k)})\big)$ holds in SS_k.*

Proof. Without loss of generality, we may assume $i=1$. By Axiom (6), there exist $q_1,\dots,q_k$ with $0\leq q_1,\dots,q_k\leq n-1$ such that $n\mid x+q_1^{(1)}+\cdots+q_k^{(k)}$ holds. Thus, there exist y,z such that we have $-1^{(k)}<2z<1^{(k)}$ and $x+q_1^{(1)}+\cdots+q_k^{(k)}=ny+z$. Suppose for a contradiction that $q_1\neq 0$.

Let $2y<1^{(1)}$ hold. Then, by $2ny<1^{(1)}$, we have

$$\begin{aligned}2(x+q_2^{(2)}+\cdots+q_k^{(k)}-z)&=2ny-2q_1^{(1)}\\&<1^{(1)}-2q_1^{(1)}\leq -1^{(1)}.\end{aligned}$$

Now $2(x+q_2^{(2)}+\cdots+q_k^{(k)}-z)>-1^{(1)}$ holds, a contradiction.

Let $1^{(1)} < 2y$ hold. Then, we have $2(1^{(1)} - y) < 1^{(1)}$. Hence, we have $2n^{(1)} - 1^{(1)} < 2ny$. Thus it follows

$$\begin{aligned} 2(x + q_2^{(2)} + \cdots + q_k^{(k)} - z) &= 2ny - 2q_1^{(1)} \\ &> 2n^{(1)} - 1^{(1)} - 2q_1^{(1)} \geq 1^{(1)}. \end{aligned}$$

Now $2(x + q_2^{(2)} + \cdots + q_k^{(k)} - z) < 1^{(1)}$ holds, a contradiction. Therefore we get $q_1 = 0$. □

Lemma 3. *Let $1 \leq i < j \leq k$ and $n > 0$. Then, the formula $\forall x(1^{(i)} < 2x \to 1^{(i)} < 2(x - n^{(j)}))$ holds in SS_k*

Proof. Suppose for a contradiction that $2(x - n^{(j)}) < 1^{(i)}$ holds. Then we have $2x < 1^{(i)} + 2n^{(j)}$. Thus, it follows $10x < 6^{(i)}$. Now, by $1^{(i)} < 2x$, we have $2(1^{(i)} - x) < 1^{(i)}$. By Axiom (4), it follows $10^{(i)} - 10x < 1^{(i)}$. Therefore $9^{(i)} < 10x$ holds, a contradiction. □

Using the above lemmas, we show the following.

Proposition 4. *Both DC_k and SC_k admit elimination of quantifiers.*

Proof. Let $\exists x\varphi$ be a formula, where φ is a quantifier-free formula. We may assume that φ is the form $\psi_1 \wedge \cdots \wedge \psi_n$, where each ψ_i is an atomic formula or the negation of an atomic formula. In addition, ψ_i is of one of the forms $t = s$, $\neg(t = s)$, $0 < t$, $\neg(0 < t)$, $n \mid t$ or $\neg(n \mid t)$. Moreover $t = s$, $\neg(t = s)$, $\neg(0 < t)$ and $\neg(n \mid t)$ are equivalent to $t - s = 0$, $0 < t - s \vee 0 < s - t$, $t = 0 \vee 0 < -t$ and $n \mid t + 1^{(1)} \vee \cdots \vee n \mid t + (n-1)^{(1)} + \cdots + (n-1)^{(k)}$, respectively. Thus, we may assume that each ψ_i is of one of the forms $t = 0$, $0 < t$ or $n \mid t$.

Now, each term t can be written in the form $px + s$ with $p \in \mathbb{Z}$ and s a term which does not contain x. Therefore $\exists x\varphi$ can be written as

$$\begin{aligned} &\exists x(p_1 x < t_1 \wedge \cdots \wedge p_i x < t_i \wedge u_1 < q_1 x \wedge \cdots \wedge u_j < q_j x \\ &\quad \wedge r_1 x = v_1 \wedge \cdots \wedge r_l x = v_l \wedge n_1 \mid s_1 x + w_1 \wedge \cdots \wedge n_m \mid s_m x + w_m), \end{aligned}$$

where $p_1, \ldots, p_i, q_1, \ldots, q_j, r_1, \ldots, r_l, s_1, \ldots, s_m, n_1, \ldots, n_m \in \mathbb{N} \setminus \{0\}$ and $t_1, \ldots, t_i, u_1, \ldots, u_j, v_1, \ldots, v_l, w_1, \ldots, w_m$ are terms which do not contain x.

Let p be the least common multiple of $p_1, \ldots, p_i, q_1, \ldots, q_j, r_1, \ldots, r_l, s_1, \ldots, s_m$. Then we may assume that $\exists x\varphi$ is equivalent to

$$\begin{aligned} &\exists x(x < t_1 \wedge \cdots \wedge x < t_i \wedge u_1 < x \wedge \cdots \wedge u_j < x \\ &\quad \wedge x = v_1 \wedge \cdots \wedge x = v_l \wedge n_1 \mid x + w_1 \wedge \cdots \wedge n_m \mid x + w_m \wedge \exists y(x = py)). \end{aligned}$$

Now, by Axiom (7), $\exists y(x = py)$ is equivalent to $p \mid x$.

If $l \geq 1$ holds, $\exists x\varphi$ is equivalent to

$$\begin{aligned} &v_1 < t_1 \wedge \cdots \wedge v_1 < t_i \wedge u_1 < v_1 \wedge \cdots \wedge u_j < v_1 \\ &\wedge v_1 = v_2 \wedge \cdots \wedge v_1 = v_l \wedge n_1 \mid v_1 + w_1 \wedge \cdots \wedge n_m \mid v_1 + w_m \wedge p \mid v_1. \end{aligned}$$

Thus, we may assume $l = 0$. Moreover we may assume $i, j \leq 1$. Let n be the least common multiple of $n_1, \ldots n_m, p$.

Suppose that $i = 0$. Let $A_{q_1,\ldots,q_k}$ be a formula

$$\begin{aligned} &n_1 \mid q_1^{(1)} + \cdots + q_k^{(k)} + u_1 + w_1 \wedge \cdots \wedge n_m \mid q_1^{(1)} + \cdots + q_k^{(k)} + u_1 + w_m \\ &\wedge p \mid q_1^{(1)} + \cdots + q_k^{(k)} + u_1, \end{aligned}$$

where $0 \leq q_1, \ldots, q_k < n$. Then, $\exists x\varphi$ is equivalent to $\bigvee_{0 \leq q_1,\ldots,q_k < n} A_{q_1,\ldots,q_k}$.

On the other hand, suppose that $i = 1$. Let $B_{q_1,\dots,q_k}$ be a formula

$$\exists y(0 < ny + q_1^{(1)} + \cdots + q_k^{(k)} < t_1 - u_1) \wedge n_1 \mid q_1^{(1)} + \cdots + q_k^{(k)} + u_1 + w_1$$
$$\wedge \cdots \wedge n_m \mid q_1^{(1)} + \cdots + q_k^{(k)} + u_1 + w_m \wedge p \mid q_1^{(1)} + \cdots + q_k^{(k)} + u_1,$$

where $0 \le q_1, \dots, q_k < n$. Then, $\exists x\varphi$ is equivalent to $\bigvee_{0 \le q_1,\dots,q_k < n} B_{q_1,\dots,q_k}$. Hence, we may show that $\exists y(0 < ny + q_1^{(1)} + \cdots + q_k^{(k)} < t)$ is equivalent to some quantifier-free formula.

Claim 1. For each k' with $1 \le k' \le k-1$, let $q_1 = \cdots = q_{k'-1} = 0$ and $q_{k'} \ne 0$. Then, $\exists y(0 < ny + q_1^{(1)} + \cdots + q_k^{(k)} < t)$ is equivalent to $2(q_{k'}^{(k')} - t) < 1^{(k')}$.

Without loss of generality, we may assume $k' = 1$.

Suppose that there exists y such that $0 < ny + q_1^{(1)} + \cdots + q_k^{(k)} < t$ holds. If $-1^{(1)} < 2y$ holds, by $-1^{(1)} < 2ny$, we have $-1^{(1)} + 2q_1^{(1)} + \cdots + 2q_k^{(k)} < 2ny + 2q_1^{(1)} + \cdots + 2q_k^{(k)} < 2t$. By $-1^{(1)} + 2q_1^{(1)} \le -1^{(1)} + 2q_1^{(1)} + \cdots + 2q_k^{(k)}$, we have $2(q_1^{(1)} - t) < 1^{(1)}$, as desired. If $2y < -1^{(1)}$ holds, by Lemma 3, we have $2(y + 1^{(2)} + \cdots + 1^{(k)}) < -1^{(1)}$. Thus $2(1^{(1)} + y + 1^{(2)} + \cdots + 1^{(k)}) < 1^{(1)}$ holds. Hence, by $n(1^{(1)} + y + 1^{(2)} + \cdots + 1^{(k)}) < 1^{(1)}$, we have $(n-1)^{(1)} + ny + n^{(2)} + \cdots + n^{(k)} < 0$. However it follows $0 < ny + q_1^{(1)} + \cdots + q_k^{(k)} < ny + (n-1)^{(1)} + n^{(2)} + \cdots + n^{(k)} < 0$, a contradiction.

On the other hand, suppose that $2(q_1^{(1)} - t) < 1^{(1)}$ holds. If $q_1^{(1)} - t \le 0$ holds, we have $0 < -n^{(2)} + q_1^{(1)} + \cdots + q_k^{(k)} < t$, as desired. If $0 < q_1^{(1)} - t$ holds, we have $0 < 2(q_1^{(1)} - t) < 1^{(1)}$. Hence, by Lemma 2, there exist $\alpha_2, \dots, \alpha_k$ with $0 \le \alpha_2, \dots, \alpha_k \le n-1$ such that $n \mid q_1^{(1)} - t + \alpha_2^{(2)} + \cdots + \alpha_k^{(k)}$ holds. Thus, there exists y such that we have $q_1^{(1)} - t + \alpha_2^{(2)} + \cdots + \alpha_k^{(k)} = ny$. Hence we obtain $q_1^{(1)} - ny = t - \alpha_2^{(2)} - \cdots - \alpha_k^{(k)}$. Now, by $1^{(1)} < 2t$, we have $0 < t - \alpha_2^{(2)} - \cdots - \alpha_k^{(k)} - n^{(2)} + q_2^{(2)} + \cdots + q_k^{(k)} = q_1^{(1)} - ny - n^{(2)} + q_2^{(2)} + \cdots + q_k^{(k)} < q_1^{(1)} - ny \le t$. Hence it follows $0 < n(-y - 1^{(2)}) + q_1^{(1)} + \cdots + q_k^{(k)} < t$, as desired. Therefore we prove Claim 1.

Claim 2. Let $q_1 = \cdots = q_k = 0$. Then, in DC_k or SC_k, $\exists y(0 < ny + q_1^{(1)} + \cdots + q_k^{(k)} < t)$ is equivalent to $n^{(k)} < t$ or $0 < t$, respectively.

It is clear for DC_k.

We consider in the case of SC_k. If $\exists y(0 < ny < t)$ holds, we obtain $0 < t$. On the other hand, let $0 < t$ hold. In the case of $1^{(k)} \le t$, by Axiom (9), there exists y such that $0 < 2y < 1^{(k)}$ holds. Then we have $0 < ny < 1^{(k)}$. Thus, it follows $0 < ny < t$, as desired. In the case of $0 < t < 1^{(k)}$, we have $0 < 1^{(k)} - t < 1^{(k)}$. If $t < 1^{(k)} - t$ holds, then $0 < 2t < 1^{(k)}$ holds. Thus, there exists $y > 0$ such that $t = (n+1)y$ holds. Hence it follows $0 < ny = t - y < t$, as desired. If $1^{(k)} - t < t$ holds, then $0 < 2(1^{(k)} - t) < 1^{(k)}$ holds. Thus, there exists $y > 0$ such that $ny = 1^{(k)} - t$ holds. Hence it follows $0 < ny < t$, as desired. Therefore we prove Claim 2.

Claim 3. Let $q_1 = \cdots = q_{k-1} = 0$ and $q_k \ne 0$. Then, in DC_k or SC_k, $\exists y(0 < ny + q_1^{(1)} + \cdots + q_k^{(k)} < t)$ is equivalent to $q_k^{(k)} < t$ or $2(q_k^{(k)} - t) < 1^{(k)}$, respectively.

First, we consider in the case of DC_k. If $q_k^{(k)} < t$ holds, clearly $\exists y(0 < ny + q_k^{(k)} < t)$ holds. On the other hand, suppose that there exists y such that $0 < ny + q_k^{(k)} < t$ holds. If $y \ge 0$ holds, then $q_k^{(k)} \le ny + q_k^{(k)} < t$ holds, as desired. If $y < 0$ holds, by Axiom (8), we have $y \le -1^{(k)}$. Thus, it follows $0 < ny + q_k^{(k)} \le -n^{(k)} + q_k^{(k)} < 0$, a contradiction. Hence, in the case of DC_k, Claim 3 holds.

Next, we consider in the case of SC_k.

Suppose that there exists y such that $0 < ny + q_k^{(k)} < t$ holds. If $-1^{(k)} < 2y$ holds, by $-1^{(k)} < 2ny$, we have $-1^{(k)} + 2q_k^{(k)} < 2ny + 2q_k^{(k)} < 2t$. Thus it follows $2(q_k^{(k)} - t) < 1^{(k)}$, as desired. If $2y < -1^{(k)}$ holds, we have $2(y+1^{(k)}) < 1^{(k)}$. Hence, by $n(y+1^{(k)}) < 1^{(k)}$, we have $ny+(n-1)^{(k)} < 0$. However, $0 < ny+q_k^{(k)} \le ny+(n-1)^{(k)} < 0$ holds, a contradiction.

On the other hand, suppose that $2(q_k^{(k)} - t) < 1^{(k)}$.

Let $q_k^{(k)} \le t$ hold. By Axiom (9), there exists y such that $-1^{(k)} < 2y < 0$ holds. Hence, by $-1^{(k)} < ny < 0$, we have $0 \le -1^{(k)} + q_k^{(k)} < ny + q_k^{(k)} < q_k^{(k)} \le t$, as desired.

Let $t < q_k^{(k)}$ hold. By $0 < 2(q_k^{(k)} - t) < 1^{(k)}$, there exists y with $0 < y < q_k^{(k)} - t < 1^{(k)}$ such that $q_k^{(k)} - t = (n-1)y$ holds. Again, by $2(q_k^{(k)} - t) < 1^{(k)}$, we have $ny < 1^{(k)}$. Thus, we obtain $0 < 1^{(k)} - ny \le q_k^{(k)} - ny = t - y < t$, as desired. Therefore we prove Claim 3.

By Claims 1 through 3, $\exists y(0 < ny+q_1^{(1)}+\cdots+q_k^{(k)} < t)$ is equivalent to some quantifier-free formula. Therefore, $\exists x\varphi$ is equivalent to some quantifier-free formula, as desired. □

Proposition 5. *Both DC_k and SC_k are complete.*
Namely, $DC_k = \mathrm{Th}_L(\mathbb{Z}^k)$ and $SC_k = \mathrm{Th}_L(\mathbb{Z}^k \times \mathbb{Q})$.

Proof. Let M be a model of DC_k. Suppose that $f : \mathbb{Z}^k \to M$ by $f(n_1, \dots, n_k) = n_1^{(1)^M} + \cdots + n_k^{(k)^M}$. Then f is an embedding. Thus, by Proposition 4, DC_k is complete. Similarly SC_k is complete. □

Lemma 6. *Let $\psi(x)$ be a quantifier-free formula with one free variable x. Suppose that $M \models SS_k$. Then*

(i) *either $M \models \psi(a)$ for each a with $0 < 2a < 1^{(k)}$, or $M \models \neg\psi(a)$ for each $0 < 2a < 1^{(k)}$;*
(ii) *either $M \models \psi(a)$ for each a with $-1^{(k)} < 2a < 0$, or $M \models \neg\psi(a)$ for each $-1^{(k)} < 2a < 0$.*

Proof. (i) Let $\psi(x)$ be a quantifier-free formula with one free variable x. The formula $\psi(x)$ is equivalent to a boolean combination of formulas which is of the forms $px = q_1^{(1)}+\cdots+q_k^{(k)}$, $px < q_1^{(1)}+\cdots+q_k^{(k)}$, $q_1^{(1)}+\cdots+q_k^{(k)} < px$ or $m \mid px+q_1^{(1)}+\cdots+q_k^{(k)}$, where $p, m \in \mathbb{N}\setminus\{0\}$ and $q_1, \dots, q_k \in \mathbb{Z}$.

Let $M \models pa = q_1^{(1)} + \cdots + q_k^{(k)}$ for some $0 < 2a < 1^{(k)}$. Then we have $0 < pa < 1^{(k)}$, a contradiction.

Let $M \models pa < q_1^{(1)} + \cdots + q_k^{(k)}$ for some $0 < 2a < 1^{(k)}$. Then, by $0 < pa$, we have $1^{(k)} \le q_1^{(1)} + \cdots + q_k^{(k)}$. Thus, $M \models pa < q_1^{(1)} + \cdots + q_k^{(k)}$ for each $0 < 2a < 1^{(k)}$.

Let $M \models q_1^{(1)} + \cdots + q_k^{(k)} < pa$ for some $0 < 2a < 1^{(k)}$. Then, by $pa < 1^{(k)}$, we have $q_1^{(1)} + \cdots + q_k^{(k)} \le 0$. Thus, $M \models q_1^{(1)} + \cdots + q_k^{(k)} < pa$ for each $0 < 2a < 1^{(k)}$.

Let $M \models m \mid pa + q_1^{(1)} + \cdots + q_k^{(k)}$ for some $0 < 2a < 1^{(k)}$. Then, by $0 < pa < 1^{(k)}$, there exist $n_1, \dots, n_k \in \mathbb{Z}$ such that $q_1 = mn_1, \dots, q_k = mn_k$. Thus, we have $M \models m \mid pa + q_1^{(1)} + \cdots + q_k^{(k)}$ for each $0 < 2a < 1^{(k)}$. This completes the proof of (i). (ii) Similarly, we can prove this. □

We show the converse of Weispfenning's results.

Theorem 7. *Let M be a model of SS_k. Suppose that $\mathrm{Th}(M)$ admits elimination of quantifiers. Then M is a model of either DC_k or SC_k. Namely, we have either $M \equiv \mathbb{Z}^k$ or $M \equiv \mathbb{Z}^k \times \mathbb{Q}$.*

Proof. First, suppose that Axiom (8) holds in M. Then Axiom (7) holds in M. Thus, M is a model of DC_k.

Secondly, suppose that Axiom (9) holds in M. Let $n \in \mathbb{N} \setminus \{0\}$. Because $\mathrm{Th}(M)$ admits elimination of quantifiers, there exists a quantifier-free formula $\psi_n(x)$ such that

$$\mathrm{Th}(M) \models \forall x[(-1^{(k)} < 2x < 1^{(k)} \to \exists y(x = ny)) \leftrightarrow \psi_n(x)].$$

Now $M \models \psi_n(a)$ if $2a < -1^{(k)}$, $1^{(k)} < 2a$ or $a = 0$. Let $0 < 2a < 1^{(k)}$ hold. Then $M \models \psi_n(na)$. Let $-1^{(k)} < 2a < 0$ hold. Then $M \models \psi_n(na)$. Hence, by Lemma 6, $M \models \psi(a)$ if $-1^{(k)} < 2a < 0$ or $0 < 2a < 1^{(k)}$. It follows from this that $M \models \psi(a)$ for each $a \in M$. Thus Axiom (7) holds in M. Therefore, M is a model of SC_k. □

4. Exchange principle

In this section, we show that for each model M of either DC_k or SC_k, algebraic closure over M satisfies the Exchange Principle.

Let $\mathcal{L}$ be a language and M an $\mathcal{L}$-structure. Finite tuples of variables are denoted by $\overline{x}, \overline{y}, \dots$. Finite tuples of elements from M are denoted by $\overline{a}, \overline{b}, \dots$. For a tuple $\overline{a} = (a_1, \dots, a_n)$ from M, we simply write $\overline{a} \in M$ instead of $\overline{a} \in M^n$.

Let $A \subseteq M$. We say that $a \in M$ is *algebraic* over A if there exists an $\mathcal{L}$-formula $\varphi(x, \overline{y})$ and $\overline{b} \in A$ such that $M \models \varphi(a, \overline{b})$ and $\{c \in M \mid M \models \varphi(c, \overline{b})\}$ is finite. For $A \subseteq M$, the *algebraic closure* of A in M, denoted $\mathrm{acl}(A)$, is given by $\{a \in M \mid a \text{ is algebraic over } A\}$.

Definition 8 (Exchange Principle)**.** *Let $\mathcal{L}$ be a language and M an $\mathcal{L}$-structure. We say that the algebraic closure over M satisfies the* Exchange Principle *if $A \subseteq M$, $a, b \in M$ and $a \in \mathrm{acl}(A \cup \{b\}) \setminus \mathrm{acl}(A)$, then $b \in \mathrm{acl}(A \cup \{a\})$.*

Let M be a model of either DC_k or SC_k. Let $A \subseteq M$. Suppose that $\langle A \rangle := \{\alpha \in M \mid$ there exists $\overline{a} \in A$, a term $t(\overline{x})$ and $m \in \mathbb{N}$ such that $m\alpha = t(\overline{a})\}$.

We first prove the following lemma.

Lemma 9. *Let M be a model of either DC_k or SC_k, and let $A \subseteq M$. Then $\langle A \rangle = \mathrm{acl}(A)$.*

Proof. We have $\langle A \rangle \subseteq \mathrm{acl}(A)$. We show that $\mathrm{acl}(A) \subseteq \langle A \rangle$. As both DC_k and SC_k admit elimination of quantifiers, an $L(A)$-formula with one free variable x is equivalent to a boolean combination of the forms $mx = t(\overline{a})$, $t_1(\overline{a}) < m'x < t_2(\overline{a})$ or $n \mid lx + s(\overline{a})$, where $l, m, m', n \in \mathbb{N} \setminus \{0\}$, $\overline{a} \in A$ and t, t_1, t_2, s are terms which do not contain x.

First, let M be a model of DC_k and A a subset of M.

Claim 1. Let $D := \{x \in M \mid t_1(\overline{a}) < m'x < t_2(\overline{a})\}$ be finite and α an element of D. Then α is an element of $\langle A \rangle$.

Since D is finite, there exists $p \in \mathbb{N}$ such that we have $m'\alpha = t_2(\overline{a}) - p^{(k)}$. Hence it follows $\alpha \in \langle A \rangle$.

Claim 2. Let D be infinite. Then $E := \{x \in M \mid t_1(\overline{a}) < m'x < t_2(\overline{a}) \wedge n \mid lx + s(\overline{a})\}$ is empty or infinite.

Let E be non-empty. Let α be an element of E. Suppose that p is a multiple of n. Then we have $\alpha + p^{(k)} \in E$. Thus, E is infinite.

By Claims 1 and 2, if $\alpha \in \mathrm{acl}(A)$, then $\alpha \in \langle A \rangle$.

Secondly, let M be a model of SC_k and A a subset of M.

Claim 3. E is empty or infinite.

Let E be non-empty. Let α be an element of E. Without loss of generality, we may assume $m' = n$. Now, there exists $\beta \in M$ with $0 < 2\beta < 1^{(k)}$ such that we have $n\alpha < n\alpha + n^2\beta < t_2(\overline{a})$. Then, we obtain $n \mid l(\alpha + n\beta) + s(\overline{a})$. Thus, $\alpha + n\beta$ is an element of E. Iterating this process, it follows that E is infinite.

By Claim 3, if $\alpha \in \mathrm{acl}(A)$, then $\alpha \in \langle A \rangle$. □

Theorem 10. *Let M be a model of either DC_k or SC_k Then the algebraic closure over M satisfies the Exchange Principle.*

Proof. Let M be a model of either DC_k or SC_k. Let $A \subseteq M$, $a, b \in M$ and $a \in \mathrm{acl}(A \cup \{b\}) \setminus \mathrm{acl}(A)$. By Lemma 9, there exists $\overline{a} \in A$, a term $t(\overline{x})$ and $m, n \in \mathbb{Z} \setminus \{0\}$ such that $ma = t(\overline{a}) + nb$ holds. Thus $nb = ma - t(\overline{a})$ holds. It follows $b \in \mathrm{acl}(A \cup \{a\})$. □

References

[1] Y. Komori, Completeness of two theories on ordered abelian groups and embedding relations, Nagoya Math. J., **77** (1980), 33–39.

[2] G. Kreisel and J. L. Krivine, Elements of mathematical logic, North-Holland, Amsterdam, 1967.

[3] D. Marker, Model theory: an introduction, GTM **217**, Berlin Heidelberg New York, Springer, 2002.

[4] N. Suzuki, Quantifier elimination results for products of ordered abelian groups, Tsukuba J. Math., **28** (2004), 291–301.

[5] K. Tanaka, On the theory of ordered groups, Kobe J. Math., **5** (1988), 117–122.

[6] H. Tanaka and H. Yokoyama, Quantifier elimination of the products of ordered Abelian groups, Tsukuba J. Math., to appear.

[7] V. Weispfenning, Elimination of quantifiers for certain ordered and lattice-ordered abelian groups, Bulletin de la Société Mathématique de Belgique, Ser. B, **33** (1981), 131–155.

Department of Mathematics, Faculty of Science, Okayama University 1-1, Naka 3-chome, Tsushima, Okayama 700-8530, Japan

E-mail address: htanaka@math.okayama-u.ac.jp