COOPER'S APPROACH TO CHAOTIC OPERATOR MEANS

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Received July 25, 2005

ABSTRACT. We introduce a new class of operator means, called chaotic ones, which is a proper extension of that of Kubo and Ando [9]. From this point of view, we define chaotically quasi-arithmetic means and discussed inequalities in this class like Cooper's classical results [2].

1 Introduction.

It is known that the power mean

$$M_r(a,b) = \left(\frac{a^r + b^r}{2}\right)^{1/r}$$

is increasing for real numbers r, which was discussed systematically by R.Cooper [2]. He also gave conditions for ordering among so-called quasi-arithmetic means:

$$Q_f(a,b) = f^{-1}\left(\frac{f(a)+f(b)}{2}\right).$$

In this note, we consider similar conditions for ordering among operator means to Cooper's.

A typical theory of operator means is established by Kubo and Ando [9]. For positive operators on a Hilbert space, a binary operation m is called *Kubo-Ando mean* if m satisfies the following axioms:

monotonicity: $A_1 \le A_2$ and $B_1 \le B_2$ imply $A_1 m B_1 \le A_2 m B_2$.

semi-continuity: $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n m\ B_n \downarrow A m B$.

transformer inequality: $T^*(AmB)T \le (T^*AT)m(T^*BT)$.

normalization: $AmA = A$.

It is easy to show the transformer equality if T is invertible. In particular, we have:

homogeneity: $\alpha(AmB) = (\alpha A)m(\alpha B)$ for every positive number α.

For an operator mean m, the corresponding numerical function $f_m(x) = 1 m x$ is operator monotone:

$$0 \le A \le B \qquad \text{implies} \qquad f_m(A) \le f_m(B).$$

This correspondence $m \mapsto f_m$ is bijective. In fact, if f is a continuous nonnegative operator monotone function on $[0,\infty)$ with $f(1) = 1$, then a binary operation m defined by

$$AmB = A^{1/2}\left(I m A^{-1/2}BA^{-1/2}\right)A^{1/2} = A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}$$

for positive invertible operators A and B induces an operator mean. As typical Kubo-Ando means, the power ones $\sharp_{r,t}$ are defined by

$$A \sharp_{r,t} B = A^{1/2}\left((1-t)I + t(A^{-1/2}BA^{-1/2})^r\right)^{1/r} A^{1/2}$$

2000 *Mathematics Subject Classification.* 47A64, 47A63.
Key words and phrases. operator mean, chaotic order, quasi-arithmetic mean, Cooper's inequality.

for $0 \leqq t \leqq 1$ and $-1 \leqq r \leqq 1$ (For $|r| > 1$, they cannot be Kubo-Ando means and are often denoted by $\natural_{r,t}$). These means $\sharp_{r,t}$ are monotone increasing for r in the usual operator order since the ordering for Kubo-Ando means is obtained by that of the numerical functions. Thus we obtain the Kubo-Ando geometric mean

$$A \sharp_t B = \lim_{r \to 0} A \sharp_{r,t} B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$$

as a monotone convergence limit. Thus, the Kubo-Ando mean is one of natural extensions of numerical ones and is closely related to the usual operator order, but its ordering is obtained by the numerical case.

On the other hand, the chaotic order $\gg$, that is, $\log A \geq \log B$, has been deserves our attention even for the operator inequality for the usual order [3, 4, 5]. So we consider another class of operator means including the *chaotically geometric mean*:

$$A \lozenge_t B = \exp\big((1-t)\log A + t \log B\big)$$

which is a special case of *chaotic power means*:

$$A \lozenge_{r,t} B = \big((1-t)A^r + tB^r\big)^{1/r}.$$

(The reason why we use this name is that they are chaotic means in the sense in the next section.) Moreover this class should include *chaotically quasi-arithmetic means:*

$$A \mathbf{m}_{f,t} B = f^{-1}\big((1-t)f(A) + tf(B)\big)$$

for a monotone function f with some conditions and $0 \leqq t \leqq 1$.

So we introduce a class of operator means, called *chaotic means*, for positive (invertible) operators on a Hilbert space and show ordering results of Cooper's type.

2 Chaotic means.

A sequence $\{A_n\}$ of positive (invertible) operators is called *chaotically decreasing* and denoted by $A_n \downdownarrows$ if $A_n \gg A_{n+1}$ for all n. If a chaotically decreasing sequence $\{A_n\}$ is lower bounded; $\log A_n \geq c$ for some scalar c, then it converges to some positive (invertible) operator A, which is denoted by $A_n \downdownarrows A$. Now, following the Kubo-Ando theory, we define a *chaotic mean* $\mathbf{m}$ as a binary operation on positive operators satisfying:

monotonicity: $A \leq C$ and $B \leq D$ imply $A\mathbf{m}B \ll C\mathbf{m}D$.

semi-continuity: $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n \mathbf{m} B_n \downdownarrows A\mathbf{m}B$.

normalization: $A\mathbf{m}A = A$.

Clearly, all Kubo-Ando means are chaotic ones.

A real function f is called *chaotically monotone* if

$$0 \leq A \leq B \quad \text{implies} \quad f(A) \ll f(B).$$

We define chaotically quasi-arithmetic means so that they should belong to chaotic ones: Let f (resp. $-f$) be a non-constant operator monotone function on $(0, \infty)$ such that f^{-1} (resp. $1/f^{-1}$) is chaotically monotone. Then, for $0 \leqq t \leqq 1$,

$$A \mathbf{m}_{f,t} B = f^{-1}\big((1-t)f(A) + tf(B)\big)$$

is called *chaotically quasi-arithmetic means*. This class of means is considered as a subclass of (numerical) quasi-arithmetic ones. Hardy, Littlewood and Pólya [8] showed that *only homogeneous quasi-arithmetic means are power ones*, which is the reason we cannot add the transformer inequality to the above axioms. They also showed the invariance under affine transformations for quasi-arithmetic means, which shows immediately:

Lemma 2.1. *All chaotically quasi-arithmetic means* $\mathbf{m}_{f,t}$ *are invariant under all affine transformations for* f*:*

$$\mathbf{m}_{\alpha f+\beta,t} = \mathbf{m}_{f,t}$$

for all real numbers $\alpha \neq 0$ *and* β*.*

Thus we may assume that $f(1) = 1$ if necessary. Moreover they are chaotic means as we desired:

Lemma 2.2. *All chaotically quasi-arithmetic means* $\mathbf{m}_{f,t}$ *are chaotic ones.*

Proof. We have only to show the monotonicity and the normalization. Suppose $A \leq C$ and $B \leq D$. Since f (resp. $-f$) is operator monotone, we have

$$(1-t)f(A) + tf(B) \leq (1-t)f(C) + tf(B) \quad \big(\text{resp.} \geq (1-t)f(C) + tf(B)\big),$$

and hence, by the operator monotonicity of $\log f^{-1}$ (resp. $-\log f^{-1}$),

$$\log f^{-1}\big((1-t)f(A) + tf(B)\big) \leq \log f^{-1}\big((1-t)f(C) + tf(B)\big),$$

which implies the monotonicity. The normalization follows from

$$A\,\mathbf{m}_{f,t}A = f^{-1}\big((1-t)f(A) + tf(A)\big) = f^{-1}\big(f(A)\big) = A. \qquad \square$$

The following result is already shown:

Corollary 2.3. *A function* $G(r) = A\Diamond_{r,t}B$ *is chaotically monotone,* [6]*:*

$$A\Diamond_{r,t}B \ll A\Diamond_{s,t}B \quad \text{if } r < s.$$

In particular, $G(r)$ *is monotone increasing in the usual order for* $|r| \geqq 1$, [1].

Thereby, as a monotone convergence limit, we have easily

$$A\Diamond_{0,t}B \equiv \lim_{r\to 0} A\Diamond_{r,t}B = \exp\big((1-t)\log A + t\log B\big) = A\Diamond_t B.$$

At the end of this section, we give an example of nonhomogeneous chaotically quasi-arithmetic means, which is induced by the function $F_r(x) = x^r + 2x^{r/2}$ for $-1 \leqq r \leqq 1$. Since

$$F_r^{-1}(x) = \big(\sqrt{1+x} - 1\big)^{2/r},$$

and $\log(F_r^{-1}(x)) = \frac{2}{r}\log(\sqrt{1+x} - 1)$ is operator monotone, we have

$$A\,\mathbf{m}_{F_r,t}B = \left(\sqrt{I + (1-t)(A^r + 2A^{r/2}) + t(B^r + 2B^{r/2})} - I\right)^{2/r}$$

is a quasi-arithmetic operator mean.

3 Monotonicity. Now we observe of ordrings of Cooper's type. We also have the similar result to Corollary 2.3 by the following general property:

Theorem 3.1. *If one of the following conditions is satisfied, then* $A\mathbf{m}_{f,t}B \ll A\mathbf{m}_{g,t}B$*:*

(i) g *is operator monotone and* $g\circ f^{-1}$ *is operator convex.*

(i') $-g$ *is operator monotone and* $g\circ f^{-1}$ *is operator concave.*

(ii) f *is operator monotone and* $f\circ g^{-1}$ *is operator concave.*

(ii') $-f$ *is operator monotone and* $f\circ g^{-1}$ *is operator convex.*

Proof. Suppose (i). Then, by the assumption of the quasi-arithmetic operator mean, $\log\circ g^{-1}$ is operator monotone. It follows from Jensen's operator inequality (see [7] in detail) that

$$g\circ f^{-1}\left((1-t)f(A)+tf(B)\right)\le (1-t)g\circ f^{-1}(f(A))+tg\circ f^{-1}(f(B)=(1-t)g(A)+tg(B).$$

Thereby we have

$$\begin{aligned}\log(A\mathbf{m}_{g,t}B) &= \log\circ g^{-1}\left((1-t)g(A)+tg(B)\right)\\ &\ge \log\circ g^{-1}\left(g\circ f^{-1}\left((1-t)f(A)+tf(B)\right)\right)=\log(A\mathbf{m}_{f,t}B),\end{aligned}$$

and hence $A\mathbf{m}_{f,t}B \ll A\mathbf{m}_{g,t}B$. We also show the other cases. □

Corollary 3.2. *A function*

$$H(r)=A\mathbf{m}_{F_r,t}B=\left(\sqrt{I+(1-t)(A^r+2A^{r/2})+t(B^r+2B^{r/2})}-I\right)^{2/r}$$

is chaotically monotone:

$$A\mathbf{m}_{F_r,t}B \ll A\mathbf{m}_{F_s,t}B \quad \textit{if} \quad r<s\in[-1,1].$$

Proof. Let $-1\leqq r<s<0$. Then, by (i') of the above theorem, it suffices to show $F_s\circ F_r^{-1}$ is operator concave for since $-F_s$ is operator monotone. We have

$$F_s(F_r^{-1}(x))=(\sqrt{1+x}-1)^{2s/r}+2(\sqrt{1+x}-1)^{s/r}=\left((\sqrt{1+x}-1)^{s/r}+1\right)^2-1.$$

By $0<s/r<1$, the function $h(y)=\left((\sqrt{y}-1)^{s/r}+1\right)^2$ is operator monotone on $(0,\infty)$ since

$$\operatorname{Arg}(\sqrt{z}-1)^{s/r}\leqq \operatorname{Arg}(\sqrt{z}-1)$$

for $\operatorname{Im} z>0$, which implies that $\operatorname{Arg} h(z)\leqq \operatorname{Arg} z$. Consequently h leaves the upper half plane invariant. Therefore h is also operator concave, and so is $F_s\circ F_r^{-1}$. Similarly we can show the case $0<r<s\leqq 1$ by (ii) of the above theorem. □

4 Concluding Remarks. Finally we consider *quasi-arithmetic Kubo-Ando means*: Let f be a nonnegative (nonconstant) operator monotone function on $(0,\infty)$ with $f(1)=1$. Put

$$Am_{f,t}B=A^{1/2}f^{-1}\bigl(1-t+tf(A^{-1/2}BA^{-1/2})\bigr)A^{1/2}.$$

Then $Am_{f,t}B$ is a Kubo-Ando mean by the following confirmation:

Lemma 4.1. $\tilde{f}_t(x)=f^{-1}\bigl(1-t+tf(x)\bigr)$ *is operator monotone.*

Proof. Considering analytic continuation to the upper half plane $U = \{z|\mathrm{Im}\, z > 0\}$, we have $0 < \mathrm{Arg}\, f(z) < \pi$ for all $z \in U$. Since

$$\mathrm{Arg}\, f(z) \geqq \mathrm{Arg}\big(1 - t + tf(x)\big) > 0,$$

we have

$$1 - t + tf(z) \in f(U)$$

and consequently $\tilde{f}_t(z) \in U$ for all $z \in U$, which implies $\tilde{f}_t$ is operator monotone. □

Since an operator $(1-t)I + tf(A^{-1/2}BA^{-1/2})$ belongs to a commutative algebra generated by $A^{-1/2}BA^{-1/2}$, we easily have the order $m_{f,t} \le m_{g,t}$ holds, that is,

$$A m_{f,t} B \le A m_{g,t} B \qquad \text{for all } A \text{ and } B \ge 0$$

if and only if

$$\tilde{f}_t(x) \leqq \tilde{g}_t(x) \qquad \text{for all } x > 0.$$

So we have the following ordering of Cooper's type immediately:

Theorem 4.2. *$m_{f,t} \le m_{g,t}$ if and only if $f(g^{-1}(x))$ is concave (, or $g(f^{-1}(x))$ is convex).*

Proof. Suppose $m_{f,t} \le m_{g,t}$, that is,

$$\tilde{f}_t(x) \leqq \tilde{g}_t(x)$$

for all $x > 0$. Then

$$(1-t)f(g^{-1}(1)) + tf(g^{-1}(g(x))) = 1 - t + tf(x) \leqq f\Big(g^{-1}\big(1 - t + tg(x)\big)\Big),$$

which implies $f(g^{-1}(x))$ is concave. □

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