A CHARACTERIZATION OF THE HARMONIC OPERATOR MEAN AS AN EXTENSION OF ANDO’S THEOREM

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Abstract. We show that the (weighted) harmonic operator mean is characterized as an operator mean \( m \) satisfying \( F(\text{AmB}) \leq F(A)mF(B) \) for every operator monotone function \( F \) on \((0, \infty)\) based on the numerical means. We also show the non-affine representing function \( f_m(x) = 1/m \cdot x \) of an operator mean \( m \) is an extreme point of the set of representing functions \( F \) with \( F \circ f_m \leq f_m \circ F \).

1 Introduction. Let us consider the arithmetic operator mean \( A\nabla B = (A + B)/2 \) for a pair of positive (invertible) operators \( A \) and \( B \) acting on a Hilbert space \( H \). Then a real function \( F \) is operator concave if

\[ F(A \nabla B) \geq F(A) \nabla F(B) \]

holds. It is known that every operator monotone function \( f \) on \((0, \infty)\) satisfying

\[ f(A) \leq f(B) \quad \text{whenever} \quad 0 \leq A \leq B \]

is operator concave. The harmonic operator mean \(!\) is defined by

\[ A ! B = \left( \frac{A^{-1} + B^{-1}}{2} \right)^{-1} \]

and T. Ando [1, Theorem III.5] showed the contrastive result to the above:

**Theorem** (Ando)1. If \( F \) is positive operator monotone, then

\[ F(A ! B) \leq F(A) ! F(B). \]

In this note, based on this inequality, we discuss when

\[ F(\text{AmB}) \leq F(A) m F(B) \]

holds not only for numerical means but also operator means in the sense of Kubo and Ando [5] which can be constructed as

(1) \[ A m B = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2} \]

for a positive operator monotone function \( f \) on \((0, \infty)\) with \( f(1) = 1 \).

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2 Numerical mean. Let $M(a, b)$ be a positive homogeneous mean for positive numbers $a$ and $b$. According to the Kubo-Ando theory [5], the operations $M^\circ(a, b) = M(b, a)$ and $M^*(a, b) = M(1/a, 1/b)^{-1}$ are called the transpose and adjoint for $M$ respectively.

The symbols $\nabla_w, \#_w$ and $!_w$ denote the arithmetic, geometric and harmonic means respectively for $0 < w < 1$:

$$\nabla_w(a, b) = (1 - w)a + wb, \quad \#_w(a, b) = a^{1-w}b^w \quad \text{and} \quad !_w(a, b) = \frac{ab}{wa + (1 - w)b}.$$  

Then

$$\nabla_w^* = !_w, \quad !_w^* = \nabla_w \quad \text{and} \quad \#_w^* = \#_w$$  

and these means are all symmetric for $w = 1/2$, i.e., $M^\circ = M$.

These operations $^*$ and $^\circ$ are also applied to the representing function $f_M(x) = M(1, x)$ for $M$:

$$f^*(x) = \frac{1}{f\left(\frac{1}{x}\right)} \quad \text{and} \quad f^\circ(x) = xf\left(\frac{1}{x}\right).$$  

Note that the normalized condition $M(a, a) = a$ is equivalent to $f_M(1) = 1$. By homogeneity, such means are reconstructed by the representing functions:

$$M(a, b) = af_M(b/a) = bf^*_M(a/b).$$  

Here we assume that $f_M$ is positive, monotone-increasing and concave. Then so is $f^*_M$. In fact, it is clear that $f^*_M$ is positive and monotone-increasing. The concavity follows from

$$f^*_M((1 - w)x + wy) = ((1 - w)x + wy)f_M\left(\frac{1}{(1 - w)x + wy}\right)$$  

$$= ((1 - w)x + wy)f_M\left(\frac{(1 - w)x + wy1}{1 - w}x + wy\right)$$  

$$\geq (1 - w)xf_M\left(\frac{1}{x}\right) + wyf_M\left(\frac{1}{y}\right) = (1 - w)f^*_M(x) + wf^*_M(y).$$  

The adjoint $f^*_M$ is also positive and monotone-increasing, but it is not always concave as in the following example:

Example 1. Put $F(x) = \sqrt{x} \land x$. Then $F(1/x) = (1/\sqrt{x}) \land (1/x)$ and hence

$$F^*(x) = \sqrt{x} \lor x,$$

which is not concave in a neighborhood of 1.

Moreover the concavity of $F^*$ is equivalent to Ando’s type theorem:

Lemma 2.1. Let $F$ be a positive monotone-increasing concave function on $(0, \infty)$. Then $F^*$ is concave if and only if

$$F(!_w(a, b)) \leq !_w(F(a), F(b))$$  

for all $a, b > 0$.  

Proof. The concavity of $F^*$ is written by

$$\frac{1}{F\left(\frac{1}{(1-w)x+wy}\right)} = F^*((1-w)x+wy) \geq (1-w)F^*(x) + wF^*(y) = (1-w)\frac{1}{F\left(\frac{1}{x}\right)} + w\frac{1}{F\left(\frac{1}{y}\right)}$$

By putting $a = 1/x$ and $b = 1/y$, it is equivalent to

$$F(1_w(a, b)) = F\left(\frac{1}{1-w}x + wy\right) \leq \frac{F(a)F(b)}{(1-w)F(b) + wF(a)} = 1_w(F(a), F(b)).$$

Thus the equivalence is shown. \(\square\)

Here we restrict ourselves to the homogeneous numerical means $M$ with the representing functions $f_M$ satisfying

(i) $f_M$, $f_M^*$ and $f_M^\circ$ are positive monotone-increasing concave functions.

(ii) $f_M$ is normalized: $f_M(1) = 1$ (i.e., $M(a, a) = a$).

Note that (i) implies that the above means do not include trivial means: $M_1(a, b) = a$ and $M_\ell(a, b) = b$.

Next we consider when

$$F(M(a, b)) \leq M(F(a), F(b))$$

holds. Note that it is equivalent to

$$(2') \quad F^*(M^*(a, b)) \geq M^*(F^*(a), F^*(b)).$$

In spite of the above situation, it holds for a special pair of a mean $M \neq 1_w$ and a function $F$. In fact, putting $F(x) = \sqrt{x}$ and $M(a, b) = \sqrt{ab}$, the geometric mean. Then

$$F(M(a, b)) = \sqrt[1+w]{ab} = M(F(a), F(b)).$$

But, considering the case that $F^*$ is affine, we can characterize the harmonic mean in such means, which is an extension of Ando’s theorem:

**Theorem 2.2.** A homogeneous mean $M$ in the above sense is the harmonic one if and only if

$$F(M(a, b)) \leq M(F(a), F(b))$$

for all positive monotone-increasing concave functions $F$ on $(0, \infty)$ with the concave adjoint $F^*$ and positive numbers $a$ and $b$.

*Proof. It follows from the above lemma that (2) holds for $M = 1_w$. Suppose $M \neq 1_w$ and (2) holds. Then $M^* \neq \nabla_w$, so that there exists $x$ with

$$\frac{M^*(1, 1) + M^*(1, x)}{2} = \frac{1 + f_M(x)}{2} < f_M^*\left(\frac{1 + x}{2}\right) = M^*\left(1, \frac{1 + x}{2}\right).$$

Applying $F^*(x) = (1 + x)/2$ to $(2')$, we have

$$\frac{1 + f_M^*(x)}{2} = F(M^*(1, x)) \geq M\left(1, \frac{1 + x}{2}\right) = f_M\left(\frac{1 + x}{2}\right).$$

This contradiction shows $M^* = \nabla_w$, that is, $M = 1_w$. \(\square\)
3 Operator mean. Next we discuss the case of operators. The harmonic operator mean with a weight $w$ is defined by

$$A \! _w B = ((1 - w)A^{-1} + wB^{-1})^{-1}$$

and the arithmetic one with $w$ is $A \! \Delta_w B = (1 - w)A + wB$ for positive invertible operators $A$ and $B$ on a Hilbert space. In general, operator means here stand for the Kubo-Ando operator means defined by (1). Note that the representing function $f_m(x) = 1 - mx$ of a nontrivial operator mean $m$ is a positive monotone-increasing concave function and so are $f_m^*$ and $f_m^\circ$. Now we have a characterization of the harmonic operator mean $!_w$:

**Theorem 3.1.** A nontrivial operator mean $m$ is the (weighted) harmonic (resp. arithmetic) one if and only if

$$(3) \quad F(A \! m B) \leq F(A) m F(B) \quad \text{(resp. } F(A \! m B) \geq F(A) m F(B)\text{)}$$

for all positive operator monotone functions $F$ and positive operators $A$ and $B$.

Finally we observe noncommutative examples. As we state above, for commuting operators $A$ and $B$, we have

$$\sqrt{A \# B} = \sqrt{AB} = \sqrt{A \# \sqrt{B}},$$

where $\#$ is the geometric operator mean

$$A \# B = A^{1/2} \sqrt{A^{-1/2}BA^{-1/2}A^{1/2}}.$$

But it does not hold in general and moreover we can give examples:

$$\sqrt{A \# B} \leq \sqrt{A \# \sqrt{B}} \quad \text{and} \quad \sqrt{C \# B} \geq \sqrt{C \# \sqrt{B}}.$$

Recall the following formula in [2]:

$$(x \ y \ z) \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \implies \quad S \# P = \sqrt{\frac{xz - |y|^2}{z}}P.$$  

Put

$$S_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad A = S_1^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \quad \text{and} \quad B = P.$$  

Then $S_1 \# B = P$ and $S_1^2 \# P = \frac{1}{\sqrt{2}}P$, and hence

$$\sqrt{A \# B} = \sqrt{S_1^2 \# B} = \frac{1}{\sqrt{2}}P \leq P = S_1 \# P = \sqrt{A \# \sqrt{B}}.$$  

Next, put

$$S_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad C = S_2^2 = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \quad \text{and} \quad B = P.$$  

Then we have $S_2 \# P = \sqrt{\frac{3}{2}}P$ and $S_2^2 \# P = \sqrt{\frac{15}{2}}P$, so that

$$\sqrt{C \# B} = \sqrt{S_2^2 \# P} = \frac{2}{\sqrt{5}}P \geq \sqrt{\frac{3}{2}}P = S_2 \# P = \sqrt{C \# \sqrt{B}}.$$  

These examples show the difficulty to discuss the class of functions $F$ satisfying (3). So we discuss another related class in the next section.
4 A class of functions. Finally we consider the set \( MF \) of all representing functions of operator means, that is, positive operator monotone functions \( f_m \) on \((0, \infty)\) with \( f_m(1) = 1 \). Note that \( !_w \) and \( \nabla_w \) belong to the boundary of \( MF \), which corresponds to [3, Theorem 8] and [5, Theorem 4.5]:

Theorem 4.1. The weighted arithmetic means \( \nabla_w \) (resp. harmonic ones \( !_w \)) are the largest (resp. smallest) operator means whose representing functions satisfy \( f_m(1) = w \).

Proof. Every representing function \( f_m \) is concave and differentiable, so we have

\[
f_m(x) \leq f_m'(1)(x - 1) + f_m(1) = 1 - f_m'(1) + f_m'(1)x = f_{\nabla_w}(x)
\]

for all \( x > 0 \), which shows \( m \leq \nabla_w \). Therefore \( !_w = \nabla_w^* \) are the smallest since \( m \leq n \) implies \( m^* \geq n^* \) for all operator means \( m \) and \( n \). \( \square \)

Let \( S(m) = S(f_m) \) be the set of all \( F \in MF \) satisfying

\[
(4) \quad F \circ f_m \leq f_m \circ F,
\]

which is derived from the case \( A = 1 \) in (3):

\[
F(f_m(B)) = F(1 \ m B) \leq F(1) \ m F(B) = 1 \ m F(B) = f_m(F(B)).
\]

Then \( S(m) \) is a closed convex subset of \( MF \) with the maximal extreme points \( \nabla_w \) by the above theorem. Since the equality in (4) holds, we have \( f_m \) itself belongs to \( S(m) \). This suggests that \( m \) occupies an extremal position in \( S(m) \). The above argument shows that \( S(\nabla_w) \) coincides with \( MF \) and \( S(\nabla_w) = \{ f_{\nabla_w} : 0 < w < 1 \} \). In other words, by Theorem 4, the smallest class of \( S(\nabla_w) \) and \( S(\nabla_w) \) is \( \{ f_{\nabla_w} \} \) and \( \{ f_{\nabla_w} \} \) respectively. In particular, these means are extreme points of \( MF \). Moreover it is valid in general, which is another variation of Ando’s theorem:

Theorem 4.2. If \( f_m \) be the non-affine representing function for an operator mean \( m \), then it is an extreme point of \( S(m) \):

\[
f_m \in \text{ext} \ S(m).
\]

Proof. Let \( (F_1 + F_2)/2 = f_m \) for \( F_k \in S(m) \). Then, putting \( y = f_m(x) \) for each \( x > 0 \), we have

\[
f_m(y) = \frac{F_1 + F_2}{2}(y) = \frac{F_1 + F_2}{2}(f_m(x)) = \frac{F_1(f_m(x)) + F_2(f_m(x))}{2} \leq f_m(F_1(x) + F_2(x)) \leq f_m(F_1(x) + F_2(x)) = f_m(f_m(x)) = f_m(y).
\]

Therefore, the equality holds and hence \( F_1(x) = F_2(x) \) by the strict concavity of \( f_m \). Consequently \( F_1 = F_2 = f_m \), which implies \( f_m \in \text{ext} \ S(m) \). \( \square \)

Moreover we conjecture that \( f_m \) is a minimal function for \( S(m) \), that is, for all totally ordered path of representing functions \( f_m \), passing through \( f_m \), (see [4])

\[
f_m = \min \{ f_m \ : f_m \in S(m) \}.
\]

Though it is valid for \( m = !_w \) and \( \nabla_w \), it is an open problem in general.

Recall that for the power mean \( m_{r,w} \) for \(|r| \leq 1 \), the representing function

\[
f_{m_{r,w}}(x) = (1 - w + wx^r)^{1/r},
\]
is operator monotone and hence the representing one of an operator mean. For a fixed weight $w$, it is monotone increasing for $r$ (while the power operator mean $A_{mr,w}B$ is not always monotone increasing in the usual order for operators). For $r \to 0$, we obtain the geometric operator mean $A_{#w}$ with a weight $w$:

$$A_{#w}B = A^{1/2} \left( A^{-1/2}BA^{-1/2} \right)^w A^{1/2}.$$ 

Now we can verify that the representing function $f_{#w}(x) = x^w$ is the smallest one in the power ones in $S(#w)$. In fact, the monotonicity of power means shows

$$(1 - w + wx^{-r})^{-1/r} \leq (1 - w + wx^{-w})^{-1/(wr)} \leq (1 - w + wx^{wr})^{1/(wr)} \leq (1 - w + wx^r)^{1/r}$$

for all $0 < r \leq 1$. This is equivalent to

$$(1 - w + wx^{-r})^{-w/r} \leq (1 - w + wx^{-w})^{-1/r} \leq (1 - w + wx^{wr})^{1/r} \leq (1 - w + wx^r)^{w/r},$$

which shows $f_{#w} = \min \{ f_{mr,w} | f_{mr,w} \in S(#w) \}$.

REFERENCES


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