ON INTUITIONISTIC $Q$-FUZZY IDEALS OF SEMIGROUPS

KYUNG HO KIM

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Abstract. In this paper, for a set $Q$, the notion of intuitionistic $Q$-fuzzy ideals of semigroups is introduced, and some properties of such ideals are investigated.

1 Introduction Zadeh, in his classic paper [10], introduced the notion of fuzzy sets and fuzzy set operations. Since then the fuzzy set theory developed by Zadeh and others has evoked great interest among researchers working in different branches of mathematics. Kuroki [4, 5] have studied sever properties of fuzzy left (right) ideals, fuzzy bi-ideal and fuzzy interior ideals in semigroups. Yuan and Shen [8] introduced a new fuzzy subgroup, called an $S$-fuzzy subgroup, of a group over a set $S$ and they showed that a fuzzy subgroup of a group can be seen as an $S$-fuzzy subgroup. The idea of “intuitionistic $Q$-fuzzy set” was first published by Atanassov [1, 2], as a generalization of the notion of fuzzy set. In this paper, using the Atanassov’s idea, we establish the intuitionistic $Q$-fuzzification of the concept of ideals in semigroups. For a set $Q$, the notions of intuitionistic $Q$-fuzzy subsemigroup, intuitionistic $Q$-fuzzy left (right) ideal and intuitionistic $Q$-fuzzy interior ideal of a semigroup $X$ are given, and some properties of such ideals are investigated.

2 Preliminaries Let $X$ be a semigroup. By a subsemigroup of $X$ we mean a non-empty subset $A$ of $X$ such that $A^2 \subseteq A$, and by a left (resp. right) ideal of $X$ we mean a non-empty subset $A$ of $G$ such that $GA \subseteq A$ (resp. $AX \subseteq A$). By two-sided ideal or simply ideal, we mean a non-empty subset of $X$ which is both a left and a right ideal of $X$.

A fuzzy set in $X$ is a function $\mu$ from $X$ into the unit interval $[0, 1]$. A fuzzy set $\mu$ in $X$ is called a fuzzy subsemigroup of $X$ if it satisfies

$$\forall x, y \in X \ (\mu(xy) \geq \min\{\mu(x), \mu(y)\}),$$

and is called a fuzzy left (resp. right) ideal of $X$ if

$$\forall x, y \in X \ (\mu(xy) \geq \mu(y) \ (\text{resp. } \mu(xy) \geq \mu(x))).$$

If $\mu$ is both a fuzzy left and a fuzzy right ideal of $X$, we say that $\mu$ is a fuzzy ideal of $X$.

In what follows let $X$ and $Q$ denote a semigroup and a non-empty set, respectively, unless otherwise specified. A mapping $f : X \times Q \rightarrow [0, 1]$ is called a $Q$-fuzzy set in $X$.

A $Q$-fuzzy set $f : X \times Q \rightarrow [0, 1]$ is called a fuzzy subsemigroup of $X$ over $Q$ (briefly, $Q$-fuzzy subsemigroup of $X$) if $f(xy, q) \geq \min\{f(x, q), f(y, q)\}$ for all $x, y \in X$ and $q \in Q$.

Example 2.1. Let $X = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

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Let $f : X \times Q \to [0, 1]$ be an $Q$-fuzzy set in $X$ defined by $f(a, q) = 0.9, f(b, q) = 0.4, f(c, q) = 0.7, f(d, q) = 0.5$, for all $q \in Q$. By routine calculations one show that $f$ is an $Q$-fuzzy subsemigroup of $X$.

**Definition 2.2.** An $Q$-fuzzy set $f$ in $X$ is called an $Q$-fuzzy interior ideal of $X$ if it satisfies

(i) $f$ is an $Q$-fuzzy subsemigroup of $X$,

(ii) $f(xwy, q) \geq f(w, q)$ for all $w, x, y \in X$ and $q \in Q$.

**Example 2.3.** Let $S = \{0, e, f, a, b\}$ be a set with the following Cayley table:

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Then $X$ is a semigroup. Let $h : X \times Q \to [0, 1]$ be an $Q$-fuzzy set in $X$ defined by

$h(0, q) = h(e, q) = h(f, q) = 1, h(a, q) = h(b, q) = 0$ for all $q \in Q$. By routine calculations one show that $h$ is an $Q$-fuzzy interior ideal of $X$.

### 3 Intuitionistic $Q$-fuzzy ideals

**Definition 3.1.** An intuitionistic $Q$-fuzzy set (IQFS for short) $A$ is an object having the form

$A = \{(x, \mu_A(x, q), \gamma_A(x, q)) : x \in X, q \in Q\}$

where the functions $\mu_A : X \times Q \to [0, 1]$ and $\gamma_A : X \times Q \to [0, 1]$ denote the degree of membership (namely $\mu_A(x, q)$) and the degree of nonmembership (namely $\gamma_A(x, q)$) of each element $(x, q) \in X \times Q$ to the set $A$, respectively, and $0 \leq \mu_A(x, q) + \gamma_A(x, q) \leq 1$ for all $x \in X$ and $q \in Q$.

For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the IQFS $A = \{(x, \mu_A(x, q), \gamma_A(x, q)) : x \in X, q \in Q\}$.

**Definition 3.2.** An IQFS $A = (\mu_A, \gamma_A)$ in $X$ is called an intuitionistic $Q$-fuzzy subsemigroup of $X$ if

(IQFS 1) $\mu_A(xy, q) \geq \min\{\mu_A(x, q), \mu_A(y, q)\}$

(IQFS 2) $\gamma_A(xy, q) \leq \max\{\gamma_A(x, q), \gamma_A(y, q)\}$

for all $x, y \in X$ and $q \in Q$.

**Example 3.3.** Let $X = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

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Let $A = (\mu_A, \gamma_A)$ be an IQFS in $X$ defined by
\[
\mu_A(a, q) = 0.9, \mu_A(b, q) = 0.4, \mu_A(c, q) = 0.5, \mu_A(d, q) = 0.7
\]
\[
\gamma_A(a, q) = 0.8, \gamma_A(b, q) = 0.3, \gamma_A(c, q) = 0.5, \gamma_A(d, q) = 0.5
\]
for all $q \in Q$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic $Q$-fuzzy subsemigroup of $X$.

**Proposition 3.4.** If IQFS $A = (\mu_A, \gamma_A)$ is an intuitionistic $Q$-fuzzy subsemigroup of $X$, then the set
\[
X_A = \{ x \in X \mid \mu_A(x, q) = \mu_A(0, q), \gamma_A(x, q) = \gamma_A(0, q), q \in Q \}
\]
is a subsemigroup of $X$.

**Proof.** Let $x, y \in X$ and $q \in Q$. Then $\mu_A(x, q) = \mu_A(y, q) = \mu_A(0, q)$ and $\gamma_A(x, q) = \gamma_A(y, q) = \gamma_A(0, q)$. Since $A = (\mu_A, \gamma_A)$ is an intuitionistic $Q$-fuzzy subsemigroup of $X$, it follows that
\[
\mu_A(xy, q) \geq \min\{\mu_A(x, q), \mu_A(y, q)\} = \mu_A(0, q),
\]
\[
\gamma_A(xy, q) \leq \max\{\gamma_A(x, q), \gamma_A(y, q)\} = \gamma_A(0, q)
\]
so that $\mu_A(xy, q) = \mu_A(0, q)$ and $\gamma_A(xy, q) = \gamma_A(0, q)$. Thus $xy \in X_A$, and consequently $X_A$ is a subsemigroup of $X$. \qed

Let $A = (\mu_A, \gamma_A)$ be an IQFS in a set $X$ and let $\alpha + \beta \in [0, 1]$ be such that $\alpha + \beta \leq 1$. Then we define the set
\[
X_A^{(\alpha, \beta)} = \{ x \in X \mid \mu_A(x, q) \geq \alpha, \gamma_A(x, q) \leq \beta, q \in Q \}.
\]

**Theorem 3.5.** Let $A = (\mu_A, \gamma_A)$ be an intuitionistic $Q$-fuzzy subsemigroup of $X$. Then $X_A^{(\alpha, \beta)}$ is a subsemigroup of semigroup $X$ for every $(\alpha, \beta) \in \text{Im} (\mu_A) \times \text{Im} (\gamma_A)$ with $\alpha + \beta \leq 1$.

**Proof.** Let $x, y \in X_A^{(\alpha, \beta)}$ and $q \in Q$. Then $\mu_A(x, q) \geq \alpha, \gamma_A(x, q) \leq \beta, \mu_A(y, q) \geq \alpha, \gamma_A(y, q) \leq \beta$ which imply that
\[
\mu_A(xy, q) \geq \min\{\mu_A(x, q), \mu_A(y, q)\} \geq \alpha
\]
\[
\gamma_A(xy, q) \leq \max\{\gamma_A(x, q), \gamma_A(y, q)\} \leq \beta.
\]
Thus $x - y \in X_A^{(\alpha, \beta)}$ Therefore $X_A^{(\alpha, \beta)}$ is a subsemigroup of semigroup $X$. \qed

A semigroup $X$ is said to be a *monoid* if there exists an identity element $e \in X$ such that $xe = ex = x$ for all $x \in X$.

**Definition 3.6.** An intuitionistic $Q$-fuzzy subsemigroup IQFS $A = (\mu_A, \gamma_A)$ of a monoid $M$ is said to be *intuitionistic $Q$-fuzzy submonoid* if
\[
\mu_A(e, q) \geq \mu_A(x, q) \text{and} \gamma_A(e, q) \leq \gamma_A(x, q)
\]
for all $x \in M$, where $e$ is the identity element of $M$.

Let $f : M \rightarrow M'$ be a homomorphism of monoids $M$ and $M'$. For any IQFS $A = (\mu_A, \gamma_A)$ in $Y$, we define a new IQFS $F = (\mu_A^f, \gamma_A^f)$ in $X$ by
\[
\mu_A^f(x, q) = \mu_A(f(x), q), \gamma_A^f(x, q) = \gamma_A(f(x), q)
\]
for all $x$ and $q$. 
Theorem 3.7. Let $f : M \to M'$ be a homomorphism of monoids $M$ and $M'$. If an IQFS $A = (\mu_A, \gamma_A)$ is an intuitionistic $Q$-fuzzy submonoid of $M'$, then $IQFS^f = (\mu_A^f, \gamma_A^f)$ is an intuitionistic $Q$-fuzzy submonoid in $M$.

Proof. Let $f : M \to M'$ be a homomorphism from $M$ to $M'$ and IQFS $A = (\mu_A, \gamma_A)$ be an intuitionistic $Q$-fuzzy submonoid of $M'$. Then the IQFS $A' = (\mu_A^f, \gamma_A^f)$ of $M$ satisfies

$$\mu_A^f(e, q) = \mu_A(f(e), q) = \mu_A(e', q) \geq \mu_A(f(x), q) = \mu_A^f(x, q)$$

and

$$\gamma_A^f(e, q) = \gamma_A(f(e), q) = \gamma_A(e', q) \leq \gamma_A(f(x), q) = \gamma_A^f(x, q)$$

where $e, e'$ are identity elements of $M$ and $M'$. Now we have

$$\mu_A^f(xy, q) = \mu_A(f(xy), q) = \mu_A(f(x)f(y), q) \geq \min\{\mu_A(f(x), q), \mu_A(f(y), q)\} = \min\{\mu_A^f(x, q), \mu_A^f(y, q)\}$$

and

$$\gamma_A^f(xy, q) = \gamma_A(f(xy), q) = \gamma_A(f(x)f(y), q) \leq \max\{\gamma_A(f(x), q), \gamma_A(f(y), q)\} = \max\{\gamma_A^f(x, q), \gamma_A^f(y, q)\}.$$ 

Hence $A' = (\mu_A^f, \gamma_A^f)$ is an intuitionistic $Q$-fuzzy submonoid in $M$. □

Definition 3.8. An IQFS $A = (\mu_A, \gamma_A)$ is called an intuitionistic $Q$-fuzzy left ideal of $X$ if $\mu_A(xy, q) \geq \mu_A(y, q)$ and $\gamma_A(xy, q) \leq \gamma_A(y, q)$ for all $x, y \in X$ and $q \in Q$.

Definition 3.9. An IQFS $A = (\mu_A, \gamma_A)$ is called an intuitionistic $Q$-fuzzy right ideal of $X$ if $\mu_A(xy, q) \geq \mu_A(x, q)$ and $\gamma_A(xy, q) \leq \gamma_A(x, q)$ for all $x, y \in X$ and $q \in Q$.

An IQFS $A = (\mu_A, \gamma_A)$ in $X$ is called an intuitionistic $Q$-fuzzy ideal of $X$ if it is both an intuitionistic $Q$-fuzzy left ideal and intuitionistic $Q$-fuzzy right ideal of $X$.

Note that every intuitionistic $Q$-fuzzy left (resp. right) ideal of $X$ is an intuitionistic $Q$-fuzzy subsemigroup of $X$. But the converse is not true. By Example 3.2, it is observed that $\mu_A(ca, q) = \mu_A(c, q) = 0.7 \leq \mu_A(a, q) = 0.9$ for all $q \in Q$.

Lemma 3.10. Let IQFS $A = (\mu_A, \gamma_A)$ be an intuitionistic $Q$-fuzzy subgroup of $X$ such that $\mu_A(x, q) \geq \mu_A(y, q)$ (or $\mu_A(y, q) \geq \mu_A(x, q)$) and $\gamma_A(x, q) \leq \gamma_A(y, q)$ (or $\gamma_A(y, q) \geq \gamma_A(x, q)$) for all $x, y \in X$ and $q \in Q$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic $Q$-fuzzy left (or right) ideal of $X$.

Proof. Let $\mu_A(x, q) \geq \mu_A(y, q)$ and $\gamma_A(x, q) \leq \gamma_A(y, q)$ for all $x, y \in X$ and $q \in Q$. Then we have

$$\mu_A(xy, q) \geq \min\{\mu_A(x, q), \mu_A(y, q)\} = \mu_A(y, q)$$

$$\gamma_A(xy, q) \leq \max\{\gamma_A(x, q), \gamma_A(y, q)\} = \gamma_A(y, q).$$

Hence $A = (\mu_A, \gamma_A)$ is an intuitionistic $Q$-fuzzy left ideal of $X$. Similarly, if we let $\mu_A(y, q) \geq \mu_A(x, q)$ and $\gamma_A(y, q) \leq \gamma_A(x, q)$ for all $x, y \in X$ and $q \in Q$, then it is easy to prove that $A = (\mu_A, \gamma_A)$ is an intuitionistic $Q$-fuzzy right ideal of $X$. □
Theorem 3.16. If an IQFS $\text{Im}(\mu_A) \cap \text{Im}(\gamma_A) \subseteq [0, 1]$, then $\mu_A(x, y, q) \geq \gamma_A(x, y, q) \geq 1$ for all $x, y, q \in Q$.

Proof. Let $\alpha \in \text{Im}(\mu_A) \cap \text{Im}(\gamma_A) \subseteq [0, 1]$ and let $x, y \in \mu_A^{\geq \alpha}$. Then $\mu_A(x, y, q) \geq \alpha$ and $\mu_A(y, q) \geq \alpha$ for all $q \in Q$. It follows from (IQFS 1) that $\mu_A(x, y, q) \geq \min\{\mu_A(x, q), \mu_A(y, q)\} \geq \alpha$ so that $xy \in \mu_A^{\geq \alpha}$.

Example 3.12. Let $X = \{0, e, f, a, b\}$ be a semigroup with the following Cayley table:

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Then $X$ is a semigroup. Define an IQFS $A = (\mu_A, \gamma_A)$ in $X$ by $\mu_A(0, q) = \mu_A(e, q) = \mu_A(f, q) = 1, \mu_A(a, q) = \mu_A(b, q) = 0, \gamma_A(0, q) = \gamma_A(e, q) = \gamma_A(f, q) = 0, \gamma_A(a, q) = \gamma_A(b, q) = 1$. By routine calculations we know that $A = (\mu_A, \gamma_A)$ is an intuitionistic $Q$-fuzzy subsemigroup of $X$ and an intuitionistic $Q$-fuzzy interior ideal of $X$.

Theorem 3.13. If $\{A_i\}$ is a family of intuitionistic $Q$-fuzzy interior ideals of $X$, then $\bigcap A_i$ is an intuitionistic $Q$-fuzzy interior ideal of $X$, where $\bigcap A_i = (\bigvee A_i, \bigwedge A_i)$ and $\bigwedge A_i$ and $\bigvee A_i$ are defined as follows:

$\forall A_i = \{A_i(x, q) \mid i \in I, x \in X\}$ and $\bigwedge A_i = \sup\{A_i(x, q) \mid i \in I, x \in X\}$ for all $q \in Q$.

Proof. It is straightforward. \qed

Theorem 3.14. If an IQFS $A = (\mu_A, \gamma_A)$ in $X$ is an intuitionistic $Q$-fuzzy interior ideal of $X$, then so is $\Box A$, where $\Box A = \{(x, \mu_A(x, q), 1 - \mu_A(x, q)) : x \in X, q \in Q\}$.

Proof. It is sufficient to show that $\Box A$ satisfies the conditions (IQFS 2) and (IQFS 4). For any $a, x, y \in X$, we have

$$\overline{\mu_A}(x, y, q) = 1 - \mu_A(x, y, q) \leq 1 - \min\{\mu_A(x, q), \mu_A(y, q)\} = \max\{1 - \mu_A(x, q), 1 - \mu_A(y, q)\} = \max\{\overline{\mu_A}(x, q), \overline{\mu_A}(y, q)\}$$

and

$$\overline{\mu_A}(x, y, q) = 1 - \mu_A(x, y, q) \leq 1 - \mu_A(a, q) = \overline{\mu_A}(a, q).$$

Therefore $\Box A$ is an intuitionistic $Q$-fuzzy interior ideal of $X$. \qed

Definition 3.15. Let $A = (\mu_A, \gamma_A)$ be an IQFS in $X$ and let $\alpha \in [0, 1]$. Then the set $\mu_A^{\geq \alpha} := \{x \in X : \mu_A(x, q) \geq \alpha \text{ for all } q \in Q\}$ is called a $\mu$-level $\alpha$-cut and $\gamma_A^{\leq \alpha} := \{x \in X : \gamma_A(x, q) \leq \alpha \text{ for all } q \in Q\}$ is called a $\gamma$-level $\alpha$-cut of $A$.

Theorem 3.16. If an IQFS $A = (\mu_A, \gamma_A)$ in $X$ is an intuitionistic $Q$-fuzzy interior ideal of $X$, then the $\mu$-level $\alpha$-cut $\mu_A^{\geq \alpha}$ and the $\gamma$-level $\alpha$-cut $\gamma_A^{\leq \alpha}$ of $A$ are interior ideals of $X$ for every $\alpha \in \text{Im}(\mu_A) \cap \text{Im}(\gamma_A) \subseteq [0, 1]$.}

Proof. Let $\alpha \in \text{Im}(\mu_A) \cap \text{Im}(\gamma_A) \subseteq [0, 1]$ and let $x, y \in \mu_A^{\geq \alpha}$. Then $\mu_A(x, y, q) \geq \alpha$ and $\mu_A(y, q) \geq \alpha$ for all $q \in Q$. It follows from (IQFS 1) that $\mu_A(x, y, q) \geq \min\{\mu_A(x, q), \mu_A(y, q)\} \geq \alpha$ so that $xy \in \mu_A^{\geq \alpha}$.\qed

Definition 3.11. An intuitionistic $Q$-fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ of $X$ is called is an intuitionistic $Q$-fuzzy interior ideal of $X$ if

(IQFS 3) $\mu_A(xy, q) \geq \mu_A(a, q)$

(IQFS 4) $\gamma_A(xy, q) \leq \gamma_A(a, q)$

for all $x, y, a \in X$ and $q \in Q$.\qed
If \( x, y \in \gamma_{A,\alpha}^\leq \), then \( \gamma_A(x, q) \leq \alpha \) and \( \gamma_A(y, q) \leq \alpha \), and so
\[
\gamma_A(xy, q) \leq \max\{\gamma_A(x, q), \gamma_A(y, q)\} \leq \alpha, \text{ i.e., } xy \in \gamma_{A,\alpha}^\leq.
\]
Hence \( \mu_{A,\alpha}^\geq \) and \( \gamma_{A,\alpha}^\leq \) are subsemigroups of \( X \). Now let \( x, y \in X \) and \( a \in \mu_{A,\alpha}^\geq \). Then \( \mu_A(xy, q) \geq \mu_A(a, q) \geq \alpha \) and so \( xy \in \gamma_{A,\alpha}^\leq \). Hence the condition (IQFS 1) is true. The proof of
Theorem 3.20. An IQFS \( A = (\mu_A, \gamma_A) \) is an intuitionistic \( Q \)-fuzzy interior ideal of \( X \) if and only if the fuzzy sets \( \mu_A \) and \( \gamma_A \) are \( Q \)-fuzzy interior ideals of \( X \).
**Proof.** Let \( A = (\mu_A, \gamma_A) \) be an intuitionistic \( Q \)-fuzzy interior ideal of \( X \). Then clearly \( \mu_A \) is an \( \mu_A \)-fuzzy interior ideal of \( X \). Let \( \gamma_A \) be an intuitionistic \( Q \)-fuzzy interior ideal of \( X \). Let \( x, a, y \in X \) and \( q \in Q \). Then

\[
\gamma_A(x, q) = 1 - \gamma_A(x, q) \\
\geq 1 - \max\{\gamma_A(x, q), \gamma_A(y, q)\} \\
= \min\{1 - \gamma_A(x, q), 1 - \gamma_A(y, q)\} \\
= \min\{\gamma_A(x, q), \gamma_A(y, q)\}, \quad \text{and}
\]

\[
\gamma_A(xay, q) = 1 - \gamma_A(xay, q) \geq 1 - \gamma_A(a, q) = \gamma_A(a, q).
\]

Hence \( \gamma_A \) is a \( \gamma_A \)-fuzzy interior ideal of \( X \).

Conversely suppose that \( \mu_A \) and \( \gamma_A \) are \( \mu_A \)-fuzzy interior ideals of \( X \). Let \( x, a, y \in X \) and \( q \in Q \). Then

\[
1 - \gamma_A(x, q) = \gamma_A(x, q) \geq \min\{\gamma_A(x, q), \gamma_A(y, q)\} \\
= \min\{1 - \gamma_A(x, q), 1 - \gamma_A(y, q)\} \\
= 1 - \max\{\gamma_A(x, q), \gamma_A(y, q)\}, \quad \text{and}
\]

\[
1 - \gamma_A(xay, q) = \gamma_A(xay, q) \geq \gamma_A(a, q) = 1 - \gamma_A(a, q),
\]

which imply that \( \gamma_A(x, q) \leq \max\{\gamma_A(x, q), \gamma_A(y, q)\} \) and \( \gamma_A(xay, q) \leq \gamma_A(a, q) \). This completes the proof.

**Corollary 3.21.** An IQFS \( A = (\mu_A, \gamma_A) \) is an intuitionistic \( Q \)-fuzzy interior ideal of \( X \) if and only if \( \square A = (\mu_A, \gamma_A) \) and \( \Diamond A = (\gamma_A, \gamma_A) \) are intuitionistic \( Q \)-fuzzy interior ideals of \( X \).

**Proof.** It is straightforward by Theorem 3.20.

Let \( f \) be a map from a set \( X \) to a set \( Y \). If \( A = (\mu_A, \gamma_A) \) and \( B = (\mu_B, \gamma_B) \) are IQFSs in \( X \) and \( Y \) respectively, then the preimage of \( B \) under \( f \), denoted by \( f^{-1}(B) \), is an IQFS in \( X \) defined by

\[
f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B)), \text{ where } f^{-1}(\mu_B) = \mu_B(f).
\]

**Theorem 3.22.** Let \( f : X \to Y \) be a homomorphism of semigroups. If \( B = (\mu_B, \gamma_B) \) is an intuitionistic \( Q \)-fuzzy interior ideal of \( Y \), then the preimage \( f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B)) \) of \( B \) under \( f \) is an intuitionistic \( Q \)-fuzzy interior ideal of \( X \).

**Proof.** Assume that \( B = (\mu_B, \gamma_B) \) is an intuitionistic \( Q \)-fuzzy interior ideal of \( Y \) and let \( x, y \in X \) and \( q \in Q \). Then

\[
f^{-1}(\mu_B)(x, q) = \mu_B(f(x), q) \\
= \mu_B(f(x)f(y), q) \\
\geq \min\{\mu_B(f(x), q), \mu_B(f(y), q)\} \\
= \min\{f^{-1}(\mu_B(x, q)), f^{-1}(\mu_B(y, q))\}, \quad \text{and}
\]

\[
f^{-1}(\gamma_B)(x, q) = \gamma_B(f(x), q) \\
= \gamma_B(f(x)f(y), q) \\
\leq \max\{\gamma_B(f(x), q), \gamma_B(f(y), q)\} \\
= \max\{f^{-1}(\gamma_B(x, q)), f^{-1}(\gamma_B(y, q))\}.
\]
Hence $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$ is an intuitionistic $Q$-fuzzy subsemigroup of $X$. For any $a, x, y \in X$ and $q \in Q$, we have

$$f^{-1}(\mu_B)(xay, q) = \mu_B(f(xay), q) = \mu_B(f(x)f(a)f(y), q) \geq \mu_B(f(a), q) = f^{-1}(\mu_B(a, q)),$$  and

$$f^{-1}(\gamma_B)(xay, q) = \gamma_B(f(xay), q) = \gamma_B(f(x)f(a)f(y), q) \leq \gamma_B(f(a), q) = f^{-1}(\gamma_B(a, q)).$$

Therefore $f^{-1}(B) = (f^{-1}(\mu_B), f^{-1}(\gamma_B))$ is an intuitionistic $Q$-fuzzy interior ideal of $X$.  

References


K. H. Kim, Department of Mathematics, Chungju National University, Chungju 380-702, Korea ghkim@chungju.ac.kr