NOTES ON COMMUTATORS OF FRACTIONAL INTEGRAL OPERATROS ON GENERALIZED MORREY SPACES

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ABSTRACT. We show that $b \in BMO(\mathbb{R}^n)$ if and only if the commutator $[b, I_\alpha]$ of the multiplication operator by b and the fractional integral operator I_α is bounded from generalized Morrey spaces $L^{p,\varphi}(\mathbb{R}^n)$ to $L^{q,\varphi^{q/p}}(\mathbb{R}^n)$, where φ is non-decreasing, and $1 < p < \infty$, $0 < \alpha < n$ and $1/q = 1/p - \alpha/n$.

1. Introduction

The author [7] proved that $b \in BMO$ if and only if the commutator $[b, I_\alpha]$ defined by

$$[b, I_\alpha]f(x) := b(x)I_\alpha f(x) - I_\alpha(bf)(x),$$

is bounded from the classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$ with appropriate indices p, q, λ and μ, where I_α is the fractional integral operator of order α, that is,

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad 0 < \alpha < n.$$

Y. Komori and T. Mizuhara [2], E. Nakai [4], C. Zorko [8] and many authors considered various generalized Morrey spaces. In particular, Y. Komori and T. Mizuhara [2] proved by using a factorization theorem for Hardy space $H^1(\mathbb{R}^n)$ that if the commutator $[b, T]$ is bounded on generalized Morrey spaces, then b is in $BMO(\mathbb{R}^n)$, where T is a Caldeón-Zygmund operator.

The purpose of this paper is to show that $b \in BMO$ if and only if the commutator $[b, I_\alpha]$ is bounded from generalized Morrey spaces $L^{p,\varphi}(\mathbb{R}^n)$ to $L^{q,\varphi^{q/p}}(\mathbb{R}^n)$. Our proof is direct, but Y. Komori and T. Mizuhara have used a factorization theorem. Also we can apply our method to obtain the boundedness of the higher order commutator on generalized Morrey spaces.

2. Definitions and Notation

Throughout this paper all notation is standard or will be defined as needed. All cubes are assumed to have their sides parallel to the coordinate axes. $Q = Q(x_0, r)$ denotes the cube centered at x_0 with side length r. Given a Lebesgue measurable set E, χ_E will denote the characteristic function of E and $|E|$ is the Lebesgue measure of E. The letter C will be used for various constants, and may change from one occurrence to another.

Definition 1 (generalized Morrey space)**.** Let $1 < p < \infty$ and let $\varphi\colon (0,\infty) \to (0,\infty)$ such that φ be non-decreasing and satisfy the following condition:

$$\int_r^\infty \frac{\varphi(s)}{s}\frac{ds}{s} \le C\frac{\varphi(r)}{r}. \tag{2.1}$$

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We define a generalized Morrey space by

$$L^{p,\varphi}(\mathbb{R}^n) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)\colon \|f\|_{L^{p,\varphi}} < \infty\},$$

where

$$\|f\|_{L^{p,\varphi}} := \sup_{\substack{x\in\mathbb{R}^n\\ r>0}} \left(\frac{1}{\varphi(|Q(x,r)|)}\int_{Q(x,r)} |f(y)|^p\, dy\right)^{1/p}.$$

Note that if $\varphi(t) = t^{\lambda/n}$, $0 < \lambda < n$, then $L^{p,\varphi}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n)$. Another typical example of φ satisfying (2.1) is $\varphi(t) = t^{\alpha}(\log(1+t))^{\beta}$ with $0 < \alpha < 1$ and $\alpha + \beta \geq 0$.

Remark 1. Y. Komori and T. Mizuhara [2] defined a generalized Morrey space $L^{p,\psi}(\mathbb{R}^n)$ by

$$L^{p,\psi}(\mathbb{R}^n) = \Bigg\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)\colon \|f\|_{L^{p,\psi}} = \sup_{\substack{x\in\mathbb{R}^n\\ r>0}} \frac{1}{\psi(|Q(x,r)|)}\left(\frac{1}{|Q(x,r)|}\int_{Q(x,r)} |f(y)|^p\, dy\right)^{1/p} < \infty\Bigg\},$$

where $\psi\colon (0,\infty) \to (0,\infty)$ such that $\psi(t)$ is non-increasing and $t^{1/p}\psi(t)$ is non-decreasing. Our definition of a generalized Morrey space is identified with their definition by $\varphi^{1/p} \sim \psi \cdot |Q|^{1/p}$.

Definition 2 (John-Nirenberg space)**.** $BMO(\mathbb{R}^n)$ is the John-Nirenberg space. That is, $BMO(\mathbb{R}^n)$ is a Banach space, modulo constants, with the norm $\|\cdot\|_*$ defined by

$$\|b\|_* := \sup_{\substack{x\in\mathbb{R}^n\\ r>0}} \frac{1}{|Q(x,r)|}\int_{Q(x,r)} |b(y) - b_Q|\, dy,$$

where

$$b_Q := \frac{1}{|Q(x,r)|}\int_{Q(x,r)} b(y)\, dy.$$

Definition 3 (Lipschitz space)**.** We define the Lipschitz space of order β, $0 < \beta < 1$, by

$$\dot{\Lambda}_\beta(\mathbb{R}^n) = \{f\colon\ |f(x) - f(y)| \leq C\,|x-y|^{\beta}\}$$

and the smallest constant $C > 0$ is the Lipschitz norm $\|\cdot\|_{\dot{\Lambda}_\beta}$. Then $\dot{\Lambda}_\beta(\mathbb{R}^n)$ is a Banach space, modulo constants, with the norm $\|\cdot\|_{\dot{\Lambda}_\beta}$.

Generalized blocks and the spaces generated by generalized blocks were introduced by Y. Komori and T. Mizuhara [2].

Definition 4. Let $1 < q < \infty$ and $1/q + 1/q' = 1$. A function $g(x)$ on $\mathbb{R}^n$ is called a (φ,q)-block, if there exists a cube $Q(x_0,r)$ such that

$$\mathrm{supp}(g) \subset Q(x_0,r) \quad \text{and} \quad \|g\|_{L^q} \leq \varphi(|Q(x_0,r)|)^{-1/q'}.$$

Remark 2. Let φ be non-decreasing. If g is a (φ,q)-block and $\mathrm{supp}(g) \subset Q(x_0,r)$, then

$$\|g\|_{L^q} \leq \varphi(1)^{-1/q'} \quad \text{if } r > 1 \quad \text{and} \quad \|g\|_{L^1} \leq \varphi(1)^{-1/q'} \quad \text{if } r \leq 1.$$

Definition 5. Let $1 < q < \infty$ and φ be non-decreasing. We define the space generated by generalized blocks by

$$h^{\varphi,q}(\mathbb{R}^n) := \Big\{ f = \sum_{j=1}^{\infty} m_j g_j + \sum_{j=1}^{\infty} \tilde{m}_j \tilde{g}_j \colon \|f\|_{h_{\varphi,q}} = \inf \sum_{j=1}^{\infty} (|m_j| + |\tilde{m}_j|) < \infty,$$
$$g_j \text{ are } (\varphi, q)\text{-blocks and } \operatorname{supp}(g_j) \subset Q(x_j, r_j) \text{ where } r_j > 1,$$
$$\tilde{g}_j \text{ are } (\varphi, q)\text{-blocks and } \operatorname{supp}(\tilde{g}_j) \subset Q(x_j, r_j) \text{ where } r_j \leq 1 \Big\},$$

where the infimum is taken over all possible representations f.

3. Theorems

Theorem 1. *Let $0 < \alpha < n$, $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. Suppose that φ is non-decreasing and satisfy the condition* (2.1)*. Then the following two conditions are equivalent*:
(a) $b \in BMO(\mathbb{R}^n)$.
(b) $[b, I_\alpha]$ *is bounded from* $L^{p,\varphi}(\mathbb{R}^n)$ *to* $L^{q,\varphi^{q/p}}(\mathbb{R}^n)$.

Furthermore we get the following results when $\alpha < n(1/p - 1/q)$.

Theorem 2. *Let $1 < p < q < \infty$, $0 < \alpha < n$, $0 < \beta < 1$ and $0 < \alpha + \beta = n(1/p - 1/q) < n$. Suppose that φ is non-decreasing satisfy the condition* (2.1)*. Then the following conditions are equivalent*:
(a) $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.
(b) $[b, I_\alpha]$ *is bounded from* $L^{p,\varphi}(\mathbb{R}^n)$ *to* $L^{q,\varphi^{q/p}}(\mathbb{R}^n)$.

Remark 3. In that definition of Y. Komori and T. Mizuhara [2], both of (b) in our theorems can be rewrited as follows:
(b)′ $[b, I_\alpha]$ is bounded from $L^{p,\psi}(\mathbb{R}^n)$ to $L^{q,\psi_\alpha}(\mathbb{R}^n)$, whose norm is defined by

$$\|f\|_{L^{q,\psi_\alpha}} = \sup_{Q} \frac{1}{\psi(|Q(x,r)|) \cdot r^\alpha} \left(\frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y)|^q \, dy \right)^{1/q},$$

where $\psi_\alpha(|Q|) = \psi(|Q(x,r)|) \cdot r^\alpha$.

4. Technical Lemmas

We need some lemmas to prove our theorems. Lemmas 1 and 2 are similar to the result due to Y. Komiri and T. Mizuhara [2]. Since the definition of generalized Morrey spaces differ from their definition, we give a proof for completeness.

Lemma 1. *Let $1 < p < q < \infty$ and $1/q + 1/q' = 1$. Then we have*

$$\|\chi_{Q(x_0,r)}\|_{L^{p,\varphi}} \leq \varphi(|Q(x_0,r)|)^{-1/p} |Q(x_0,r)|^{1/p},$$
$$\|\chi_{Q(x_0,r)}\|_{h^{\varphi^{q/p},q'}} \leq \varphi(|Q(x_0,r)|)^{1/p} |Q(x_0,r)|^{1/q'}.$$

Proof. Fix $Q(x_0, r) = Q$. For any cube Q', we get

$$\left(\frac{1}{\varphi(|Q'|)} \int_{Q'} \chi_Q(y)^p \, dy \right)^{1/p} = \left(\frac{|Q' \cap Q|}{\varphi(|Q'|)} \right)^{1/p}$$
$$\leq \left(\frac{|Q|}{\varphi(|Q'|)} \right)^{1/p} \leq \left(\frac{|Q|}{\varphi(|Q|)} \right)^{1/p}.$$

Next we estimate the norm of $\chi_{Q(x_0,r)}$ on $h^{\varphi^{q/p},q'}$. It follows immediately that

$$\left\|\chi_{Q(x_0,r)}\right\|_{L^{q'}} = |Q|^{1/q'} = |Q|^{1/q'}\varphi(|Q|)^{-\frac{1}{q}\cdot\frac{q}{p}}\varphi(|Q|)^{\frac{1}{q}\cdot\frac{q}{p}}.$$

Therefore $|Q|^{-1/q'}\varphi(|Q|)^{-\frac{1}{q}\cdot\frac{q}{p}}\chi_Q(x)$ is $(\varphi^{q/p},q')$-blocks. Hence we have

$$\left\|\chi_{Q(x_0,r)}\right\|_{h^{\varphi^{q/p},q'}} \leq \varphi(|Q(x_0,r)|)^{1/p}|Q(x_0,r)|^{1/q'}.$$

□

Lemma 2. *If $g \in L^{q,\varphi^{q/p}}(\mathbb{R}^n)$ and b is a $(\varphi^{q/p},q')$-blocks, then*

$$\left|\int_{\mathbb{R}^n} g(x)b(x)\,dx\right| \leq \|g\|_{L^{q,\varphi^{q/p}}}.$$

Proof. Suppose that $\operatorname{supp}(b) \subset Q$. Then we have

$$\begin{aligned}\left|\int g(x)b(x)\,dx\right| &\leq \left(\int_Q |g(x)|^q\,dx\right)^{1/q}\left(\int_Q |b(x)|^{q'}\,dx\right)^{1/q'}\\ &= \left(\frac{1}{\varphi(|Q|)^{q/p}}\int_Q |g(x)|^q\,dx\right)^{1/q}\varphi(|Q|)^{\frac{1}{q}\cdot\frac{q}{p}}\|b\|_{L^{q'}} \leq \|g\|_{L^{q,\varphi^{q/p}}}.\end{aligned}$$

□

Lemma 3 (cf. Paluszyński [5])**.** *For $0 < \beta < 1$ and $1 \leq q \leq \infty$, we have*

$$\|f\|_{\dot{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}}\int_Q |f - f_Q| \approx \sup_Q \frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q |f - f_Q|^q\right)^{1/q},$$

for $q = \infty$ the formula should be interpreted appropriately.

5. Proof of theorems

Proof of Theorem 1. (a) ⇒ (b): T. Mizuhara have proved in [3].

(b) ⇒ (a): We use the same argument as Janson [1]. Choose $0 \neq z_0 \in \mathbb{R}^n$ such that $0 \notin Q(z_0,2)$. Then for $x \in Q(z_0,2)$, $|x|^{n-\alpha} \in C^\infty(Q(z_0,2))$. Hence, considering a cut function on the cube $Q(z_0,2+\delta)$ for sufficiently small $\delta > 0$, $|x|^{n-\alpha}$ can be written as the absolutely convergent Fourier series;

$$|x|^{n-\alpha} = \sum_{m\in\mathbb{Z}^n} a_m e^{i\langle v_m,x\rangle}$$

with $\sum_m |a_m| < \infty$, where the exact form of the vectors v_m is unrelated.

For any $x_0 \in \mathbb{R}^n$ and $r > 0$, let $Q = Q(x_0,r)$ and $Q^{z_0} = Q(x_0 + z_0 r, r)$. Let $s(x) = \overline{\operatorname{sgn}(\int_{Q^{z_0}}(b(x) - b(y))\,dy)}$. If $x \in Q$ and $y \in Q^{z_0}$, then $(y-x)/r \in Q(z_0,2)$. Hence we get

$$\begin{aligned}&\frac{1}{|Q|}\int_Q |b(x) - b_{Q^{z_0}}|\,dx\\ &= \frac{1}{|Q|}\frac{1}{|Q^{z_0}|}\int_Q \left|\int_{Q^{z_0}}(b(x) - b(y))\,dy\right|dx\\ &= \frac{1}{r^{2n}}\int_Q s(x)\left(\int_{Q^{z_0}}(b(x) - b(y))\,|x-y|^{\alpha-n}|x-y|^{n-\alpha}\,dy\right)dx\\ &= \frac{r^{n-\alpha}}{r^{2n}}\int_Q s(x)\left(\int_{Q^{z_0}}(b(x) - b(y))\,|x-y|^{\alpha-n}\left|\frac{x-y}{r}\right|^{n-\alpha}\,dy\right)dx\end{aligned}$$

$$
\begin{aligned}
&= r^{-n-\alpha} \sum_{m\in\mathbb{Z}^n} a_m \int_Q s(x) \\
&\quad \times \left(\int_{Q^{z_0}} (b(x)-b(y))\, |x-y|^{\alpha-n}\, e^{i\langle v_m, y/r\rangle}\, dy \right) e^{-i\langle v_m, x/r\rangle}\, dx \\
&\le r^{-n-\alpha} \left| \sum_{m\in\mathbb{Z}^n} a_m \int_{\mathbb{R}^n} s(x)[b,I_\alpha](\chi_{Q^{z_0}} e^{i\langle v_m, \cdot/r\rangle})(x)\chi_Q(x) e^{-i\langle v_m, x/r\rangle}\, dx \right| \\
&\le r^{-n-\alpha} \sum_{m\in\mathbb{Z}^n} |a_m| \left\| [b,I_\alpha](\chi_{Q^{z_0}} e^{i\langle v_m, \cdot/r\rangle}) \right\|_{L^{q,\varphi^{q/p}}} \cdot \left\| \chi_Q \right\|_{h^{\varphi^{q/p}, q'}} \\
&\le r^{-n-\alpha} \sum_{m\in\mathbb{Z}^n} |a_m|\, \|[b,I_\alpha]\|_{L^{p,\varphi}\to L^{q,\varphi^{q/p}}} \cdot \left\| \chi_{Q^{z_0}} \right\|_{L^{p,\varphi}} \cdot \left\| \chi_Q \right\|_{h^{\varphi^{q/p}, q'}} \\
&\le r^{-n-\alpha} \sum_{m\in\mathbb{Z}^n} |a_m|\, \|[b,I_\alpha]\|_{L^{p,\varphi}\to L^{q,\varphi^{q/p}}} \\
&\quad \times \varphi(|Q(x_0+z_0 r, r)|)^{-1/p} |Q(x_0+z_0 r, r)|^{1/p} \varphi(|Q(x_0,r)|)^{1/p} |Q(x_0,r)|^{1/q'} \\
&= C\|[b,I_\alpha]\|_{L^{p,\varphi}\to L^{q,\varphi^{q/p}}}.
\end{aligned}
$$

The second inequality follows from Lemma 2, the fourth inequality follows from Lemma 1. Therefore we get

$$
\begin{aligned}
\frac{1}{|Q|}\int_Q |b(x)-b_Q|\, dx &\le \frac{2}{|Q|}\int_Q |b(x)-b_{Q^{z_0}}|\, dx \\
&\le 2C\|[b,I_\alpha]\|_{L^{p,\varphi}\to L^{q,\varphi^{q/p}}}.
\end{aligned}
$$

This implies that $b \in BMO(\mathbb{R}^n)$ and $\|b\|_* \le C\|[b,I_\alpha]\|_{L^{p,\varphi}\to L^{q,\varphi^{q/p}}}$, and the proof of the theorem is completed. □

The part of (a) ⇒ (b) in Theorem 2 was proved by T. Mizuhara [3]. By using Lemma 3, the proof of (b) ⇒ (a) in Theorem 2 is the same argument as the proof of Theorem 1.

6. Generalization to higher order commutator

In this section we will consider a higher order commutator operator defined by

$$
[b,I_\alpha]^k f(x) := \int_{\mathbb{R}^n} \frac{\Delta_h^k b(x) f(h)}{|h|^{n-\alpha}}\, dh,
$$

where

$$
\begin{aligned}
\Delta_h^1 b(x) &= \Delta_h b(x) = b(x+h) - b(x), \\
\Delta_h^{k+1} b(x) &= \Delta_h^k b(x+h) - \Delta_h^k b(x), \quad k \ge 1.
\end{aligned}
$$

Let $0<\beta<k\le n$, k an integer and n be the dimension of the whole space. We now try to define the Lipschitz space $\dot{\Lambda}_\beta(\mathbb{R}^n)$ again. For $\beta>0$, we say $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ if

$$
\|b\|_{\dot{\Lambda}_\beta} = \sup_{\substack{x,h\in\mathbb{R}^n\\ h\ne 0}} \frac{|\Delta_h^k b(x)|}{|h|^\beta} < \infty, \quad k\ge 1.
$$

Theorem 3. *Suppose the same condition as Theorem 2. The following conditions are equivalent*:
(a) $b = b_1 + P$, *where* $b_1 \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ *and* P *is a polynomial of degree less than* k.
(b) $[b,I_\alpha]^k$ *is bounded from* $L^{p,\varphi}(\mathbb{R}^n)$ *to* $L^{q,\varphi^{q/p}}(\mathbb{R}^n)$.

If $k = [\beta] + 1$, then (a) of theorem says that $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

The proof of (a) $\Rightarrow$ (b) will be omitted since we can prove the same argument as Theorem 2. The part of (b) $\Rightarrow$ (a) is based on the following results for the Besov spaces.

Remark 4. It is difficult to prove the part of (b) $\Rightarrow$ (a) by a factorization theorem due to Y. Komori and T. Mizuhara [2].

Lemma 4 (Paluszyński and Taibleson [6]). *Let $0 < \beta < k$, with k an integer. Suppose $f \in \mathcal{S}' \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The following conditions are equivalent*:
(a) *$f = f_1 + P$, where $f_1 \in \dot{B}^{\beta,\infty}_\infty(\mathbb{R}^n)(= \dot{\Lambda}_\beta(\mathbb{R}^n))$ and P is a polynomial of degree less than k.*
(b) *There exists $z_0 \in \mathbb{R}^n$ such that*

$$\sup_{r>0} r^{-\beta} \sup_{x_0\in\mathbb{R}^n} \frac{1}{|Q|} \frac{1}{|Q^{z_0}|} \left(\int_Q \left| \int_{Q^{z_0}} (\Delta^k_{(y-x)/k} f(x))\, dy \right| dx \right) \le C < \infty,$$

where $Q = Q(x_0, r)$, and $Q^{z_0} = Q(x_0 + z_0 r, r)$.

If these conditions hold then $\|f\|_{\dot{B}^{\beta,\infty}_\infty}$ is comparable with the best possible C in (b).

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References

1. S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Math. **16** (1978), 263–270.
2. Y. Komori and T. Mizuhara, *Factorization of functions in $H^1(\mathbb{R}^n)$ and generalized Morrey spaces*, to appear in Math. Nachr.
3. T. Mizuhara, *Commutators of singular integral operators on Morrey spaces with general growth functions*, Harmonic analysis and nonlinear partial differential equations (Kyoto, 1998). Sūrikaisekikenkyūsho Kōkyūroku, **1102** (1999), 49–63.
4. E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166** (1994), 95–103.
5. M. Paluszyński, *Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss*, Indiana Univ. Math. J. **44** (1995), 1–17.
6. M. Paluszyński and M. Taibleson, *Characterization of the Besov spaces*, Fourier analysis and partial differential equations(Miraflores de la Sierra, 1992), 251–269, Stud. Adv. Math., CRC, Boca Raton, FL, 1995.
7. S. Shirai, *Necessary and sufficient conditions for boundedness of commutators of fractional integral operators on classical Morrey spaces*, to appear in Hokkaido Math. J.
8. C. Zorko, *Morrey space*, Proc. Amer. Math. Soc. **98** (1986), no. 4, 586–592.

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