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STRONG CONVERGENCE THEOREM BY THE HYBRID AND EXTRAGRADIENT METHODS FOR MONOTONE MAPPINGS AND COUNTABLE FAMILIES OF NONEXPANSIVE MAPPINGS

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Abstract. In this paper we introduce an iterative process for finding a common element of the set of common fixed points of a countable family of nonexpansive mappings and the set of solutions of the variational inequality problem for a monotone, Lipschitz continuous mapping. The iterative process is based on two known methods - hybrid and extragradient. We obtain a strong convergence theorem for three sequences generated by this process. Based on this theorem, we construct an iterative process for solving the generalized lexicographic variational inequality problem.

1 Introduction Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C. A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0$$

for all $u, v \in C$. The *variational inequality problem* is to find a $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0$$

for all $v \in C$. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$. A mapping A of C into H is called *α-inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha \|Au - Av\|^2$$

for all $u, v \in C$; see [1], [6]. It is obvious that any α-inverse-strongly-monotone mapping A is monotone and Lipschitz-continuous. A mapping T of C into itself is called *nonexpansive* if

$$\|Tu - Tv\| \leq \|u - v\|$$

for all $u, v \in C$; see [15]. We denote by $F(T)$ the set of fixed points of T. For finding an element of $VI(C, A)$ under the assumption that a set $C \subset H$ is closed and convex and a mapping A of C into H is α-inverse-strongly-monotone, Iiduka, Takahashi and Toyoda [3] introduced the following iterative scheme by the hybrid method:

$$\begin{cases} x_1 = x \in C \\ y_n = P_C(x_n - \lambda_n A x_n) \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\} \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

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for every $n = 1, 2, ...,$ where $\lambda_n \subset [a, b]$ for some $a, b \in (0, 2\alpha)$. They showed that if $VI(C, A)$ is nonempty, then the sequence $\{x_n\}$, generated by this iterative process, converges strongly to $P_{VI(C,A)}x$. On the other hand, for solving the variational inequality problem in the finite-dimensional Euclidean space $\mathbb{R}^n$ under the assumption that a set $C \subset \mathbb{R}^n$ is closed and convex and a mapping A of C into $\mathbb{R}^n$ is monotone and k-Lipschitz-continuous, Korpelevich [5] introduced the following so-called extragradient method:

$$\begin{cases} x_1 = x \in C \\ \overline{x}_n = P_C(x_n - \lambda A x_n) \\ x_{n+1} = P_C(x_n - \lambda A \overline{x}_n) \end{cases} \tag{1.1}$$

for every $n = 1, 2, ...,$ where $\lambda \in (0, 1/k)$. He showed that if $VI(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{\overline{x}_n\}$, generated by (1.1), converge to the same point $z \in VI(C, A)$.

Let $T_1, T_2, ...$ be a countable family of mappings of C into itself and let $\alpha_1, \alpha_2, ...$ be real numbers such that $0 \leq \alpha_i \leq 1$ for all $n = 1, 2, ...$ For any $n \in$ N, Takahashi [13] defined the mapping W_n of C into itself as follows:

$$\begin{aligned} U_{n,n+1} &= I, \\ U_{n,n} &= \alpha_n T_n U_{n,n+1} + (1 - \alpha_n) I, \\ U_{n,n-1} &= \alpha_{n-1} T_{n-1} U_{n,n} + (1 - \alpha_{n-1}) I, \\ &\vdots \\ U_{n,k} &= \alpha_k T_k U_{n,k+1} + (1 - \alpha_k) I, \\ U_{n,k-1} &= \alpha_{k-1} T_{k-1} U_{n,k} + (1 - \alpha_{k-1}) I, \\ &\vdots \\ U_{n,2} &= \alpha_2 T_2 U_{n,3} + (1 - \alpha_2) I, \\ W_n = U_{n,1} &= \alpha_1 T_1 U_{n,2} + (1 - \alpha_1) I. \end{aligned}$$

Such mappings W_n are called W–mappings generated by $T_n, T_{n-1}, ..., T_1$ and $\alpha_n, \alpha_{n-1}, ...,$ α_1. Shimoji and Takahashi [12] also defined mappings $U_{\infty,k}$ and U of C into itself as follows:

$$U_{\infty,k}x = \lim_{n\to\infty} U_{n,k}x$$

$$Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$$

for every $x \in C$. Such a U is called the W-mapping generated by $T_1, T_2, ...$ and $\alpha_1, \alpha_2, ...$; see [12] for more details.

This paper is motivated by the idea of combining hybrid and extragradient methods. We introduce an iterative process for finding a common element of the set of common fixed points of a countable family of nonexpansive mappings and the set of solutions of the variational inequality problem for a monotone, Lipschitz continuous mapping in a real Hilbert space. Then we obtain a strong convergence theorem for three sequences generated by this process. We also consider three applications of this theorem. As a corrolary of our theorem we get the theorem proved by Kikkawa and Takahashi for W-mappings [4] . We also construct iterative process for solving the generalized lexicographic variational inequality problem. Furthermore, we obtain a strong convergence theorem for a pseudocontractive mapping and a countable family of nonexpansive mappings in a Hilbert space.

2 Preliminaries Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that $\{x_n\}$ converges strongly to x. For every point $x \in H$ there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. P_C is called *the metric projection* of H onto C. We know that P_C is a nonexpansive mapping of H onto C. It is also known that P_C is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, P_C x - y \rangle \geq 0. \tag{2.1}$$

Further, we know that

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.2}$$

for all $x \in H$ and $y \in C$; see [15] for more details. Let A be a monotone mapping of C into H. In the context of variational inequality problem this implies

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda A u), \quad \forall \lambda > 0.$$

It is also known that H satisfies Opial's condition [9], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

A set-valued mapping $T : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is *maximal* if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone, $k-$Lipschitz-continuous mapping of C into H and $N_C v$ be the normal cone to C at $v \in C$, i.e. $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [11].

3 Strong Convergence Theorem In this section we prove a strong convergence theorem for a countable family of nonexpansive mappings and a monotone, Lipschitz continuous mapping. To prove it, we need two lemmas which were proved by Shimoji and Takahashi [12] in a strictly convex Banach space.

Lemma 3.1. *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} F(T_i)$ is nonempty, and let b and $\alpha_1, \alpha_2, ...$ be real numbers such that $0 < \alpha_i \leq b < 1$ for any $i \in \mathrm{N}$. Then, for every $x \in C$ and $k \in \mathrm{N}$, $U_{\infty,k} x = \lim_{n\to\infty} U_{n,k} x$ exists.*

Lemma 3.2. *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} F(T_i)$ is nonempty, and let b and $\alpha_1, \alpha_2, ...$ be real numbers such that $0 < \alpha_i \leq b < 1$ for any $i \in \mathrm{N}$. Then $F(U) = \cap_{i=1}^{\infty} F(T_i)$.*

We are now ready to prove our main strong convergence theorem.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz-continuous mapping of C into H and $T_1, T_2, ...$ be a countable family of nonexpansive mappings of C into itself such that $\left(\cap_{i=1}^{\infty} F(T_i)\right) \cap VI(C,A) \neq \emptyset$. Let c, d and $\alpha_1, \alpha_2, ...$ be real numbers such that $0 < c \leq \alpha_i \leq d < 1$ for every $i \in \mathrm{N}$. Let W_n, $n = 1, 2, ...$ be the W-mappings of C into itself generated by $T_n, T_{n-1}, ..., T_1$ and $\alpha_n, \alpha_{n-1}, ..., \alpha_1$ and let U be the W-mapping of C into itself generated by $T_1, T_2, ...$ and $\alpha_1, \alpha_2, ...$, i.e. $Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x$ for every $x \in C$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in C \\ y_n = P_C(x_n - \lambda_n A x_n) \\ z_n = W_n P_C(x_n - \lambda_n A y_n) \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\} \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 1, 2, ...$, where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $P_{F(U)\cap VI(C,A)}x$.

Proof. It is obvious that C_n is closed and Q_n is closed and convex for every $n = 1, 2, ...$ As $C_n = \left\{z \in C : \|z_n - x_n\|^2 + 2\langle z_n - x_n, x_n - z\rangle \leq 0\right\}$, we also have C_n is convex for every $n = 1, 2, ...$ Put $t_n = P_C(x_n - \lambda_n A y_n)$ for every $n = 1, 2, ...$ Let $u \in F(U) \cap VI(C,A)$. From (2.2), monotonicity of A and $u \in VI(C,A)$, we have

$$\begin{aligned} \|t_n - u\|^2 &\leq \|x_n - \lambda_n A y_n - u\|^2 - \|x_n - \lambda_n A y_n - t_n\|^2 \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, u - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 \\ &\quad + 2\lambda_n \left(\langle A y_n - Au, u - y_n\rangle + \langle Au, u - y_n\rangle + \langle A y_n, y_n - t_n\rangle\right) \\ &\leq \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n \langle A y_n, y_n - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 \\ &\qquad + 2\lambda_n \langle A y_n, y_n - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 \\ &\qquad + 2\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle. \end{aligned}$$

Further, since $y_n = P_C(x_n - \lambda_n A x_n)$ and A is k-Lipschitz-continuous, we have

$$\begin{aligned} &\langle x_n - \lambda_n A y_n - y_n, t_n - y_n\rangle \\ &\quad = \langle x_n - \lambda_n A x_n - y_n, t_n - y_n\rangle + \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\ &\quad \leq \langle \lambda_n A x_n - \lambda_n A y_n, t_n - y_n\rangle \\ &\quad \leq \lambda_n k \|x_n - y_n\| \|t_n - y_n\|. \end{aligned}$$

So, we have

$$
\begin{aligned}
&\|t_n-u\|^2\\
&\le \|x_n-u\|^2-\|x_n-y_n\|^2-\|y_n-t_n\|^2+2\lambda_n k\,\|x_n-y_n\|\,\|t_n-y_n\|\\
&\le \|x_n-u\|^2-\|x_n-y_n\|^2-\|y_n-t_n\|^2+\lambda_n^2k^2\,\|x_n-y_n\|^2+\|y_n-t_n\|^2\\
&\le \|x_n-u\|^2+\left(\lambda_n^2k^2-1\right)\|x_n-y_n\|^2 \qquad (3.1)\\
&\le \|x_n-u\|^2.
\end{aligned}
$$

Therefore from $z_n=W_nt_n$ and $u=Su$, we have

$$\|z_n-u\|=\|W_nt_n-W_nu\|\le\|t_n-u\|\le\|x_n-u\| \qquad (3.2)$$

for every $n=1,2,...$ and hence $u\in C_n$. So, $F(U)\cap VI(C,A)\subset C_n$ for every $n=1,2,....$ Next, let us show by mathematical induction that $\{x_n\}$ is well-defined and $F(U)\cap VI(C,A)\subset C_n\cap Q_n$ for every $n=1,2,....$ For $n=1$ we have $Q_1=C$. Hence we obtain $F(U)\cap VI(C,A)\subset C_1\cap Q_1$. Suppose that x_k is given and $F(U)\cap VI(C,A)\subset C_k\cap Q_k$ for some $k\in N$. Since $F(U)\cap VI(C,A)$ is nonempty, $C_k\cap Q_k$ is a nonempty closed convex subset of C. So, there exists a unique element $x_{k+1}\in C_k\cap Q_k$ such that $x_{k+1}=P_{C_k\cap Q_k}x$. It is also obvious that $\langle x_{k+1}-z,x-x_{k+1}\rangle\ge 0$ for every $z\in C_k\cap Q_k$. Since $F(U)\cap VI(C,A)\subset C_k\cap Q_k$, we have $\langle x_{k+1}-z,x-x_{k+1}\rangle\ge 0$ for $z\in F(U)\cap VI(C,A)$ and hence $F(U)\cap VI(C,A)\subset Q_{k+1}$. Therefore, we obtain $F(U)\cap VI(C,A)\subset C_{k+1}\cap Q_{k+1}$.

Let $t_0=P_{F(U)\cap VI(C,A)}x$. From $x_{n+1}=P_{C_n\cap Q_n}x$ and $t_0\in F(U)\cap VI(C,A)\subset C_n\cap Q_n$, we have

$$\|x_{n+1}-x\|\le\|t_0-x\| \qquad (3.3)$$

for every $n=1,2,....$ Therefore, $\{x_n\}$ is bounded. We also have

$$\|z_n-u\|=\|W_nt_n-W_nu\|\le\|t_n-u\|\le\|x_n-u\|$$

for $u\in F(U)\cap VI(C,A)$. So, $\{z_n\}$ and $\{t_n\}$ are bounded. Since $x_{n+1}\in C_n\cap Q_n\subset Q_n$ and $x_n=P_{Q_n}x$, we have

$$\|x_n-x\|\le\|x_{n+1}-x\|$$

for every $n=1,2,....$ Therefore, there exists $c=\lim_{n\to\infty}\|x_n-x\|$. Using $x_n=P_{Q_n}x$ and $x_{n+1}\in Q_n$ again, we have also

$$
\begin{aligned}
\|x_{n+1}-x_n\|^2&=\|x_{n+1}-x\|^2+\|x_n-x\|^2+2\langle x_{n+1}-x,x-x_n\rangle\\
&=\|x_{n+1}-x\|^2-\|x_n-x\|^2-2\langle x_n-x_{n+1},x-x_n\rangle\\
&\le\|x_{n+1}-x\|^2-\|x_n-x\|^2
\end{aligned}
$$

for every $n=1,2,....$ This implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

Since $x_{n+1}\in C_n$, we have $\|z_n-x_{n+1}\|\le\|x_n-x_{n+1}\|$ and hence

$$\|x_n-z_n\|\le\|x_n-x_{n+1}\|+\|x_{n+1}-z_n\|\le 2\|x_{n+1}-x_n\|$$

for every $n = 1, 2, \ldots$. From $\|x_{n+1} - x_n\| \to 0$, we have $\|x_n - z_n\| \to 0$.

For $u \in F(U) \cap VI(C, A)$, from (3.1) and (3.2) we obtain

$$\|z_n - u\|^2 \le \|t_n - u\|^2 \le \|x_n - u\|^2 + \left(\lambda_n^2 k^2 - 1\right) \|x_n - y_n\|^2 .$$

Therefore, we have

$$\begin{aligned} \|x_n - y_n\|^2 &\le \frac{1}{1 - \lambda_n^2 k^2} \left(\|x_n - u\|^2 - \|z_n - u\|^2\right) \\ &= \frac{1}{1 - \lambda_n^2 k^2} \left(\|x_n - u\| - \|z_n - u\|\right)\left(\|x_n - u\| + \|z_n - u\|\right) \\ &\le \frac{1}{1 - \lambda_n^2 k^2} \left(\|x_n - u\| + \|z_n - u\|\right) \|x_n - z_n\| . \end{aligned}$$

Since $\|x_n - z_n\| \to 0$, we obtain $x_n - y_n \to 0$. From (3.1) and (3.2) we also have

$$\begin{aligned} \|z_n - u\|^2 &\le \|t_n - u\|^2 \\ &\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k \|x_n - y_n\| \|t_n - y_n\| \\ &\le \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \|x_n - y_n\|^2 \\ &\qquad + \lambda_n^2 k^2 \|y_n - t_n\|^2 \\ &\le \|x_n - u\|^2 + \left(\lambda_n^2 k^2 - 1\right) \|y_n - t_n\|^2 . \end{aligned}$$

Therefore we have

$$\begin{aligned} \|t_n - y_n\|^2 &\le \frac{1}{1 - \lambda_n^2 k^2} \left(\|x_n - u\|^2 - \|z_n - u\|^2\right) \\ &= \frac{1}{1 - \lambda_n^2 k^2} \left(\|x_n - u\| - \|z_n - u\|\right)\left(\|x_n - u\| + \|z_n - u\|\right) \\ &\le \frac{1}{1 - \lambda_n^2 k^2} \left(\|x_n - u\| + \|z_n - u\|\right) \|x_n - z_n\| . \end{aligned}$$

Since $\|x_n - z_n\| \to 0$, we obtain $t_n - y_n \to 0$. Since A is k-Lipschitz-continuous, we have $Ay_n - At_n \to 0$.

Using the Eberlein–Smulian theorem on weak compactness (see, e.g., [2], p. 430), as $\{x_n\}$ is bounded, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to some u. We can obtain that $u \in F(S) \cap VI(C, A)$. First, we show $u \in VI(C, A)$. Since $x_n - t_n \to 0$ and $x_n - y_n \to 0$, we have $\{t_{n_i}\} \rightharpoonup u$ and $\{y_{n_i}\} \rightharpoonup u$. Let

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [11]. Let $(v, w) \in G(T)$. Then, we have $w \in Tv = Av + N_C v$ and hence $w - Av \in N_C v$. So, we have $\langle v - t, w - Av \rangle \ge 0$ for all $t \in C$. On the other hand, from $t_n = P_C(x_n - \lambda_n A y_n)$ and $v \in C$ we have

$$\langle x_n - \lambda_n A y_n - t_n, t_n - v \rangle \ge 0$$

and hence

$$\left\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + A y_n \right\rangle \ge 0.$$

Therefore from $w - Av \in N_C v$ and $t_{n_i} \in C$, we have

$$\begin{aligned}\langle v - t_{n_i}, w\rangle &\geq \langle v - t_{n_i}, Av\rangle \\ &\geq \langle v - t_{n_i}, Av\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ay_{n_i}\right\rangle \\ &= \langle v - t_{n_i}, Av - At_{n_i}\rangle + \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle \\ &\qquad - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle \\ &\geq \langle v - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle.\end{aligned}$$

Hence, we obtain $\langle v - u, w\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $u \in T^{-1}0$ and hence $u \in VI(C, A)$.

Let us show $u \in F(U)$. Assume $u \notin F(U)$. From Opial's condition, we have

$$\begin{aligned}\liminf_{i\to\infty} \|t_{n_i} - u\| &< \liminf_{i\to\infty} \|t_{n_i} - Uu\| \\ &\leq \liminf_{i\to\infty} \left(\|t_{n_i} - W_{n_i} t_{n_i}\| + \|W_{n_i} t_{n_i} - W_{n_i} u\| + \|W_{n_i} u - Uu\|\right) \\ &\leq \liminf_{i\to\infty} \left(\|t_{n_i} - z_{n_i}\| + \|t_{n_i} - u\| + \|W_{n_i} u - Uu\|\right) \\ &\leq \liminf_{i\to\infty} \|t_{n_i} - u\|.\end{aligned}$$

This is a contradiction. So, we obtain $u \in F(U)$. This implies $u \in F(U) \cap VI(C, A)$.

From $t_0 = P_{F(U)\cap VI(C,A)}x$, $u \in F(U) \cap VI(C, A)$ and (3.3), we have

$$\|t_0 - x\| \leq \|u - x\| \leq \liminf_{i\to\infty} \|x_{n_i} - x\| \leq \limsup_{i\to\infty} \|x_{n_i} - x\| \leq \|t_0 - x\|.$$

So, we obtain

$$\lim_{i\to\infty} \|x_{n_i} - x\| = \|u - x\|.$$

From $x_{n_i} - x \rightharpoonup u - x$ we have $x_{n_i} - x \to u - x$ and hence $x_{n_i} \to u$. Since $x_n = P_{Q_n}x$ and $t_0 \in F(U) \cap VI(C, A) \subset C_n \cap Q_n \subset Q_n$, we have

$$-\|t_0 - x_{n_i}\|^2 = \langle t_0 - x_{n_i}, x_{n_i} - x\rangle + \langle t_0 - x_{n_i}, x - t_0\rangle \geq \langle t_0 - x_{n_i}, x - t_0\rangle.$$

As $i \to \infty$, we obtain $-\|t_0 - u\|^2 \geq \langle t_0 - u, x - t_0\rangle \geq 0$ by $t_0 = P_{F(U)\cap VI(C,A)}x$ and $u \in F(U) \cap VI(C, A)$. Hence we have $u = t_0$. This implies that $x_n \to t_0$. It is easy to see $y_n \to t_0$, $z_n \to t_0$. □

4 Applications. In this section, we shall apply Theorem 3.1 to construct iterative sequences which converge strongly to a common fixed point for various countable families of mappings. The following result was obtained by Kikkawa and Takahashi [4].

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2, \ldots$ be a countable family of nonexpansive mappings of C into itself such that $\cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let a, b and $\alpha_1, \alpha_2, \ldots$ be real numbers such that $0 < a \leq \alpha_i \leq b < 1$ for every $i \in$ N. Let W_n, $n = 1, 2, \ldots$ be the W-mappings of C into itself generated by $T_n, T_{n-1}, \ldots, T_1$ and $\alpha_n, \alpha_{n-1}, \ldots, \alpha_1$ and let U be the W-mapping of C into itself generated* *by $T_1, T_2, ...$ and $\alpha_1, \alpha_2, ...$, i.e. $Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x$ for every $x \in C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in C \\ z_n = W_n x_n \\ C_n = \{z \in C : \|z_n - z\| \le \|x_n - z\|\} \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \ge 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 1, 2, ...$. Then $F(U) = \cap_{i=1}^{\infty} F(T_i)$ and the sequence $\{x_n\}$ converges strongly to $P_{F(U)}x$.

Proof. Putting $A = 0$, by Theorem 3.1, we obtain the desired result. □

Lexicografic variational inequality problem in the finite-dimensional Euclidean space R^n is formulated as follows (see, e.g., [10]). Let C be a closed convex subset of R^n. Let $A_0, A_1, ..., A_m$ be finite mappings from C into R^n. We are to obtain an element of the set C_m, where the sets C_i, $i = 1, 2, ..., m$ are given by

$$C_0 = C, \quad C_i = VI(C_{i-1}, A_{i-1}).$$

The set of solutions of the lexicografic variational inequality problem is denoted by $LVI(C, A_0, A_1, ..., A_m) = C_m = \cap_{i=0}^{m} C_i$.

Motivated by this problem, we formulate the *generalized lexicographic variational inequality problem* in a real Hilbert space. Let C be a closed convex subset of a real Hilbert space H. Let $A_0, A_1, A_2, ...$ be a countable family of mappings from C into H. We are to obtain some element $x \in C$ such that $x \in C_i$ for all $i \in \mathrm{N}$, where the sets C_i, $i = 1, 2, ...$ are given by

$$C_0 = C, \quad C_i = VI(C_{i-1}, A_{i-1}).$$

We denote the set of solutions of the generalized lexicographic variational inequality problem by $GLVI(C, A_0, A_1, A_2, ...) = \cap_{i=0}^{\infty} C_i$.

For solving the lexicographic variational inequality problem for monotone and continuous mappings in the finite-dimensional space R^n we require some additional restrictions of regularity or compactness type. Let us consider an iterative process for solving the generalized variational inequality problem for monotone, Lipschitz continuous and inverse-strongly monotone mappings in a real Hilbert space without any additional restrictions. To prove the strong convergence of this iterative process, we need the following lemma. This lemma was proved by Matsushita and Kuroiwa ([7], Proposition 2.2).

Lemma 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let T be a nonexpansive mapping of C into H. If $F(T) \neq \emptyset$, then $F(P_C T) = F(T)$.*

Now we state a strong convergence theorem.

Theorem 4.2. *Let C be a closed convex subset of a real Hilbert space H. Let A_0 be a monotone and k-Lipschitz-continuous mapping of C into H and $A_1, A_2, ...$ be a countable family of mappings of C into H such that every mapping A_i is γ_i-inverse-strongly-monotone, $i = 1, 2, ...$. Suppose that the set of solutions of the generalized lexicographical variational inequality problem $GLVI(C, A_0, A_1, ...)$ is not empty. Denote by W_n, $n = 1, 2, ...$ the W-mappings of C into itself generated by $T_n, T_{n-1}, ..., T_1$ and $\alpha_n, \alpha_{n-1}, ..., \alpha_1$, where $T_i =$*

$P_C(I-\mu_i A_i)$, where $0<\mu_i\le 2\gamma_i$; c,d and $\alpha_1,\alpha_2,\ldots$ are real numbers such that $0<c\le \alpha_i\le d<1$ for every $i\in \mathrm{N}$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by

$$\begin{cases} x_1=x\in C \\ y_n=P_C(x_n-\lambda_n A_0 x_n) \\ z_n=W_nP_C(x_n-\lambda_n A_0 y_n) \\ D_n=\{z\in C:\|z_n-z\|\le\|x_n-z\|\} \\ Q_n=\{z\in C:\langle x_n-z,x-x_n\rangle\ge 0\} \\ x_{n+1}=P_{D_n\cap Q_n}x \end{cases}$$

for every $n=1,2,\ldots$, where $\{\lambda_n\}\subset[a,b]$ for some $a,b\in(0,1/k)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to some $x\in GLVI(C,A_0,A_1,\ldots)$.

Proof. If a mapping A_i from C into H is γ_i-inverse-strongly-monotone, then for all $u,v\in C$ and $\mu_i>0$, $n=1,2,\ldots$, we have

$$\begin{aligned} &\|P_C(u-\mu_iA_iu)-P_C(v-\mu_iA_iv)\|^2 \\ &\quad\le\|(I-\mu_iA_i)u-(I-\mu_iA_i)v\|^2=\|(u-v)-\mu_i(A_iu-A_iv)\|^2 \\ &\quad=\|u-v\|^2-2\mu_i\langle u-v,A_iu-A_iv\rangle+\mu_i^2\|A_iu-A_iv\|^2 \\ &\quad\le\|u-v\|^2+\mu_i(\mu_i-2\gamma_i)\|A_iu-A_iv\|^2 . \end{aligned}$$

So, if $\mu_i\le 2\gamma_i$, then $T_i=P_C(I-\mu_iA_i)$, $n=1,2,\ldots$ are nonexpansive mappings from C into itself. It is obvious that for $T_i=P_C(I-\mu_iA_i)$ we have $F(T_i)=VI(C,A_i)$. It is also obvious that the mappings $S_i=I-\mu_iA_i$, $n=1,2,\ldots$, are nonexpansive mappings from C into H. Then from Lemma 4.1 for any closed convex subset C_i of C we have $F(T_i)\cap C_i=VI(C_i,A_i)$. From the definition of C_i in the generalized lexicographical variational inequality problem we have

$$\begin{aligned} C_n=VI(C_{n-1},A_{n-1})&=F(T_{n-1})\cap C_{n-1}=F(T_{n-1})\cap F(T_{n-2})\cap C_{n-2} \\ &=\ldots=\big(\cap_{i=1}^{n-1}F(T_i)\big)\cap VI(C,A_0). \end{aligned}$$

By Theorem 3.1, we obtain the desired result. □

A mapping $S:C\to C$ is called *pseudocontractive* if

$$\|Sx-Sy\|^2\le\|x-y\|^2+\|(I-S)x-(I-S)y\|^2$$

for all $x,y\in C$, or, equivalently,

$$\langle Sx-Sy,x-y\rangle\le\|x-y\|^2 \tag{4.1}$$

for all $x,y\in C$.

Theorem 4.3. *Let C be a closed convex subset of a real Hilbert space H. Let S be a pseudocontractive, m-Lipschitz-continuous mapping of C into itself and $T_1,T_2,\ldots$ be a countable family of nonexpansive mappings of C into itself such that $\big(\cap_{i=1}^{\infty}F(T_i)\big)\cap F(S)\neq\emptyset$. Let c,d and $\alpha_1,\alpha_2,\ldots$ be real numbers such that $0<c\le\alpha_i\le d<1$ for every $i\in\mathrm{N}$. Let W_n, $n=1,2,\ldots$ be the W-mappings of C into itself generated by $T_n,T_{n-1},\ldots,T_1$ and $\alpha_n,\alpha_{n-1},\ldots,\alpha_1$ and let U be the W-mapping of C into itself generated by $T_1,T_2,\ldots$ and*

$\alpha_1, \alpha_2, ...,$ *i.e.* $Ux = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x$ *for every* $x \in C$. *Let* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *be sequences generated by*

$$\begin{cases} x_1 = x \in C \\ y_n = x_n - \lambda_n (x_n - Sx_n) \\ z_n = W_n P_C (x_n - \lambda_n (y_n - Sy_n)) \\ C_n = \{z \in C : \|z_n - z\| \le \|x_n - z\|\} \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \ge 0\} \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for every $n = 1, 2, ...,$ *where* $\{\alpha_n\} \subset [a, b]$ *for some* a, b *with* $0 < a < b < \frac{1}{m+1}$.
Then $F(U) = \cap_{i=1}^{\infty} F(T_i)$ *and the sequences* $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *converge strongly to* $P_{F(U)\cap F(S)} x$.

Proof. Let $A = I - S$. Let us show the mapping A is monotone and $(m+1)$-Lipschitz-continuous. From the definition of the mapping A and (4.1), we have

$$\begin{aligned} \langle Ax - Ay, x - y \rangle &= \langle x - y - Sx + Sy, x - y \rangle \\ &= \|x - y\|^2 - \langle Sx - Sy, x - y \rangle \ge \|x - y\|^2 - \|x - y\|^2 = 0. \end{aligned}$$

So, A is monotone. We also have

$$\begin{aligned} \|Ax - Ay\|^2 &= \|(I - S)x - (I - S)y\|^2 \\ &= \|x - y\|^2 + \|Sx - Sy\|^2 - 2\langle x - y, Sx - Sy \rangle \\ &\le \|x - y\|^2 + m^2 \|x - y\|^2 + 2\|x - y\| \|Sx - Sy\| \\ &\le \|x - y\|^2 + m^2 \|x - y\|^2 + 2m \|x - y\|^2 = (m+1)^2 \|x - y\|^2. \end{aligned}$$

So, we have $\|Ax - Ay\| \le (m+1)\|x - y\|$ and A is $(m+1)$-Lipschitz-continuous. Now let us show $F(S) = VI(C, A)$. In fact, we have, for $\lambda > 0$,

$$\begin{aligned} u \in VI(C, A) &\Leftrightarrow \langle y - u, Au \rangle \ge 0 \quad \forall y \in C \\ &\Leftrightarrow \langle u - y, u - \lambda Au - u \rangle \ge 0 \quad \forall y \in C \\ &\Leftrightarrow u = P_C (u - \lambda Au) \\ &\Leftrightarrow u = P_C (u - \lambda u + \lambda Su) \\ &\Leftrightarrow \langle u - \lambda u + \lambda Su - u, u - y \rangle \ge 0 \quad \forall y \in C \\ &\Leftrightarrow \langle u - Su, u - y \rangle \le 0 \quad \forall y \in C \\ &\Leftrightarrow u = Su. \end{aligned}$$

By Theorem 3.1 we obtain the desired result. □

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