THE $(E.R.T.)$-INTEGRAL AND FOURIER TRANSFORM

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Received October 25, 2006; revised November 20, 2006

Abstract. We show the relation between the $(E.R.T)$-integral and the improper Riemann integral, and define a restricted Fourier's integral formula.

1 Introduction In our previous paper [11], we described the $(E.R.T)$-integral defined independently of points where the integrand have the infinity. It is defined on the space $\Gamma_0(I) \bigoplus M_0(I)$ of generalized functions over an interval I of $\mathbf{R}$. The set $\Gamma_0(I)$ is the singular part of $\Gamma_0(I) \bigoplus M_0(I)$ in the sense that it contains the δ-function and it's heigher derivatives, and the set $M_0(I)$ consists of all Lebesgue measurable functions on I, which is the regular part of $\Gamma_0(I) \bigoplus M_0(I)$.

In this paper, we do not consider $\Gamma_0(I)$. That is, we consider only the $(E.R.T)$-integral for generalized function in $0 \bigoplus M_0(I)$ (namely, $M_0(I)$).

In Section 2, we recall some terminologies and notations containing the definition of the $(E.R.T)$-integral in $M_0(I)$.

In Section 3, we will show the relations between the $(E.R.T)$-integral and the improper Riemann integral as well as the Lebesgue integral.

From these relations, it is easy to see that the main fundamental theorems for Fourier integrals hold also for the $(E.R.T)$-integrals. In Section 4, we see that the theorem of Riemann-Lebesgue, Fourier's integral formula, and Fourier's single-integral formula hold also for the $(E.R.T)$-integrals.

In Section 5, we define a restricted Fourier's integral formula for some functions which are neither Lebesgue integrable nor of bounded variation on $\mathbf{R}$.

2 Terminologies and notations Let I be a finite or infinite open interval in $\mathbf{R}$, and $M_0(I)$ the set of all real valued Lebesgue measurable functions defined on I.

We recall some terminologies and notations used in the definition of the $(E.R.T)$-integral in $M_0(I)$ ([11]).

In what follows, we suppose that the set $M_0(I)$ is classified by the usual equivalence relation $f(x) = g(x)$ *a.e.* We denote a class in $M_0(I)$ and it's representative by the same symbol $f(x)$ or f, and call also the class a function. For each Lebesgue measurable subset A of I and $\epsilon > 0$, we difine a pre-neighbourhood $V(f, \epsilon, A)$ as

$$V(f, \epsilon, A) = \{g \in M_0(I); \int_A |f(x) - g(x)|dx \leq \epsilon\}.$$

We denote $V(f, \epsilon, A)$ or $V(f)$ for short.

A sequence $(V(f_n)) = (V(f_n, \epsilon_n, A_n))$ is called a Cauchy sequence if (i) $V(f_1) \supseteq V(f_2) \supseteq \cdots$, and (ii) $\epsilon_n \to 0$.

Let $\Lambda = (\lambda_n)$ be a sequence of finite measures on $\mathbf{R}$ such that (1) any Lebesgue measurable set is λ_n-measurable and (2) $\mathrm{m}(A) = 0$ if and only if $\lambda_n(A) = 0$.

2000 *Mathematics Subject Classification.* 46Fxx.

Key words and phrases. integral, generalized function, fourier transform.

A Cauchy sequence $(V(g_n, \epsilon_n, A_n))$ is called an L_0-Cauchy sequence for Λ if it satisfies the following three conditions on I:

(K_1) if B is a Lebesgue measurable subset of I with $\lambda_n(I \setminus A_n) \geq \lambda_n(B)$, then

$$m(B \cap [-1/\epsilon_n, 1/\epsilon_n]) \leq \epsilon_n.$$

(K_2) if m$(I \setminus A_n) > 0$ for all n, there exist $k, k' > 0$ such that

$$k \leq \lambda_n(I \setminus A_n) \leq k'$$

for all n.

(K_3) if B is a Lebesgue measurable subset of I with $\lambda_n(I \setminus A_n) \geq \lambda_n(B)$, then

$$\int_B |g_n(x)| dx \leq \epsilon_n.$$

Let $\mathbf{F}_0(\Lambda)$ be the set of L_0-Cauchy sequences for Λ, and $L_0(\Lambda)$ the set of sequences (g_n) in $L_1(I)$ such that there exists an L_0-Cauchy sequence $(V(g_n)) \in \mathbf{F}_0(\Lambda)$.[1] A sequence $(V(g_n)) \in \mathbf{F}_0(\Lambda)$ is called an L_0-Cauchy sequence for Λ and g, or for g, if $\bigcap_{n=1}^{\infty} V(g_n) = \{g\}$.

We fix two increasing sequences (α_n) and (β_n) of real numbers with $\lim_{n\to\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \beta_n = \infty$, and a decreasing sequence (J_n) of measurable subsets with $J_n \subseteq [-\beta_n, \beta_n]$ and $\lim_{n\to\infty} m(J_n) = 0$.

Let ν_n be an absolutely continuous measure on $\mathbf{R}$ such that

$$\nu_n(E_n) = \exp(-\alpha_n) = \nu_n(J_n)$$

for $E_n = \mathbf{R} \setminus [-\beta_n, \beta_n]$ and non empty J_n.

Denote $J_n + a = \{x + a; x \in J_n\}$ by J_n^a. For any measurable subset E of $\mathbf{R}$ and for any different points $a_1, a_2, ..., a_l \in I$, we set

$$\mu_n^0(E) = \sum_{i=1}^{l} \nu_n((E \cap J_n^{a_i}) - a_i) + \nu_n(E \cap E_n) + m(E \cap (CE_n \setminus \bigcup_{i=1}^{l} J_n^{a_i})).^2 \tag{2.1}$$

Let

$$\mu_n = \mu_n^0 / \exp(-\alpha_n) \qquad (n = 1, 2, ...). \tag{2.2}$$

Then (μ_n) is called a sequence of measures defined for $a_1, a_2, ..., a_l$. We denote (μ_n) by $T((a_i)_1^l)$ or $T(a_1, a_2, ..., a_l)$. If $J_{n_0} = \phi$ for some $n_0 \in \mathbf{N}$, the measure μ_n for each $n \geq n_0$ is independent of the choice of points $a_1, a_2, ..., a_l$.

We fix the sequence (ν_n) in the following.

If (g_n) and (f_n) are sequence in $L_0(T((a_i)_1^l))$ with L_0-Cauchy sequences for g, then

$$\limsup_{n\to\infty} \int_I f_n(x) dx = \limsup_{n\to\infty} \int_I g_n(x) dx,$$

and

$$\liminf_{n\to\infty} \int_I f_n(x) dx = \liminf_{n\to\infty} \int_I g_n(x) dx.$$

Hence we can define an integral as follows.

[1] The set $L_1(I)$ is the set of Lebesgue integrable functuions on I.

[2] $CE_n = R \setminus E_n$

Definition 1 *Let (g_n) be a sequence in $L_0(T((a_i)_1^l))$ with an L_0-Cauchy sequence for g. If*

$$\limsup_{n\to\infty} \int_I g_n(x)dx = \liminf_{n\to\infty} \int_I g_n(x)dx,$$

this common value is denoted by

$$I(g, T((a_i)_1^l)) = (E.R.T((a_i)_1^l)) \int_I g(x)dx$$

and $I(g, T((a_i)_1^l))$ is called the $(E.R.T((a_i)_1^l))$-integral of g on I. If $-\infty < I(g, T((a_i)_1^l)) < \infty$, g is called to be $(E.R.T((a_i)_1^l))$-integrable on I.

Definition 2 *A sequence (g_n) of functions in $M_0(I)$ is said to satisfy $(*)$-condition for $a_1, a_2, ..., a_l$ if*

$$\lim_{n\to\infty} \int_{J_n^a \cap I} |g_n(x)|dx = 0$$

for any $a \in I$ with $a \neq a_i (i = 1, 2, ..., l)$.

Let $L_0^*(T((a_i)_1^l))$ be the set of all sequences (g_n) in $L_0(T((a_i)_1^l))$ with $(*)$-condition for $a_1, a_2, ..., a_l$.

We define a translation invariant integral in $M_0(I)$.

Definition 3 *Let $g \in M_0(I)$ be a function such that ,for some sequence $T((a_i)_1^l)$ of measures, there exists a sequence $(g_n) \in L_0^*(T((a_i)_1^l))$ with an L_0-Cauchy sequence $(V(g_n))$ for g. If the $(E.R.T((a_i)_1^l))$-integral of g exists, the $(E.R.T)$-integral*

$$(E.R.T) \int_I g(x)dx$$

of g is defined to be the $(E.R.T((a_i)_1^l))$-integral of g, where the $(E.R.T)$-integral of g may be finite or infinite. If the $(E.R.T)$-integral of g is finite, g is said to be $(E.R.T)$-integrable.

Remark 1 *We take sequences $(\alpha_n), (\beta_n), (T_n)$, and (ν_n) with the above conditions arbitrarily and fix there. We make some particular choices for the situations there.*

Remark 2 *In the above assertions, an open interval I can be replaced by a semiclosed or closed interval. That is, the $(E.R.T((a_i)_1^l))$-integral and $(E.R.T)$-integral can be defined on a semiclosed or closed interval.*

3 Relations to improper Riemann integral and Lebesgue integral First, we consider the relation between the $(E.R.T)$-integral and the improper Riemann integral. In this Section, let $E_n = \mathbf{R} \setminus [-\mathrm{n}, \mathrm{n}]$, $\beta_n = n$, and $J_n = [-1/(2\mathrm{n}), 1/(2\mathrm{n})]$ for $n = 1, 2, \cdots$. In the following, a function f is assumed to be measurable on an interval I.

Theorem 1 *Let $I = (a, b]$ be a finite interval. If f is a bounded Riemann integrable function on $[\mathrm{c}, \mathrm{b}]$ for every $c \in (\mathrm{a}, \mathrm{b})$ and*

$$\lim_{c\to a+0} \int_c^b f(x)dx \tag{3.1}$$

is finite, then f is $(E.R.T)$-integrable on I, and the value of the integral is given by the same value as (3.1).

Proof. Let n_0 be an integer with $n_0 > (2(b-a))^{-1}$. Put $A_n = [a + 1/(2n), b]$ for $n = n_0, n_0 + 1, \cdots$. Let (α_{n}) be an increasing sequence with

$$\alpha_n \geq \max(\sup_{\mathrm{A_n}} |\mathrm{f(x)}|, \mathrm{n}),$$

and (ν_n) a sequence of measures on $\mathbf{R}$ defined by

$$\nu_n(E) = \int_E k_n(x) dx,$$

where

$$k_n(x) = \begin{cases} (\alpha_n/(4nx^2)) \ \exp(-\alpha_n/(2n|x|)), & \text{on} \quad J_n \\ (\alpha_n/(2n)) \ \exp(-\alpha_n|x|/n), & \text{on} \quad E_n \\ 1, & \text{on} \quad \mathbf{R} \setminus (\mathbf{J_n} \cup \mathbf{E_n}). \end{cases} \tag{3.2}$$

Put $a_1 = a$, and $(\mu_n) = T(a_1)$. Let $f_n(x)$ $(n \geq n_0)$ be a function defined by $f_n(x) = f(x)$ on A_n and 0 elsewhere.

We will show that $(V(f_n))_N^\infty = (V(f_n, \epsilon_n, A_n))_N^\infty$ is an L_0-Cauchy sequence for $T(a_1)$ for sufficiently large N $(> n_0)$, where $\epsilon_n = 1/n$.

First, Since

$$\int_{A_n} |f_n(x) - f_{n+1}(x)| dx = 0,$$

$(V(f_n))_N^\infty$ is a Cauchy sequence.

Next, we show that $(V(f_n))_N^\infty$ satisfies (K_1),(K_2), and (K_3).

Let B be any Lebesgue measurable subset of I with $\mu_n^0(I \setminus A_n) \geq \mu_n^0(B)$. It follows that

$$\mu_n{}^0(I \setminus A_n) = \exp(-\alpha_n)/2 \tag{3.3}$$

and

$$\mu_n{}^0(B \cap A_n) = m(B \cap A_n). \tag{3.4}$$

From (3.3) and(3.4), we have

$$\int_B |f_n(x)| dx = \int_{B \cap A_n} |f(x)| dx \leq \alpha_n \ \exp(-\alpha_n)/2 \leq \epsilon_n$$

for sufficiently large n. Thus (K_3) is satisfied.

Moreover, we have

$$\mathrm{m}(B \cap [-1/\epsilon_n, 1/\epsilon_n]) \leq \mathrm{m}((I \setminus A_n) \cap B) + \mathrm{m}(A_n \cap B)$$

$$\leq 1/(2n) + \exp(-\alpha_n)/2 \leq \epsilon_n$$

for sufficiently large n. Thus (K_1) is satisfied.

Since $\mu_n(I \setminus A_n) = \mu_n^0(I \setminus A_n)/\exp(-\alpha_n) = 1/2$, (K_2) is satisfied.

Threrefore $(V(f_n))_N^\infty \in \mathbf{F}_0(T(a_1))$.

We see easily that (f_n) satisfies $(*)$-condition. Hence we have $(f_n) \in L_0^*(T(a_1))$ and

$$(E.R.T) \int_a^b f(x) dx = \lim_{n \to \infty} \int_a^b f_n(x) dx$$

$$= \lim_{c \to a+0} \int_c^b f(x) dx. \blacksquare$$

The approximation of the integral from the right to the left extremity a can be replaced by that from the left to the right extremity b by the same argument as Theorem 1.

Next, we consider an improper Riemann integrable function on an infinite interval $[a, \infty)$.

Theorem 2 *Suppose that f is a bounded Riemann integrable function on $[a, b)$ for every b with $a < b$ and*

$$\lim_{b\to\infty}\int_a^b f(x)dx \tag{3.5}$$

is finite. Then f is $(E.R.T)$-integrable on the interval $I = [a, \infty)$ and the integral value coinsides with the limit (3.5).

Proof. Let (α_n) and (ν_n) be sequences in Theorem 1 defined by $A_n = [a, n]$ in place of $A_n = [a + 1/(2n), n]$. A function $f_n(x)$ on $I = [a, \infty)$ is defined to be $f(x)$ on A_n and 0 elsewhere. We will show that

$$(V(f_n))_N^\infty = (V(f_n, \epsilon_n, A_n))_N^\infty \in \mathbf{F}_0(T(a_1))$$

for sufficiently large N, where $a_1 = a$ and $\epsilon_n = 1/\sqrt{n}$.

We see easily that $(V(f_n))_N^\infty$ is a Cauchy sequence. Since $\mu_n^0(I \setminus A_n) = \exp(-\alpha_n)$, (K_2) is satisfied.

Let B be any subset of I satisfying $\mu_n^0(I \setminus A_n) \geq \mu_n^0(B)$.

Since $\exp(-\alpha_n) = \mu_n^0(I \setminus A_n) \geq \mu_n^0(B)$ and $\mu_n^0(B \cap A_n) = \mathrm{m}(B \cap A_n)$, we have

$$\int_B |f_n(x)|dx \leq \int_{I\setminus A_n} |f(x)|dx + \int_{B\cap A_n} |f(x)|dx \tag{3.6}$$

$$\leq M/(2n) + \alpha_n \exp(-\alpha_n) < \epsilon_n$$

for sufficiently large n, where $M = \max\limits_{[a,a+1]} |f(x)|$. Thus (K_3) is satisfied.

Moreover, we obtain

$$\mathrm{m}(B \cap [-1/\epsilon_n, 1/\epsilon_n]) \leq \mathrm{m}(B \cap (I \setminus A_n))$$

$$+\mathrm{m}(B \cap A_n) \leq 1/(2n) + \exp(-\alpha_n) \leq \epsilon_n,$$

which means that (K_1) is satisfied. Thus $(V(f_n))_N^\infty \in \mathbf{F}_0(T(a_1))$.

Since (f_n) satisfies $(*)$-condition, f is $(E.R.T)$-integrable and

$$(E.R.T)\int_a^\infty f(x)dx = \lim_{n\to\infty}\int_a^\infty f_n(x)dx$$

$$= \lim_{b\to\infty}\int_a^b f(x)dx. \blacksquare$$

By the similar verification as Theorem 2, we can find that the assertion in Theorem 2 holds also for an interval $(-\infty, a]$ or $(-\infty, \infty)$ in place of $[a, \infty)$.

Example 1 *Since $f(x) = \sin x/x$ is improper Riemann integrable on $[0, \infty)$, f is $(E.R.T)$-integrable, and we have*

$$(E.R.T)\int_0^\infty \sin x/x \; dx = \pi/2.$$

Next we discuss with the relation of the Lebesgue integral and the $(E.R.T)$-integral.

Theorem 3 *If f is Lebesgue integrable on a finite or an infinite interval I, f is $(E.R.T)$-integrable on I and both integral values coincide.*

Proof. Let (ν_n) be a sequence of measures in Theorem 1 defined by $(\alpha_n) = (n)$. We set $(\mu_n) = T(a_1)$, where a_1 is any number in I. Put $A_n = I, f_n = f$, and $\epsilon_n = 1/n$ for $n = 1, 2, \cdots$. Since $I \setminus A_n = \phi$, it is clear that $(V(f_n, \epsilon_n, A_n)) \in \mathbf{F}_0(T(a_1))$. Since (f_n) satisfies $(*)$-condition, f is $(E.R.T)$-integrable and

$$(E.R.T)\int_I f(x)dx = \int_I f(x)dx. \blacksquare$$

Definition 4 *If $|f|$ is $(E.R.T)$-integrable on I, f is called to be absolutely $(E.R.T)$-integrable on I.*

Theorem 4 *If f is absolutely $(E.R.T)$-integrable on a finite or an infinite interval I, f is Lebesgue integrable and*

$$(E.R.T)\int_I f(x)dx = \int_I f(x)dx.$$

Proof. If $|f|$ is $(E.R.T)$-intagrable on I, there exists a Cauchy sequence $(V(g_n)) = (V(g_n, \epsilon_n, B_n))$ and a finite number of points $a_1, a_2, \cdots, a_l$ such that $\bigcap_{n=1}^{\infty} V(g_n, \epsilon_n, B_n) \ni |f|$ and $(V(g_n, \epsilon_n, B_n)) \in \mathrm{F}_0(T((a_i)_1^l))$. Hence we have

$$\int_{B_n} |g_n(x) - |f(x)||dx \leq \epsilon_n.$$

so that

$$(3.7) \qquad (E.R.T)\int_I |f(x)|dx = \lim_{n\to\infty}\int_{B_n} g_n(x)dx = \lim_{n\to\infty}\int_{B_n} |f(x)|dx.$$

Putting, ${B_n}^* = \bigcup_{j=1}^{n} B_j$, we have $\mathrm{m}({B_n}^* \setminus B_n) = 0$. Hence it follows that

$$(3.8) \qquad \int_{B_n^*} |f(x)|dx = \int_{B_n} |f(x)|dx.$$

Let $h_n(x)$ be a function defined by

$$h_n(x) = \begin{cases} |f(x)|, & \text{on} \quad B_n^* \\ 0, & \text{on} \quad I \setminus B_n^*, \end{cases}$$

Since $\mathrm{m}(I \setminus \bigcup_{j=1}^{\infty} B_j) = 0$, we obtain $h_n \uparrow |f|$ a.e. From the monotone convergence theorem, it follows that

$$\int_I |f(x)|dx = \lim_{n\to\infty}\int_I h_n(x)dx = \lim_{n\to\infty}\int_{B_n^*} |f(x)|dx.$$

Hence we have

$$\int_I |f(x)|dx = \lim_{n\to\infty}\int_{B_n} |f(x)|dx = (E.R.T)\int_I |f(x)|dx$$

from (3.7) and (3.8). Thus $|f|$ is Lebesgue integrable on I. $\blacksquare$

4 Fourier transform By virtue of Theorems 2, 3, and 4, it will be shown that the main fundamental theorems on Fourier integrals hold also for $(E.R.T)$-integral.

In this chapter, we will state three statements without proof which are the theorem of Riemann-Lebesgue, Fourier's integral formula, and Fourier's single-integral formula.

Theorem 5 *Let f be absolutely $(E.R.T)$-integrable on $\mathbf{R}$. Then it holds that*

$$\lim_{\lambda\to\infty}(E.R.T)\int_{-\infty}^{\infty} f(x)cos\lambda x dx = 0,$$

and

$$\lim_{\lambda\to\infty}(E.R.T)\int_{-\infty}^{\infty} f(x)sin\lambda x dx = 0.$$

Theorem 6 *Let f be absolutly $(E.R.T)$-integrable on $\mathbf{R}$. If f is of bounded variation in an interval including the point x, then*

$$\frac{1}{\pi}(E.R.T)\int_{0}^{\infty}(E.R.T)\int_{-\infty}^{\infty} f(t)cosu(x-t)dtdu$$

$$= \frac{1}{2}(f(x+0)+f(x-0)).$$

Theorem 7 *If $f(t)/(1+|t|)$ is absolutely $(E.R.T)$-integrable on $(-\infty,\infty)$ and f is of bounded variation in an interval including the point x, then*

$$\lim_{\lambda\to\infty}\frac{1}{\pi}(E.R.T)\int_{-\infty}^{\infty} f(t)\frac{sin\lambda(x-t)}{x-t}dt$$

$$= \frac{1}{2}(f(x+0)+f(x-0)).$$

Remark 3 *It is known that the Fourier's integral and Fourier's single-integral formulas mentioned in theorems 6 and 7 are described by using not only the Lebesgue integral but the improper Riemann integral. However, the $(E.R.T)$-integral is an extention of their integral as mentioned in Section 3, and their formulas are described by using only the $(E.R.T)$-integral.*

5 A restricted Fourier's integral formula. In this section, for some functions which are neither Lebesgue integrable nor of bounded variation on $\mathbf{R}$, we introduce a restricted Fourier transform.

For different points $c_1, c_2, \cdots, c_l$, we denote $H_n = \bigcup_{i=1}^{l}(c_i - \frac{1}{2n}, c_i + \frac{1}{2n})$ and $A_n = [-n,n]\setminus H_n$. Let (α_n) be an increasing sequence with $\alpha_n \geq n$ for $n = 1, 2, \cdots$.

In what follows, we suppose that f is a measurable function on $\mathbf{R}$ which satiisfies the following conditions:

(i) $\text{ess.sup}_{A_n}|f(x)| \leq \alpha_n$.

(ii) For each c_i $(i = 1, 2, \cdots, l)$, there exists a positive number ω_i such that

$$f(c_i + t) = -f(c_i - t)$$

for $0 < t \leq \omega_i$.

We choose any numbers ξ_1 and ξ_2 $(\xi_1 < \xi_2)$ satisfying the following two conditions:

(1) if $x \notin \{c_1, c_2, \cdots, c_l\}$, the open interval (ξ_1, ξ_2) includes x and the closed interval $[\xi_1, \xi_2]$ excludes $c_1, c_2, \cdots$, and c_l.

(2) if $x = c_i$ for some i, the open interval (ξ_1, ξ_2) includes x and the closed interval $[\xi_1, \xi_2]$ excludes any $c_j (j \neq i)$.

Let $\chi_x(t, \xi_1, \xi_2)$ be a function with

$$\chi_x(t, \xi_1, \xi_2) = \begin{cases} 1, & \text{on} \quad \xi_1 < t < \xi_2 \\ 0, & \text{on} \quad elsewhere. \end{cases}$$

We define a restricted Fourier transform

$$F(u, x) \sim (E.R.T) \int_{-\infty}^{\infty} \chi_x(t, \xi_1, \xi_2) f(t) \exp(-iut) dt,$$

and define the inverse transform by

$$f(x) \sim \frac{1}{2\pi} (E.R.T) \int_{-\infty}^{\infty} F(u, x) \exp(iux) du.$$

Let $J_n = [-1/(2n), 1/(2n)]$ and $E_n = \mathbf{R} \setminus [-n, n]$. Moreover, let (ν_n) be a sequence of measures on $\mathbf{R}$ defined by

$$\nu_n(E) = \int_E k_n(x) dx,$$

where $k_n(x)$ is a function defined by (3.2) in Theorem 1, where (α_n) is the sequence defined at the biginning of this section. We put $(\mu_n) = T((c_i)_1^l)$.

Lemma 1 *If $x = c_i$ for some i, then, for any positive number u,*

$$g(t) = f(t) \chi_x(t, \xi_1, \xi_2) \cos u(x - t)$$

is $(E.R.T)$-integrable on $\mathbf{R}$.

Proof. Let $g_n(t)$ be a function defined by $g(t)$ on A_n and 0 elsewhere.
We will show that

$$(V(g_n))_N^{\infty} = (V(g_n, \epsilon_n, A_n))_N^{\infty} \in \mathbf{F}_0(T((c_i)_1^l)$$

for sufficiently large N, where $\epsilon_n = 2l/n$. Let B be any Lebesgue measurable subset of $\mathbf{R}$ with $\mu_n^0(CA_n) \geq \mu_n^0(B)$. It follows that

$$\mu_n^0(CA_n) = (l + 1) \exp(-\alpha_n)$$

and

$$\mu_n^0(B \cap A_n) = \mathrm{m}(B \cap A_n).$$

Hence, by the similar argument as the proof of Theorem 2, we can see that

$$\int_B |g_n(t)| dt = \int_{B \cap A_n} |g(t)| dt \leq \alpha_n \mathrm{m}(B \cap A_n)$$

$$< \alpha_n (l + 1) \exp(-\alpha_n) \leq \epsilon_n$$

for sufficiently large n. Thus (K_3) is satisfied. Moreover, we have

$$\mathrm{m}(B \cap [-1/\epsilon_n, 1/\epsilon_n]) \leq \sum_{i=1}^{l} \mathrm{m}(B \cap [c_i - 1/(2n), c_i + 1/(2n)] + m(B \cap A_n)$$

$$\leq \frac{l}{n} + (l+1)\exp(-\alpha_n) < \epsilon_n$$

Thus $(K_1)$is satisfied. We can easily see that $(V(g_n))$ is a Cauchy sequence with (K_2) and (g_n) satisfies $(*)$-condition.

By virtue of condition (ii) for f, these exists a number ω_i such that $f(x+t) = -f(x-t)$ for $0 < |t| \leq \omega_i$.

First, let $(x-\omega_i, x+\omega_i) \subset (\xi_1, \xi_2)$. Putting $W = (\xi_1, x-\omega_i) \cup (x+\omega_i, \xi_2)$, $g(t)$ is essentially bounded on W. Hence g is Lebesgue integrable on W. Moreover, there exists a number n_0 such that $1/n_0 < \omega_i$. Therefore we have, for every $n > n_0$,

$$(5.1) \quad \int_{\frac{1}{n}<|x-t|<\omega_i} g(t)dt = \int_{\frac{1}{n}}^{\omega_i} (f(x-t) + f(x+t)) \cos utdt = 0$$

From (5.1) it follows that

$$(5.2) \qquad (E.R.T) \int_{-\infty}^{\infty} g(t)dt = \lim_{n\to\infty} \int_{A_n} g(t)dt = \int_W g(t)dt.$$

Thus g is $(E.R.T)$-integrable on $\mathbf{R}$.

Next, let $(x-\omega_i, x+\omega_i) \not\subset (\xi_1, \xi_2)$. If $x - \xi_1 \leq \xi_2 - x$, then

$$\int_{\xi_1}^{2x-\xi_1} g(t)dt = 0,$$

so that

$$(5.3) \qquad (E.R.T) \int_{-\infty}^{\infty} g(t)dt = \int_{2x-\xi_1}^{\xi_2} q(t)dt.$$

Since $g(t)$ is essentially bounded on $(2x-\xi_1, \xi_2)$, g is $(E.R.T)$-integrable on $\mathbf{R}$.

In the same way, if $x - \xi_1 > \xi_2 - x$, then

$$(5.4) \qquad (E.R.T) \int_{-\infty}^{\infty} g(t)dt = \int_{\xi_1}^{2x-\xi_2} g(t)dt.$$

Thus g is $(E.R.T)$-integrable on $\mathbf{R}$. ■

Lemma 2 *If $x = c_i$, there is a Cauchy sequence $(V(p_n))$ for*

$$p(u) = (E.R.T) \int_{-\infty}^{\infty} f(t)\ \chi_x(t, \xi_1, \xi_2) \cos u(x-t)dt$$

on $I = [0, \infty)$.

Proof. By the right hand side of (5.2), (5.3),and (5.4), we find that $p(u)$ is bounded.

For a function p_n defined by $p_n(u) = p(u)$ on $(1/(2n), n)$ and $p_n(u) = 0$ on elsewhere, it is easily seen that $(V(p_n, \epsilon_n, G_n))_N^{\infty} \in \mathbf{F}_0(T(a_1))$, where $a_1 = 0$, $G_n = (1/(2n), n)$ and $\epsilon_n = 1/n$. ■

Now we prove the following integral formula for the restricted Fourier transform.

Theorem 8 (1) *Suppose that $x \notin \{c_1, c_2, \cdots, c_l\}$ and f is of bounded variation in an open interval including x. Then*

$$\frac{1}{\pi}(E.R.T)\int_0^\infty (E.R.T)\int_{-\infty}^\infty f(t)\chi_x(t,\xi_1,\xi_2)cosu(x-t)dtdu \tag{5.5}$$
$$= \frac{1}{2}(f(x+0)+f(x-0)).$$

(2) *Suppose that $x = c_i$ for some i $(1 \leq i \leq l)$. Then*

$$\frac{1}{\pi}(E.R.T)\int_0^\infty (E.R.T)\int_{-\infty}^\infty f(t)\ \chi_x(t,\xi_1,\xi_2)\cos u(x-t)dtdu = 0. \tag{5.6}$$

Proof. First,we prove (1). For any $x \notin \{c_1, c_2, \cdots, c_l\}$, $f(t)$ is essentially bounded on (ξ_1, ξ_2) by the condition (i) for f. Hence, from Theorem 6, the formula (5.5) holds.

Next,we prove (2). We use the notations used in Lemmas 1 and 2.

Let $(x-\omega_i, x+\omega_i)$ be a subinterval of (ξ_1, ξ_2). Then the formula (5.2) implies that

$$p(u) = (E.R.T)\int_{-\infty}^\infty g(t)dt = \int_W g(t)dt$$
$$= \int_{\xi_1}^{x-\omega_i} f(t)\cos u(x-t)dt + \int_{x+\omega_i}^{\xi_2} f(t)\cos u(x-t)dt$$
$$= \int_{\omega_i}^{x-\xi_1} f(x-t)\cos utdt + \int_{x-\xi_2}^{-\omega_i} f(x-t)\cos utdt$$

It follows that

$$\int_{\frac{1}{2n}}^n \int_{\omega_i}^{x-\xi_1} f(x-t)\cos utdtdu = \int_{\omega_i}^{x-\xi_1} f(x-t)\frac{\sin nt - \sin(t/(2n))}{t}dt.$$

Since $f(x-t)/t$ is essentially bounded on the interval $(\omega_i, x-\xi_1)$, the formula

$$\lim_{n\to\infty}\int_{\omega_i}^{x-\xi_1} f(x-t)\frac{\sin nt}{t}dt = 0$$

holds. Moreover, since $|\sin x| \leq |x|$, we have

$$|\int_{\omega_i}^{x-\xi_1} f(x-t)\frac{\sin(t/(2n))}{t}dt| \leq \frac{1}{2n}\int_{\omega_i}^{x-\xi_1} |f(x-t)|dt.$$

Thus

$$\lim_{n\to\infty}\int_{\frac{1}{2n}}^n \int_{\omega_i}^{x-\xi_1} f(x-t)\cos utdtdu = 0 \tag{5.7}$$

By similar argument as above, we have

$$\lim_{n\to\infty}\int_{\frac{1}{2n}}^n \int_{x-\xi_2}^{-\omega_i} f(x-t)\cos utdtdu = 0. \tag{5.8}$$

It follows, from (5.7), (5.8) and Lemma 2, that

$$\frac{1}{\pi}(E.R.T)\int_0^\infty p(u)du = \frac{1}{\pi}\lim_{n\to\infty}\int_{\frac{1}{n}}^n p(u)du = 0.$$

Thus (5.6) holds.

Next, in the case of $(x-\omega_i, x+\omega_i) \not\subset (\xi_1, \xi_2)$, the formula

$$\frac{1}{\pi}(E.R.T)\int_0^\infty p(u)du = 0$$

can be shown by a similar argument as above. ■

Moreover, if an interval $[\alpha, \beta]$ does not contain x and $c_i (i = 1, 2, ..., l)$, the integral formula in Theorem 8 denoted by α, β in place of ξ_1, ξ_2 is equal to 0 as follows.

Theorem 9 *Let α, β be real numbers with $\alpha < \beta$ and $c_i \notin [\alpha, \beta]$ for $i = 1, 2, ..., l$. If $x \notin [\alpha, \beta]$, then*

$$(E.R.T)\int_0^\infty (E.R.T)\int_\alpha^\beta f(t)\cos u(x-t)dtdu = 0.$$

Proof. There is an integer n_0 with $[\alpha, \beta] \subseteq A_{n_0}$. Hence $f(x)$ is essentially bounded on $[\alpha, \beta]$ from condition (i) for f. Thus $f(t)\cos u(x-t)$ is $(E.R.T)$-integrable on $[\alpha, \beta]$ for any $u \in I = [0, \infty)$. Let $h(u)$ be the function given by

$$(E.R.T)\int_\alpha^\beta f(t)\cos u(x-t)dt.$$

Let

$$h_n(u) = \begin{cases} h(u), & \text{on} \quad G_n = [\frac{1}{2n}, n] \\ 0, & \text{on} \quad D \setminus G_n. \end{cases}$$

Then it is easily seen that $(V(h_n, \epsilon_n, G_n))_N^\infty \in \mathbf{F}_0(T(a_1))$ for sufficiently large N, where $a_1 = 0$ and $\epsilon_n = 1/n$.

Moreover, we obtain

$$\text{(5.9)} \qquad \lim_{n\to\infty}\int_0^\infty h_n(u)du = \lim_{n\to\infty}\int_{\frac{1}{2n}}^n \int_\alpha^\beta f(t)\cos u(x-t)dtdu$$

$$= \lim_{n\to\infty}\int_{\alpha-x}^{\beta-x} f(x-t)\frac{\sin(nt) - sin(t/(2n))}{t}dt.$$

By a similar argument as the proof of Theorem 8, this limit in (5.9) turns out to be 0. It follows that

$$(E.R.T)\int_0^\infty h(u)du = \lim_{n\to\infty}\int_0^\infty h_n(u)du = 0. \ ■$$

Now we apply Theorem 8 to some example.

Example 2 *Put $f(t) = t^{-(2m+1)}$ for some positive integer m. Let $A_n = \{x | 1/(2n) < |x| < n\}$ and $\alpha_n = \exp n$. Then, we have*

$$\sup_{A_n} |f(t)| \leq (2n)^{(2m+1)} \leq \exp n$$

for sufficiently large n.

We consider two cases as follows:

1. *Set $x = 0$. Let ξ_1 and ξ_2 be any real numbers with $\xi_1 < x < \xi_2$. Then*

$$\frac{1}{\pi}(E.R.T)\int_0^\infty (E.R.T)\int_{-\infty}^\infty \chi_x(t, \xi_1, \xi_2)\frac{\cos ut}{t^{2m+1}}dtdu = 0.$$

2. *Let ξ_1 and ξ_2 be any real numbers with the same sign, and x any real number with $\xi_1 < x < \xi_2$, Then*

$$\frac{1}{\pi}(E.R.T)\int_0^{\infty}(E.R.T)\int_{-\infty}^{\infty}\chi_x(t,\xi_1,\xi_2)\frac{\cos u(x-t)}{t^{2m+1}}dtdu = \frac{1}{x^{2m+1}}.$$

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