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INVERSE LIMITS AND INFINITE PRODUCTS OF EXPANDABLE SPACES

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ABSTRACT. In this paper, the followings are proved that: (1) Let X be the inverse limit of an inverse system $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ and let the projection π_α be an open and onto map for each $\alpha \in \Lambda$, if X is $|\Lambda|$-paracompact (resp. hereditarily $|\Lambda|$-paracompact) and each X_α has property $\mathcal{P}$ (resp. hereditarily property $\mathcal{P}$), then X has also property $\mathcal{P}$ (resp. hereditarily property $\mathcal{P}$). (2) Let $X=\prod_{\sigma\in\Sigma}X_\sigma$ be $|\Sigma|$-paracompact (resp. hereditarily $|\Sigma|$-paracompact), then X has property $\mathcal{P}$ (resp. hereditarily property $\mathcal{P}$) iff $\prod_{\sigma\in F}X_\sigma$ has property $\mathcal{P}$(resp. hereditarily property $\mathcal{P}$) for each $F\in[\Sigma]^{<\omega}$, where $\mathcal{P}$ denotes one of the following four properties: expanability, discrete expandability, σ-expandability, discrete σ-expandability.

In 1990, K.Chiba[1] proved the following: Let X be the inverse limit of an inverse system $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ and let the projection π_α be an open and onto map for every $\alpha \in \Lambda$, if X is $|\Lambda|$-paracompact and each X_α is normal (resp. paracompact, collectionwise normal, metacompact, subparacompact, submetacompact, paralindelof, metalindelof, σ-paralindelof, σ-metacompact, shrinking, property $\mathcal{B}$), then X is normal (resp. paracompact, collectionwise normal, metacompact, subparacompact, submetacompact, paralindelof, metalindelof, σ-paralindelof, σ-metacompact, shrinking, property $\mathcal{B}$). On the basis of this, various people ask:

Question. Is there a similar result about expanable spaces?

In this paper, we first answer this question positively. Next, we show that hereditarily expandable spaces have also similar properties. Using these, two groups of characterizations of infinite Tychonoff products of expandable spaces (resp. hereditarily expandable spaces) are obtained under the condition of $|\Sigma|$-paracompactness(resp. hereditarily $|\Sigma|$-paracompactness). And we show that both discrete expandable spaces and discrete σ-expandable spaces have also respectively similar results.

We use that $N_Y(x)$ denotes the neighourhood system of a point x of a subspace Y of a space X. Espectly, N(x) denotes $N_Y(x)$ when Y=X; |A|, clA and IntA denote respectively the cardinality, the closure and the interior of a set A; $(\mathcal{U})_x$ and $(\mathcal{U})|_A$ denote respectively $\{U\in \mathcal{U} : x \in U\}$ and $\{U\bigcap A : U\in \mathcal{U}\}$; ω and $[\Sigma]^{<\omega}$ denote, respectively, the first infinite ordinal number and the collection of all non-empty finite subsets of a non-empty set Σ. And assume that all spaces are Hausdorff spaces throughout this paper.

Definition 1. Let κ be a cardinal number, A space is κ-paracompact iff its every open cover $\mathcal{U}$ of cardinal $|\mathcal{U}| \le \kappa$ has a locally finite open refinement; A space is $|\Lambda|$-paracompact iff it is κ-paracompact, where $\kappa=|\Lambda|$.

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Definition 2. A space X is said to be expandable (resp. discrete expanable) iff its every locally finite (resp. discrete) closed family $\{F_\xi\colon \xi \in \Xi\}$ has a locally finite open family $\{U_\xi\colon \xi \in \Xi\}$ such that $F_\xi \subset U_\xi$ for every $\xi \in \Xi$; A space X is said to be σ-expandable (resp. σ-discrete expandable) iff its every locally finite (resp. discrete) closed family $\{F_\xi\colon \xi \in \Xi\}$ has a sequence $\langle\{U_{n\xi}\colon \xi \in \Xi\}\rangle_{n\in\omega}$ of locally finite open families of X such that $F_\xi \subset \bigcup_{n\in\omega} U_{n\xi}$ for every $\xi \in \Xi$.

Definition 3. A space X is said to has hereditarily property $\mathcal{P}$ iff its every subspace has property $\mathcal{P}$, where $\mathcal{P}$ denotes one of the following four properties: expanability,discrete expandability, σ-expandability, discrete σ-expandability.

It is easy to prove the following Lemma by the above Definitions:

Lemma A space X has hereditarily property $\mathcal{P}$ iff its every open subspace has property $\mathcal{P}$, where $\mathcal{P}$ is one of the following four properties: expanability,discrete expandability, σ-expandability, discrete σ-expandability.

The following are main results and their proofs of this paper:

Theorem 1. Let X be the inverse limit of an inverse system $\{X_\alpha, \pi^\alpha_\beta, \Lambda\}$ and let the projection π_α be an open and onto map for each $\alpha \in \Lambda$. If X is $|\Lambda|$-paracompact and each X_α is expandable (resp. σ-expanable), then X is expandable (resp. σ-expanable).

Proof. Let $\{F_\xi\colon \xi \in \Xi\}$ be a family of locally finite closed sets of X. For every $\alpha \in \Lambda$, put

$$V_\alpha = \bigcup\{V\colon V \text{ is open in } X_\alpha \text{ and } |\{\xi \in \Xi\colon \pi_\alpha^{-1}(V)\bigcap F_\xi \neq \phi\}| < \omega\}$$

then

(1) $\{\pi_\alpha^{-1}(V_\alpha) : \alpha \in \Lambda\}$ is an open cover of X and $\pi_\alpha^{-1}(V_\alpha) \subset \pi_\beta^{-1}(V_\beta)$ if $\alpha \le \beta$.

In fact, for every $x \in X$, there is some $W \in N(x)$ such that $\{\xi \in \Xi\colon W\bigcap F_\xi \neq \phi\}$ is a finite set since $\{F_\xi\colon \xi \in \Xi\}$ is locally finite in X. By [2,2.5.5 Proposition], there exist a $\alpha \in \Lambda$ and an open set V of X_α such that $x \in \pi_\alpha^{-1}(V) \subset W$, i.e., $|\{\xi \in \Xi\colon \pi_\alpha^{-1}(V)\bigcap F_\xi \neq \phi\}| < \omega$, then $x \in \pi_\alpha^{-1}(V) \subset \pi_\alpha^{-1}(V_\alpha)$. So, $\bigcup_{\alpha\in\Lambda} \pi_\alpha^{-1}(V_\alpha) = X$. Next, for every $x \in \pi_\alpha^{-1}(V_\alpha)$, there is some open set V of X_α such that $x \in \pi_\alpha^{-1}(V)$ and $|\{\xi \in \Xi\colon \pi_\alpha^{-1}(V)\bigcap F_\xi \neq \phi\}| < \omega$. I.e., $|\{\xi \in \Xi\colon \pi_\beta^{-1}(\pi^\beta_\alpha)^{-1}(V)\bigcap F_\xi \neq \phi\}| < \omega$ since $x \in \pi_\alpha^{-1}(V) = \pi_\beta^{-1}(\pi^\beta_\alpha)^{-1}(V)$, hence $x \in \pi_\beta^{-1}(V_\beta)$.

By [1,Lemma 2], there is an open cover $\{W_\alpha\colon \alpha \in \Lambda\}$ of X such that

(2) $\mathrm{cl}W_\alpha \subset \pi_\alpha^{-1}(V_\alpha)$ for every $\alpha \in \Lambda$, and $W_\alpha \subset W_\beta$ if $\alpha \le \beta$.

Pick $T_\alpha = X_\alpha - \pi_\alpha(X - \mathrm{cl}W_\alpha)$ for every $\alpha \in \Lambda$, then T_α is a closed set of X_α and $T_\alpha \subset V_\alpha$. Again pick $C_\alpha = \mathrm{Int}\pi_\alpha^{-1}(T_\alpha)$. Now, we prove:

(3) $\{C_\alpha : \alpha \in \Lambda\}$ is an open cover of X.

For every $x \in X$, there is some $\alpha \in \Lambda$ such that $x \in W_\alpha$ since $\{W_\alpha\colon \alpha \in \Lambda\}$ covers X. There exist a $\beta \in \Lambda$ and an open subset V of X_β such that $x \in \pi_\beta^{-1}(V) \subset W_\alpha$. Pick $\gamma \in \Lambda$ satisfying $\gamma \ge \alpha$ and $\gamma \ge \beta$, then $x \in C_\gamma$. To show this, we only assert that $\pi_\beta^{-1}(V) \subset \pi_\gamma^{-1}(T_\gamma)$. In fact, if there is some $y = (y_\delta)_{\delta\in\Lambda} \in \pi_\beta^{-1}(V) - \pi_\gamma^{-1}(T_\gamma)$, then $y_\beta \in V$ and $y_\gamma \in \pi_\gamma(X - \mathrm{cl}W_\gamma)$. There is some $z = (z_\delta)_{\delta\in\Lambda} \in X - \mathrm{cl}W_\gamma$ such that $y_\gamma = \pi_\gamma(z) = z_\gamma$, i.e., $y_\beta = \pi^\gamma_\beta(z_\gamma)$. So, $z \in \pi_\gamma^{-1}(\pi^\gamma_\beta)^{-1}(V) \subset W_\alpha \subset W_\gamma$. This contradicts to $z \in X - \mathrm{cl}W_\gamma$. Thus $x \in \pi_\beta^{-1}(V) \subset \pi_\gamma^{-1}(T_\gamma)$, then $x \in C_\gamma$.

By $|\Lambda|$-paracompactness of X, there is a locally finite open cover $\{O_\alpha\colon \alpha \in \Lambda\}$ of X such that

(4) $O_\alpha \subset C_\alpha$ for every $\alpha \in \Lambda$

Define $\mathcal{F}_\alpha = \{T_\alpha \bigcap \mathrm{cl}\pi_\alpha(F_\xi)\colon \xi \in \Xi\}$ for every $\alpha \in \Lambda$, then

(5) $\mathcal{F}_\alpha$ is a locally finite closed family of T_α.

In fact, for every $y \in T_\alpha \subset V_\alpha$, there is some open set V of X_α such that $y \in V$ and $|\{\xi \in \Xi\colon \pi_\alpha^{-1}(V)\bigcap F_\xi \neq \phi\}| < \omega$, then $|\{\xi \in \Xi\colon V\bigcap T_\alpha \bigcap \mathrm{cl}\pi_\alpha(F_\xi) \neq \phi\}| < \omega$ since $\{\xi \in \Xi\colon$

$V\bigcap T_\alpha\bigcap cl\pi_\alpha(F_\xi)\neq\phi\}\subset\{\xi\in\Xi:\pi_\alpha^{-1}(V)\bigcap F_\xi\neq\phi\}$. Hence $\mathcal{F}_\alpha$ is a locally finite family of closed subsets of T_α.

The proof for expandablity.

For every $\alpha\in\Lambda$, T_α is expandable since X_α is expandabe. There is a locally finite family $\mathcal{W}_\alpha=\{W_{\alpha\xi}: \xi\in\Xi\}$ of open sets of T_α such that

(6) $T_\alpha\bigcap cl\pi_\alpha F_\xi\subset W_{\alpha\xi}$ for every $\xi\in\Xi$.

We put $U_\xi=\bigcup_{\alpha\in\Lambda}[O_\alpha\bigcap\pi_\alpha^{-1}(W_{\alpha\xi})]$ for every $\xi\in\Xi$, then

(7) $F_\xi\subset U_\xi$ for every $\xi\in\Xi$.

In fact, for every $x\in F_\xi$, there is some $\alpha\in\Lambda$ such that $x\in O_\alpha\subset C_\alpha\subset\pi_\alpha^{-1}(T_\alpha)$, then

$$x_\alpha=\pi_\alpha(x)\in T_\alpha\bigcap\pi_\alpha(F_\xi)\subset W_{\alpha\xi}.$$

So, $x\in\pi_\alpha^{-1}(W_{\alpha\xi})\bigcap O_\alpha\subset U_\xi$.

(8) $\{U_\xi: \xi\in\Xi\}$ is a locally finite open family.

In fact, for every $x\in X$, there is some $G'\in N(x)$ such that

$$\{\alpha\in\Lambda:G'\bigcap O_\alpha\neq\phi\}=\{\alpha_0,\alpha_1,...,\alpha_k\}$$

where $k\in\omega$. Again for every $i\leq k$, there is $G_i\in N(x_{\alpha_i})$ such that $|(\mathcal{W}_{\alpha_i})_{G_i}|<\omega$. Put $G=G'\bigcap[\bigcap_{i\leq k}\pi_{\alpha_i}^{-1}(G_i)]$, then $G\in N(x)$ and

$$\{\xi\in\Xi:U_\xi\bigcap G\neq\phi\}\subset\bigcup_{i\leq k}\{\xi\in\Xi:W_{\alpha_i\xi}\bigcap G_i\neq\phi\}$$

Therefore, X is an expandable space.

The proof for σ-expandablity.

For $\alpha\in\Lambda$, if X_α is σ-expandabe, then T_α is σ-expandable. Since $\{T_\alpha\bigcap cl\pi_\alpha(F_\xi):\xi\in\Xi\}$ is a locally finite closed family of T_α, there exists a sequence $\langle\mathcal{W}_{\alpha n}=\{W_{\xi\alpha n}: \xi\in\Xi\}\rangle_{n\in\omega}$ of locally finite open families of T_α satisfying:

(6′) $T_\alpha\bigcap cl\pi_\alpha(F_\xi)\subset\bigcup_{n\in\omega}W_{\xi\alpha n}$ for every $\xi\in\Xi$.

We put $U_{\xi n}=\bigcup_{\alpha\in\Lambda}[O_\alpha\bigcap\pi_\alpha^{-1}(W_{\xi\alpha n})]$ for every $\xi\in\Xi$.

By using the ways of (7) and (8), it is easy to prove the following:

(7′) $F_\xi\subset\bigcup_{n\in\omega}U_{\xi n}$ for every $\xi\in\Xi$, and

(8′) $\{U_{\xi n}:\xi\in\Xi\}$ is a locally finite open family for every $n\in\omega$.

So, X is σ-expandable. □

Theorem 2. Let X be the inverse limit of an inverse system $\{X_\alpha,\pi_\beta^\alpha,\Lambda\}$ and let the projection π_α be an open and onto map for each $\alpha\in\Lambda$. If X is hereditarily $|\Lambda|$-paracompact and each X_α is hereditarily expandable (resp. hereditarily σ-expanable), then X is hereditarily expandable (resp. hereditarily σ-expanable).

Proof. Assume that $\{F_\xi: \xi\in\Xi\}$ is a family of locally finite closed sets of an open subspace Y of X. For every $\alpha\in\Lambda$, let us put

$$V_\alpha=\bigcup\{V: V \text{ is open in } X_\alpha,\ \pi_\alpha^{-1}(V)\subset Y \text{ and } |\{\xi\in\Xi:\ \pi_\alpha^{-1}(V)\bigcap F_\xi\neq\phi\}|<\omega\}$$

then

(1) $\bigcup_{\alpha\in\Lambda}\pi_\alpha^{-1}(V_\alpha)=Y$, and $\pi_\alpha^{-1}(V_\alpha)\subset\pi_\beta^{-1}(V_\beta)$ if $\alpha\leq\beta$.

By [1, Lemma 2], Y has an open cover $\{W_\alpha:\alpha\in\Lambda\}$ such that

(2) $cl_Y W_\alpha\subset\pi_\alpha^{-1}(V_\alpha)$ for every $\alpha\in\Lambda$, and $W_\alpha\subset W_\beta$ if $\alpha\leq\beta$.

For every $\alpha\in\Lambda$, put $E_\alpha=\bigcup\{E:E \text{ is open in } X_\alpha \text{ and } \pi_\alpha^{-1}(E)\subset W_\alpha\}$. Now, we assert that

(3) $\{\pi_\alpha^{-1}(E_\alpha):\alpha\in\Lambda\}$ is an open cover of Y , and $\pi_\alpha^{-1}(E_\alpha)\subset\pi_\alpha^{-1}(E_\beta)$ when $\alpha\leq\beta$.

In fact, for every $x\in Y$ there is some $\alpha\in\Lambda$ such that $x\in W_\alpha$, then there are both some $\beta\in\Lambda$ and some open set E in X_β such that $x\in\pi_\beta^{-1}(E)\subset W_\alpha$. Put $\gamma\in\Lambda$ such that $\gamma\geq\alpha,\beta$. Then $x\in\pi_\beta^{-1}(E)\subset W_\alpha\subset W_\gamma$. I.e., $x\in\pi_\gamma^{-1}(E_\gamma)$. So, $\bigcup_{\alpha\in\Lambda}\pi_\alpha^{-1}(E_\alpha)=Y$. The proof of the second part of (3) is trivial.

Since X is hereditarily $|\Lambda|$-paracompact, the subspace Y of X has a locally finite open cover $\{O_\alpha:\alpha\in\Lambda\}$ such that

(4) $O_\alpha\subset\pi_\alpha^{-1}(E_\alpha)$ for every $\alpha\in\Lambda$.

Let us put $Q_\alpha=clE_\alpha\bigcap(clV_\alpha\text{-}V_\alpha)$ for every $\alpha\in\Lambda$, then

(5) $\pi_\alpha^{-1}(Q_\alpha)\bigcap Y=\phi$ for every $\alpha\in\Lambda$

In fact, if there is some $x=(x_\beta)_{\beta\in\Lambda}\in\pi_\alpha^{-1}(Q_\alpha)\bigcap Y$, then $x\notin\pi_\alpha^{-1}(V_\alpha)$ since $x_\alpha\in Q_\alpha\subset(clV_\alpha$-$V_\alpha)$. On the other hand, we have $x\in cl_Y(\pi_\alpha^{-1}(E_\alpha))$. To show this, let G be an arbitrary neighborhood of x in Y. There exist a $\beta\in\Lambda$ and an open set H_β in X_β such that $x\in\pi_\beta^{-1}(H_\beta)\subset G$. Let us choose a $\gamma\in\Lambda$ such that $\gamma\geq\alpha,\beta$ and put $H_\gamma=(\pi_\beta^\gamma)^{-1}(H_\beta)$. Then $x\in\pi_\gamma^{-1}(H_\gamma)\subset G$ and $\pi_\alpha^\gamma(H_\gamma)\in N_{X_\alpha}(x_\alpha)$. Let us put $b\in\pi_\alpha^\gamma(H_\gamma)\bigcap E_\alpha$ since $\pi_\alpha^\gamma(H_\gamma)\bigcap E_\alpha\neq\phi$, then $b=\pi_\alpha^\gamma(c)$ for some $c\in H_\gamma$. There is $y=(y_\alpha)_{\alpha\in\Lambda}\in X$ such that $y_\gamma=c$ since π_γ is an onto map. Then $y_\alpha=b$ and $y\in\pi_\gamma^{-1}(H_\gamma)\bigcap\pi_\alpha^{-1}(\pi_\alpha^\gamma(H_\gamma)\bigcap E_\alpha)$. So, $\pi_\gamma^{-1}(H_\gamma)\bigcap\pi_\alpha^{-1}(E_\alpha)\neq\phi$. I.e., $G\bigcap\pi_\alpha^{-1}(E_\alpha)\neq\phi$ since $\pi_\gamma^{-1}(H_\gamma)\subset G$. So, $x\in cl_Y(\pi_\alpha^{-1}(E_\alpha))\subset\pi_\alpha^{-1}(V_\alpha)$. This is a contradiction.

(6) For every $\alpha\in\Lambda$, $\mathcal{F}_\alpha=\{(clE_\alpha\text{-}Q_\alpha)\bigcap cl\pi_\alpha(F_\xi):\xi\in\Xi\}$ is a locally finite family of closed sets in X_α-Q_α.

In fact, for every $x\in clE_\alpha\text{-}Q_\alpha=clE_\alpha\bigcap[X_\alpha\text{-}(clV_\alpha\text{-}V_\alpha)]$, we have $x\in clE_\alpha\subset clV_\alpha$ and $x\notin clV_\alpha$-V_α, then $x\in V_\alpha$. Hence there is some $V\in N_{X_\alpha}(x)$ such taht $\pi_\alpha^{-1}(V)\subset Y$ and $|\{\xi\in\Xi:\pi_\alpha^{-1}(V)\bigcap F_\xi\neq\phi\}|<\omega$, i.e.,$|\{\xi\in\Xi:V\bigcap\pi_\alpha(F_\xi)\neq\phi\}|<\omega$. Thus $\mathcal{F}_\alpha$ is locally finite in clE_α-Q_α. Again since clE_α-Q_α is closed in X_α-Q_α, then (6) is true.

The proof for hereditarily expandablity.

Assume that X_α is hereditarily expandable for every $\alpha\in\Lambda$, there exists a locally finite open family $\mathcal{K}_\alpha=\{K_{\alpha\xi}:\xi\in\Xi\}$ in X_α-Q_α such that

(7) $(clE_\alpha\text{-}Q_\alpha)\bigcap cl\pi_\alpha(F_\xi)\subset K_{\alpha\xi}$ for every $\xi\in\Xi$.

For every $\xi\in\Xi$, let us pick $U_\xi=\bigcup\{O_\alpha\bigcap\pi_\alpha^{-1}(K_{\alpha\xi}):\alpha\in\Lambda\}$. Now, we assert that

(8)$\{U_\xi:\xi\in\Xi\}$ is locally finite in Y, and $F_\xi\subset U_\xi$ for every $\xi\in\Xi$.

In fact, for every $x\in Y$, there exists a $G^*\in N_Y(x)$ such that $\triangle=\{\alpha\in\Lambda:O_\alpha\bigcap G^*\neq\phi\}$ is a finite set since $\{O_\alpha:\alpha\in\Lambda\}$ is a locally finite open cover in Y. For every $\alpha\in\triangle$, we have $\pi_\alpha(x)=x_\alpha\in X_\alpha$-$Q_\alpha$ by (5). There is a neighborhood G_α of x_α in X_α-Q_α such that $A_\alpha=\{\xi\in\Xi:G_\alpha\bigcap K_{\alpha\xi}\neq\phi\}$ is a finite set. Put $G=G^*\bigcap[\bigcap_{\alpha\in\triangle}\pi_\alpha^{-1}(G_\alpha)]$, then $G\in N(x)$ and $\{\xi\in\Xi:U_\xi\bigcap G\neq\phi\}\subset\bigcup_{\alpha\in\triangle}A_\alpha$. Next, for every $x\in F_\xi$, $x\in O_\alpha\subset\pi_\alpha^{-1}(E_\alpha)$ for some $\alpha\in\Lambda$, then $x_\alpha\in(clE_\alpha\text{- }Q_\alpha)\bigcap cl\pi_\alpha(F_\xi)\subset K_{\alpha\xi}$. Hence $x\in O_\alpha\bigcap\pi_\alpha^{-1}(K_{\alpha\xi})\subset U_\xi$. I.e., $\{U_\xi:\xi\in\Xi\}$ is a locally finite open expansion of $\{F_\xi:\xi\in\Xi\}$ in Y.

The proof for hereditarily σ-expandablity.

Let X_α be hereditarily σ-expandable for every $\alpha\in\Lambda$, there is a sequence $\langle\mathcal{K}_{\alpha n}=\{K_{\xi\alpha n}:\xi\in\Xi\}\rangle_{n\in\omega}$ of locally finite open families of X_α-Q_α satisfying:

(7′) $(clE_\alpha\text{-}Q_\alpha)\bigcap cl\pi_\alpha(F_\xi)\subset\bigcup_{n\in\omega}K_{\xi\alpha n}$ for every $\xi\in\Xi$.

Let us put $U_{\xi n}=\bigcup_{\alpha\in\Lambda}[O_\alpha\bigcap\pi_\alpha^{-1}(K_{\xi\alpha n})]$ for every $\xi\in\Xi$ and every $n\in\omega$. By using the way of (8), we have

(8′) $\{U_{\xi n}:\xi\in\Xi\}$ is locally finite for every $n\in\omega$, and $F_\xi\subset\bigcup_{n\in\omega}U_{\xi n}$ for every $\xi\in\Xi$. □

Corollary 1. Let X be the inverse limit of an inverse system $\{X_\alpha,\pi_\beta^\alpha,\Lambda\}$ and let the projection π_α be an open and onto map for each $\alpha\in\Lambda$. If X is $|\Lambda|$-paracompact and each X_α is discrete expandable (resp. discrete σ-expanable) for each $\alpha\in\Lambda$, then X is discrete expandable (resp. discrete σ-expanable).

Proof. Let $\{F_\xi:\xi\in\Xi\}$ be a family of discrete closed sets of X. For every $\alpha\in\Lambda$, we put

$$V_\alpha=\bigcup\{V:\text{ V is open in } X_\alpha \text{ and } \pi_\alpha^{-1}(V)\bigcap F_\xi\neq\phi \text{ for at most one } \xi\}$$

then it is easy to prove that X is discrete expandable (resp. discrete σ-expanable) by a similar way of Theorem 1. □

Corollary 2. Let X be the inverse limit of an inverse system $\{X_\alpha,\pi_\beta^\alpha,\Lambda\}$ and let the projection π_α be an open and onto map for each $\alpha\in\Lambda$. If X is hereditarily $|\Lambda|$-paracompact and each X_α is hereditarily discrete expandable (resp. hereditarily discrete σ-expanable), then X is hereditarily discrete expandable (resp. hereditarily discrete σ-expanable).

Proof. Assume that $\{F_\xi : \xi \in \Xi\}$ is a discrete closed family of an arbitriary subspace Y of X. For every $\alpha \in \Lambda$, let us put

$$V_\alpha = \bigcup \{V: \text{V is open in } X_\alpha,\ \pi_\alpha^{-1}(V) \subset Y \text{ and } \pi_\alpha^{-1}(V) \bigcap F_\xi \neq \phi \text{ for at most one } \xi\}.$$

By the similar way of Theorem 2, we can show that the conclusions hold respectively under the given conditions. □

Now we use that $\mathcal{P}$ denotes one of four properties: expanability, σ-expandability, discrete expandability, discrete σ-expandability, then the following hold:

Theorem 3. Let $X = \prod_{\sigma \in \Sigma} X_\sigma$ be $|\Sigma|$-paracompact, then X has property $\mathcal{P}$ iff $\prod_{\sigma \in F} X_\sigma$ has property $\mathcal{P}$ for each $F \in [\Sigma]^{<\omega}$.

Proof. ($\Leftarrow$) When $|\Sigma| < \omega$, it is obvious that $X = \prod_{\sigma \in \Sigma} X_\sigma$ has property $\mathcal{P}$ since $F = \Sigma \in [\Sigma]^{<\omega}$. Without loss of generality, we suppose $|\Sigma| \geq \omega$. Define the relation $\leq$: $F \leq E$ if and only if $F \subset E$ for any $(F,E) \in [\Sigma]^{<\omega} \times [\Sigma]^{<\omega}$. Then $[\Sigma]^{<\omega}$ is a directed set on the relation $\leq$. Let us put $X_F = \prod_{\sigma \in F} X_\sigma$ for every $F \in [\Sigma]^{<\omega}$ and define the projection:

$\pi_F^E : X_E \to X_F$ when $F \leq E$, where $\pi_F^E(x) = (x_\sigma)_{\sigma \in F} \in X_F$ for any $x = (x_\sigma)_{\sigma \in E} \in X_E$.

It is easy to prove that π_F^E is an open and onto map, then $\{X_E, \pi_F^E, [\Sigma]^{<\omega}\}$ is an inverse system of spaces X_E with bounding maps π_F^E: $X_E \to X_F (E \geq F)$.

Let X' be the inverse limit of the inverse system $\{X_E, \pi_F^E, [\Sigma]^{<\omega}\}$, by [2,2.5.3 Example], X' is homeomorphic to $X = \prod_{\sigma \in \Sigma} X_\sigma$.

Next, since $X_F = \prod_{\sigma \in F} X_\sigma$ has property $\mathcal{P}$ for every $F \in [\Sigma]^{<\omega}$, then X' has property $\mathcal{P}$ by Theorem 1 and Corollary 1. So, $X = \prod_{\sigma \in \Sigma} X_\sigma$ has property $\mathcal{P}$.

($\Leftarrow$) Assume that the product $X = \prod_{\sigma \in \Sigma} X_\sigma$ has property $\mathcal{P}$. For every $F \in [\Sigma]^{<\omega}$, pick a point $x_\sigma \in X_\sigma$ for every $\sigma \in \Sigma$-F, then the closed subspace $Y_F = \prod_{\sigma \in F} X_\sigma \times \prod_{\sigma \in \Sigma - F} \{x_\sigma\}$ of X has property $\mathcal{P}$. Therefore, $X_F = \prod_{\sigma \in F} X_\sigma$ has also property $\mathcal{P}$. □

By using Theorem 2 and the way of the proof of Theorem 3, the following holds obviously:

Corollary 3. Let $X = \prod_{\sigma \in \Sigma} X_\sigma$ be $|\Sigma|$-paracompact, then X has hereditarily property $\mathcal{P}$ iff $\prod_{\sigma \in F} X_\sigma$ has hereditarily property $\mathcal{P}$ for each $F \in [\Sigma]^{<\omega}$. □

Theorem 4. Let $X = \prod_{i \in \omega} X_i$ is countable paracompact (resp. hereditarily countable paracompact), then the following are equivalent:

(1) X has property $\mathcal{P}$ (resp. hereditarily property $\mathcal{P}$).

(2) $\prod_{i \in F} X_i$ has property $\mathcal{P}$(resp. hereditarily property $\mathcal{P}$) for each $F \in [\Sigma]^{<\omega}$.

(3) $\prod_{i \leq n} X_i$ has property $\mathcal{P}$ (resp. hereditarily property $\mathcal{P}$) for each $n \in \omega$.

Proof. The equivalence of both (1) and (2) is obvious by Theorem 3 (resp. Corollary 3). (2)$\Rightarrow$(3) hold trivially. Now we prove (3)$\Rightarrow$(2):

In fact, for every $F \in [\Sigma]^{<\omega}$, we may assume m=maxF since $F \neq \phi$. Let us pick $x_\sigma \in X_\sigma$ for each $\sigma \in \{0, 1, ..., m\} - F$, then $\prod_{\sigma \in F} X_\sigma \times \prod_{\sigma \in \{0,1,\ldots,m\} - F} \{x_\sigma\}$ is a closed set of $\prod_{i \leq m} X_i$. So, $\prod_{i \in F} X_i$ has property $\mathcal{P}$ (resp. hereditarily property $\mathcal{P}$). □

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