ON STRICTLY STAR-LINDELÖF SPACES

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Abstract. In this paper, we introduce new properties of topological spaces defined by stars of coverings, which are called strict star-Lindelöfness and strict starcompactness. Following the fundamental studies on star covering properties by E.K. van Douwen, G.M. Reed, A.W. Roscoe and I.J. Tree ([1]), there have been several related studies (see M.V. Matveev [7], [8], [9], for survey). Star-Lindelöf spaces have many nice properties. However, star-Lindelöfness is not preserved by closed subspaces ([7]). We define strict star-Lindelöfness to modify this defect and still so as to keep possible properties of star-Lindelöfness. Furthermore, we investigate relationships among these covering properties and give various examples.

1. Introduction and preliminaries. In this paper, all spaces are assumed to be $T_1$.

For a cover $U$ of a space $X$ and a subset $A$ of $X$,

$$\text{St}(A, U) = \bigcup\{U \in U | U \cap A \neq \emptyset\}$$

is called a star of $A$ (with respect to $U$). Define $\text{St}^0(A, U) = A$, $\text{St}^1(A, U) = \text{St}(A, U)$ and $\text{St}^{n+1}(A, U) = \text{St}(\text{St}^n(A, U), U)$ for $n \in \mathbb{N}$. For a singleton $A = \{x\}$, we usually write $\text{St}(x, U)$ instead of $\text{St}(\{x\}, U)$.

W.M. Fleischman [3] defined the following notion of starcompact spaces and studied its properties.

**Definition 1.1 ([3]).** A space $X$ is starcompact if for every open cover $U$ of $X$, there exists a finite subset $F$ of $X$ such that $\text{St}(F, U) = X$.

He proved in [3] that starcompactness is equivalent to countable compactness in the class of regular spaces. It was informed in [3] that R.S. Houston afterwards showed the equivalence in the class of Hausdorff spaces.

The following notion of star-Lindelöf spaces is defined by S. Ikenaga [5] originally under the name of $\omega$-star spaces, the present term was given by E.K. van Douwen, G.M. Reed, A.W. Roscoe and I.J. Tree [1].

**Definition 1.2 ([1]).** A space is star-Lindelöf if for every open cover $U$ of $X$, there exists a countable subset $A$ of $X$ such that $\text{St}(A, U) = X$.

We call such notions of topological spaces defined by taking stars of coverings star covering properties.

Later, E.K. van Douwen, G.M. Reed, A.W. Roscoe and I.J. Tree [1] established the fundamentals of star covering properties. Subsequently, there have been several related studies. M.V. Matveev [7], [8], [9] presented a nice exposition of star covering properties which contain many significant results. Among star covering properties, star-Lindelöfness has so far

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most appeared in the related studies. Many covering properties are preserved by closed subspaces, while star-Lindelöfness is not the case ([7]).

The aim of this paper is to introduce the following new notion of spaces, called strictly star-Lindelöf spaces, which will be shown to have possible properties of star-Lindelöf spaces so as to improve the above defect on closed heredity.

**Definition 1.3.** A space $X$ is strictly star-Lindelöf if for every open cover $U$ of $X$ and for every subset $A$ of $X$ satisfying $St(A, U) = X$ there exists a countable subset $B$ of $A$ such that $St(B, U) = X$.

We also define the following notion, called strictly starcompact spaces, which is related to starcompact spaces.

**Definition 1.4.** A space $X$ is strictly starcompact if for every open cover $U$ of $X$ and for every subset $A$ of $X$ satisfying $St(A, U) = X$ there exists a finite subset $F$ of $A$ such that $St(F, U) = X$.

We investigate properties of strictly star-Lindelöf spaces in Section 2., and strictly star-Lindelöfness is not the case ([7]).

2. Strictly star-Lindelöf spaces. In this section, we consider strict star-Lindelöfness. To begin with, we recall the definition of $k$-star-Lindelöfness.

For $k \in \mathbb{N} \cup \{0\}$, a space $X$ is $k$-star-Lindelöf if for every open cover $U$ of $X$, there exists a countable subset $A$ of $X$ such that $St^k(A, U) = X$. A space $X$ is $k\frac{1}{2}$-star-Lindelöf if for every open cover $U$ of $X$, there exists a countable subcollection $V$ of $U$ such that $St^k(\bigcup V, U) = X$. Moreover, a space $X$ is $\omega$-star-Lindelöf if for every open cover $U$ of $X$, there exist an $n \in \mathbb{N}$ and a countable subset $A$ of $X$ such that $St^n(A, U) = X$. Note that $\frac{1}{2}$-star-Lindelöfness and 1-star-Lindelöfness are precisely Lindelöfness and star-Lindelöfness, respectively ([7]).

Just like $k$-star-Lindelöfness, we define the following notions of spaces that are weaker than strict star-Lindelöfness.

**Definition 2.1.** Let $X$ be a space and $k \in \mathbb{N} \cup \{0\}$.

1. $X$ is strictly $k$-star-Lindelöf if for every open cover $U$ of $X$ and every subset $A$ of $X$ satisfying $St^k(A, U) = X$, there exists a countable subset $B$ of $A$ such that $St^k(B, U) = X$.

2. $X$ is strictly $k\frac{1}{2}$-star-Lindelöf if for every open cover $U$ of $X$ and every subcollection $V$ of $U$ satisfying $St^k(\bigcup V, U) = X$, there exists a countable subcollection $W$ of $V$ such that $St^k(\bigcup W, U) = X$.

3. $X$ is strictly $\omega$-star-Lindelöf if for every open cover $U$ of $X$, there exists an $n \in \mathbb{N}$ such that for every subset $A$ of $X$ satisfying $St^n(A, U) = X$, there exists a countable subset $B$ of $A$ such that $St^n(B, U) = X$.

Obviously, strict $\frac{1}{2}$-star-Lindelöfness and strict 1-star-Lindelöfness are precisely Lindelöfness and strict star-Lindelöfness, respectively.

By Definition 2.1, it is clear that every strictly $k$-star-Lindelöf space is $k$-star-Lindelöf for every $k \in \mathbb{N}$, and every strictly $\omega$-star-Lindelöf space is $\omega$-star-Lindelöf.

Furthermore, for every $k \in \mathbb{N}$ we have that every strictly $k$-star-Lindelöf space is strictly $k\frac{1}{2}$-star-Lindelöf, and every strictly $k\frac{1}{2}$-star-Lindelöf space is strictly $(k + 1)$-star-Lindelöf.
Now, recall that a space $X$ satisfies the discrete countable chain condition (DCCC, for short) if every discrete collection of non-empty open sets in $X$ is countable ([1]).

Here, we consider properties of strictly $k$-star-Lindelöf spaces. At first, similarly to the case of $k$-star-Lindelöfness ([7]), we have the following relations between strict $k$-star-Lindelöfness and the DCCC.

**Theorem 2.2.** For a regular space $X$, the following are equivalent.

(a) $X$ is strictly $2^+\star$-Lindelöf.
(b) $X$ is strictly $k$-star-Lindelöf for every $k \in \mathbb{N}$ with $k \geq 3$.
(c) $X$ is strictly $\omega$-star-Lindelöf.
(d) $X$ satisfies the DCCC.

*Proof.* The implications $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ are trivial.

$(c) \Rightarrow (d)$: This follows from the fact that every $\omega$-star-Lindelöf regular space satisfies the DCCC ([7]).

$(d) \Rightarrow (a)$: Suppose that $X$ is regular but not strictly $2^+\star$-Lindelöf. Then there exists an open cover $U$ of $X$ and a subcollection $V$ of $U$ such that $St(\bigcup \bigcup V, U) = X$ and $St(\bigcup (\bigcup A, U) \neq X$ for any countable subcollection $A$ of $V$.

Let $\alpha < \omega_1$. Suppose that there exist a subset $\{x_\beta \mid \beta < \alpha\}$ of $X$ and a subcollection $\{U_\beta \mid \beta < \alpha\}$ of $V$ satisfying $x_\beta \in X \setminus St(\bigcup_{\gamma < \beta} U_\gamma, U)$ are given for every $\beta < \alpha$. By the assumption above, we can take $x_\alpha \in X \setminus St(\bigcup_{\beta < \alpha} U_\beta, U)$. Then there exists a $U_\alpha \in V$ such that $x_\alpha \in St(\bigcup U_\alpha, U)$. Let $W = \{U_\alpha \mid \alpha < \omega_1\}$. Then one can show that $W$ is an uncountable discrete collection consisting of non-empty open sets in $X$. Therefore $X$ does not satisfy the DCCC.

Now, a space $X$ is $\omega_1$-compact if every uncountable subset of $X$ has an accumulation point. Every countably compact space and every Lindelöf space are $\omega_1$-compact, and every $\omega_1$-compact space is star-Lindelöf.

The following theorem shows that strictly star-Lindelöf spaces are located between Lindelöf spaces and $\omega_1$-compact spaces.

**Theorem 2.3.** Every strictly star-Lindelöf space is $\omega_1$-compact.

*Proof.* Suppose that $X$ is not $\omega_1$-compact. Then there exists an uncountable subset $A$ of $X$ with no accumulation points. For each $a \in A$, take a neighborhood $U_a$ of $a$ so that $U_a \cap A = \{a\}$, and define $U = \{U_a \mid a \in A\} \cup \{X \setminus A\}$. Then $U$ is an open cover of $X$. Pick $x_0 \in X \setminus A$ arbitrarily. We have $St(A \cup \{x_0\}, U) = X$, but no countable subset $B$ of $A$ satisfies $St(B \cup \{x_0\}, U) = X$. Thus $X$ is not strictly star-Lindelöf.

The converse of Theorem 2.3 need not be true (see Example 4.2 below).

It is known that every separable space is star-Lindelöf ([7]). Here we have the following theorem in the case of strictly $k$-star-Lindelöf spaces.

**Theorem 2.4.** Every separable space is strictly $1^+\star$-Lindelöf.

*Proof.* Let $X$ be a separable space and $D$ a countable dense subset of $X$. Let $U$ be an open cover of $X$ and $V$ a subcollection of $U$ satisfying $St(\bigcup V, U) = X$. For each $x \in X$, there exist a $V_x \in V$ and a $U_x \in U$ such that $x \in U_x$ and $V_x \cap U_x \neq \emptyset$. Then we can take $d_x \in D \cap (V_x \cup U_x)$ for every $x \in X$. Put $D' = \{d_x \mid x \in X\}$. Then $D'$ is a countable subset of $D$ satisfying $St(D', U) = X$. Denote $D' = \{d_n \mid n \in \mathbb{N}\}$. For each $d_n \in D'$, choose a $V_n \in \{V_x \mid x \in X\}$ so that $d_n \in V_n$. Define $W = \{V_n \mid n \in \mathbb{N}\}$. Then $W$ is a countable subcollection of $V$ satisfying $St(\bigcup W, U) = X$. Therefore, $X$ is strictly $1^+\star$-Lindelöf.
On the other hand, we cannot conclude that every separable space is strictly star-Lindelöf (see Example 4.3).

It is known that every open $F_\sigma$-set of a star-Lindelöf space is star-Lindelöf ([7]). Concerning strictly star-Lindelöf spaces, we have

**Theorem 2.5.** Every $F_\sigma$-set of a strictly star-Lindelöf space is strictly star-Lindelöf.

**Proof.** Let $X$ be a strictly star-Lindelöf space and $Y = \bigcup_{n \in \mathbb{N}} H_n$ an $F_\sigma$-set of $X$, where $H_n$ is a closed set in $X$ for every $n \in \mathbb{N}$.

Let $U$ be an open cover of $Y$ and $A$ a subset of $Y$ satisfying $\text{St}(A,U) = Y$. For each $U \in U$, take an open set $V_U$ in $X$ so that $V_U \cap Y = U$. Set $V = \{V_U \mid U \in U, U \cap A \neq \emptyset\}$.

If $\bigcup V = X$, notice that $\text{St}(A,V) = X$. Since $X$ is strictly star-Lindelöf, there exists a countable subset $B$ of $A$ such that $\text{St}(B,V) = X$. Then, we have $\text{St}(B,U) \supset \text{St}(B,V) \cap Y = Y$.

Suppose $\bigcup V \neq X$. Then, fix $n \in \mathbb{N}$. Note that $V_n = V \cup \{X \setminus H_n\}$ is an open cover of $X$. Choose $x_0 \in X \setminus \bigcup V$ and put $A' = A \cup \{x_0\}$. Then $A'$ satisfies $\text{St}(A',V_n) = X$. Since $X$ is strictly star-Lindelöf, there exists a countable subset $B'_n$ of $A'$ such that $\text{St}(B'_n,V_n) = X$. Let $B_n = B'_n \setminus \{x_0\}$. Then $B_n$ is a countable subset of $A$ satisfying $\text{St}(B_n,U) \supset \text{St}(B_n,V) \cap Y \supset H_n$.

Let us set $B = \bigcup_{n \in \mathbb{N}} B_n$. Then $B$ is a countable subset of $A$, and we have

$$\text{St}(B,U) = \bigcup_{n \in \mathbb{N}} \text{St}(B_n, U) \supset \bigcup_{n \in \mathbb{N}} H_n = Y.$$ 

Therefore $Y$ is strictly star-Lindelöf.

As opposed to star-Lindelöfness, we have

**Corollary 2.6.** Every closed subspace of a strictly star-Lindelöf space is also strictly star-Lindelöf.

On the other hand, strict $k$-star-Lindelöfness is not necessarily preserved by closed subspaces for every $k \in \mathbb{N}$ with $k \geq 1 + \frac{1}{2}$ (see Section 4).

A subset $A$ of a space $X$ is called a cozero-set if there is a continuous function $f : X \to \mathbb{R}$ such that $A = \{x \in X \mid f(x) \neq 0\}$.

**Corollary 2.7.** Every cozero-set of a strictly star-Lindelöf space is also strictly star-Lindelöf.

It is also known that every continuous image of a star-Lindelöf space is star-Lindelöf ([5],[7]). Likewise, we have

**Theorem 2.8.** Every continuous image of a strictly star-Lindelöf space is also strictly star-Lindelöf.

**Proof.** Let $X$ be a strictly star-Lindelöf space, $Y$ a space and $f : X \to Y$ a continuous mapping from $X$ onto $Y$.

Let $U$ be an open cover of $Y$ and $A$ a subset of $Y$ satisfying $\text{St}(A,U) = Y$. Put $V = \{f^{-1}(U) \mid U \in U\}$. Then $V$ is an open cover of $X$ satisfying $\text{St}(f^{-1}(A),V) = X$. Since $X$ is strictly star-Lindelöf, there is a countable subset $B$ of $f^{-1}(A)$ such that $\text{St}(B,V) = X$. Then $f(B)$ is countable and $f(B) \subset A$. We have $\text{St}(f(B),U) = Y$. Thus $Y$ is strictly star-Lindelöf.

\[ \square \]
3. **Strictly starcompact spaces.** Next, we consider properties of strictly starcompact spaces. We recall the definition of $k$-starcompactness.

For $k \in \mathbb{N} \cup \{0\}$, a space $X$ is $k$-starcompact if for each open cover $\mathcal{U}$ of $X$, there exists a finite subset $F$ of $X$ such that $\text{St}^k(F, \mathcal{U}) = X$. A space $X$ is $k^{\frac{1}{2}}$-starcompact if for each open cover $\mathcal{U}$ of $X$, there exists a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ such that $\text{St}^k(\bigcup \mathcal{V}, \mathcal{U}) = X$.

Moreover, a space $X$ is $\omega$-starcompact if for every open cover $\mathcal{U}$ of $X$, there exist an $n \in \mathbb{N}$ and a finite subset $A$ of $X$ such that $\text{St}^n(A, \mathcal{U}) = X$. Note that $\frac{1}{2}$-starcompactness and 1-starcompactness are precisely compactness and starcompactness, respectively ([7]).

Now, we define the following notions of spaces that are related to strict starcompactness.

**Definition 3.1.** Let $X$ be a space and $k \in \mathbb{N} \cup \{0\}$.

1. $X$ is strictly $k$-starcompact if for every open cover $\mathcal{U}$ of $X$ and every subset $A$ of $X$ satisfying $\text{St}^k(A, \mathcal{U}) = X$, there exists a finite subset $F$ of $A$ such that $\text{St}^k(F, \mathcal{U}) = X$.
2. $X$ is strictly $k^{\frac{1}{2}}$-starcompact if for every open cover $\mathcal{U}$ of $X$ and every subcollection $\mathcal{V}$ of $\mathcal{U}$ satisfying $\text{St}^k(\bigcup \mathcal{V}, \mathcal{U}) = X$, there exists a finite subcollection $\mathcal{A}$ of $\mathcal{V}$ such that $\text{St}^k(\bigcup \mathcal{A}, \mathcal{U}) = X$.
3. $X$ is strictly $\omega$-starcompact if for every open cover $\mathcal{U}$ of $X$, there exist an $n \in \mathbb{N}$ such that for every subset $A$ of $X$ satisfying $\text{St}^n(A, \mathcal{U}) = X$, there exists a finite subset $F$ of $A$ such that $\text{St}^n(F, \mathcal{U}) = X$.

In particular, strict $\frac{1}{2}$-starcompactness and strict 1-starcompactness are precisely compactness and strict starcompactness, respectively.

It follows from Definition 3.1 that every strictly $k$-starcompact space is $k$-starcompact for every $k \in \mathbb{N}$, and every strictly $\omega$-starcompact space is $\omega$-starcompact.

In addition, every strictly $k$-starcompact space is clearly strictly $k$-star-Lindelöf, and every strictly $\omega$-starcompact space is strictly $\omega$-star-Lindelöf.

Furthermore, for every $k \in \mathbb{N}$ we have that every strictly $k$-starcompact space is strictly $k^{\frac{1}{2}}$-starcompact, and every strictly $k^{\frac{1}{2}}$-starcompact space is strictly $(k+1)$-starcompact (see Diagram 2 in Section 4.).

Now, recall that a space $X$ satisfies the discrete finite chain condition (DFCC, for short) if every discrete collection of non-empty open sets in $X$ is finite ([1]). Similarly to the case of $k$-starcompact spaces ([7]), we have the following relations between strict $k$-starcompactness and the DFCC.

**Theorem 3.2.** For a regular space $X$, the following are equivalent.

1. $X$ is strictly $2^{\frac{1}{2}}$-starcompact.
2. $X$ is strictly $k$-starcompact for every $k \in \mathbb{N}$ with $k \geq 3$.
3. $X$ is strictly $\omega$-starcompact.
4. $X$ satisfies the DFCC.

It is known that every countably compact Lindelöf space is compact. The following result seems to be interesting in itself; the proof are easy and omitted.

**Theorem 3.3.** A space $X$ is strictly starcompact if and only if $X$ is countably compact and strictly star-Lindelöf.

Strictly starcompact spaces have the following properties similar to strictly star-Lindelöf spaces. Proofs are similar to the case of strictly star-Lindelöf spaces.

**Theorem 3.4.** Every closed subspace of a strictly starcompact space is also strictly star-compact.

**Theorem 3.5.** Every continuous image of a strictly starcompact space is also strictly star-compact.
4. Examples. In this section, we list various examples on strictly $k$-star-Lindelöf spaces and strictly $k$-starcompact spaces.

Let an infinite ordinal $\tau$ have the order topology. The symbol $\beta X$ is the Stone-Čech compactification of a completely regular space $X$.

To begin with, we give an example of a strictly star-Lindelöf space which is not Lindelöf. It shows the gap between strict $\frac{1}{2}$-star-Lindelöfness and strict 1-star-Lindelöfness.

**Example 4.1.** The space $\omega_1$ is strictly starcompact (and hence strictly star-Lindelöf).

**Proof.** Let $\mathcal{U}$ be an open cover of $\omega_1$ and $A$ a subset of $\omega_1$ satisfying $\text{St}(A, \mathcal{U}) = \omega_1$. For each $\alpha \in \omega_1$, there exist a $U_\alpha \in \mathcal{U}$ and a $\gamma_\alpha < \omega_1$ such that $(\gamma_\alpha, \alpha] \subset U_\alpha$. If $A$ is not cofinal in $\omega_1$, $A$ itself is countable. Hence we can assume that $A$ is cofinal in $\omega_1$. By the pressing-down lemma ([6]), there exist an $\alpha_0 < \omega_1$ and a cofinal subset $C$ of $\omega_1$ such that $\gamma_\alpha < \alpha_0$ for every $\alpha \in C$.

Because $A$ is cofinal in $\omega_1$, there is a $\xi \in A$ with $\alpha_0 < \xi$ such that $\gamma_\alpha < \alpha_0 < \xi$ for any $\alpha \in C \cap (\xi, \omega_1)$. Then for every $\beta \in (\xi, \omega_1)$, we can take an $\eta \in C$ so that $\xi < \beta < \eta$. Thus there exists a $U \in \mathcal{U}$ such that $\beta, \xi \in (\gamma_\eta, \eta] \subset U$. Hence $\beta \in \text{St}(\xi, \mathcal{U})$. Therefore, we have $\text{St}(\xi, \mathcal{U}) \supset (\xi, \omega_1)$.

Moreover, for each $\gamma \leq \xi$ there is an $a_\gamma \in A$ such that $\gamma \in \text{St}(a_\gamma, \mathcal{U})$. Then $\{\text{St}(a_\gamma, \mathcal{U}) \mid \gamma \leq \xi\}$ is an open cover of $[0, \xi]$. Since $[0, \xi]$ is compact, we can take finitely many $\gamma_1, \ldots, \gamma_n \leq \xi$ so that $\{\text{St}(a_{\gamma_i}, \mathcal{U}) \mid i = 1, \ldots, n\}$ covers $[0, \xi]$. Let $F = \{a_\gamma \mid i = 1, \ldots, n\} \cup \{\xi\}$. Then $F$ is a finite subset of $A$ satisfying $\text{St}(F, \mathcal{U}) = \omega_1$. Hence $\omega_1$ is strictly starcompact.

Therefore, the space $\omega_1$ is also an example of a non-compact strictly starcompact space.

**Example 4.2.** The space $\omega_1 \times (\omega_1 + 1)$ is $\omega_1$-compact but not strictly star-Lindelöf.

**Proof.** Define

$$\mathcal{U} = \{[0, \alpha] \times (\alpha, \omega_1] \mid \alpha < \omega_1\} \cup \{\omega_1 \times \omega_1\}$$

and

$$A = \{(\alpha, \beta) \in \omega_1 \times (\omega_1 + 1) \mid \alpha < \beta < \omega_1\}.$$  

Then $\mathcal{U}$ is an open cover of $\omega_1 \times (\omega_1 + 1)$ and we have $\text{St}(A, \mathcal{U}) = \omega_1 \times (\omega_1 + 1)$.

Let $C$ be an arbitrary countable subset of $A$. Define

$$\alpha_0 = \sup\{\alpha \mid (\alpha, \beta) \in C \text{ for some } \beta\} \quad \text{and} \quad \beta_0 = \sup\{\beta \mid (\alpha, \beta) \in C \text{ for some } \alpha\}.$$  

Then $\alpha_0 < \beta_0 < \omega_1$. If $\gamma > \beta_0$, then we have

$$\{U \in \mathcal{U} \mid (\gamma, \omega_1) \in U\} = \{[0, \alpha] \times (\alpha, \omega_1] \mid \alpha \geq \gamma\}.$$  

For any $\xi \geq \gamma$, $[0, \xi] \times (\xi, \omega_1]$ contains no points of $C$. Hence $\langle \gamma, \omega_1 \rangle \notin \text{St}(C, \mathcal{U})$.

Hence, the space $\omega_1 \times (\omega_1 + 1)$ is a countably compact but not strictly starcompact. In addition, we also have that $\omega_1 \times (\omega_1 + 1)$ is not strictly 2-star-Lindelöf (see Remark 4.10 below).

Moreover we obtain the following example.

**Example 4.3.** There exists a star-Lindelöf completely regular space that is strictly $1\frac{1}{2}$-star-Lindelöf but not $\omega_1$-compact.
Proof. Let $\omega$ be a countable discrete space and $A$ be a maximal almost disjoint family (m.a.d.family, for short) of infinite subsets of $\omega$. Put $\Psi = \omega \cup A$. Topologize $\Psi$ by letting $\omega$ be an open subspace of $\Psi$ and defining a local base $N(x)$ at each $x \in A$ by $N(x) = \{ \{x\} \cup (x \setminus F) \mid F \in [x]^{<\omega}\}$. This space is called a $\Psi$-space ([4]). Then it is known that $\Psi$ is a separable completely regular space which is 2-starcompact but neither 1²-star-Lindelöf nor nor 2-star-Lindelöf.

Since $\Psi$ is separable, $\Psi$ is star-Lindelöf. Moreover, $\Psi$ is strictly 1²-star-Lindelöf by Theorem 2.4.

Now, we construct the following spaces so as to obtain a 1²-star-Lindelöf space that is not strictly 2-star-Lindelöf.

For a completely regular space $X$ and an infinite cardinal $\tau$ with $\text{cf}(\tau) > \omega$, the space

$$N_\tau X = ((\tau + 1) \times \beta X) \setminus (\{\tau\} \times (\beta X \setminus X))$$

is called the Noble plank. By [7], $N_\tau X$ is 2-starcompact, and furthermore, $N_\tau X$ is 1²-star-Lindelöf.

**Example 4.4.** There exists a 1²-star-Lindelöf completely regular space that is not star-Lindelöf nor strictly 2-star-Lindelöf.

**Proof.** Let $D$ be a discrete space of size $\omega_1$. Then the Noble plank $N_{\omega_1} D$ is 1²-star-Lindelöf but not star-Lindelöf ([7]). We show that the Noble plank $N_{\omega_1} D$ is not strictly 2-star-Lindelöf.

Define $U = \{(0, \alpha) \times \beta D \mid \alpha < \omega_2\} \cup \{(\omega_2 + 1) \times \{d\} \mid d \in D\}$ and $A = \{\omega_2\} \times D$. Then $U$ is an open cover of $N_{\omega_1} D$ and we have $\text{St}^2(A, U) = N_{\omega_1} D$. However, no countable subset $B$ of $A$ satisfies $\text{St}^2(B, U) = N_{\omega_1} D$. Hence $N_{\omega_1} D$ is not strictly 2-star-Lindelöf.

For later use, we also have the another example (see Remark 4.10).

**Example 4.5.** There exists a 1²-star-Lindelöf completely regular space that is not strictly 2-star-Lindelöf.

**Proof.** Let $\Psi = \omega \cup A$ be the $\Psi$-space constructed form a m.a.d.family $A = \{a_\lambda \mid \lambda < 2^\omega\}$ of infinite subsets of $\omega$. Let $D$ be a discrete space of size $2^\omega$. Denote $D = \{y_\lambda \mid \lambda < 2^\omega\}$. Define $X = \Psi \times A(D)$, where $A(D)$ is the one-point compactification of $D$.

At first, it is easy to see that $X$ is 1²-star-Lindelöf since $\omega \times A(D)$ is Lindelöf and dense in $X$.

Next, we prove that $X$ is not strictly 2-star-Lindelöf. Define an open cover of $X$ by

$$U = \{\Psi \times \{y_\lambda\} \mid \lambda < 2^\omega\} \cup \{(n) \times A(D) \mid n \in \omega\}
\cup \{(\{a_\lambda\} \cup a_\lambda) \times (A(D) \setminus \{y_\lambda\}) \mid \lambda < 2^\omega\}. $$

Put $A = \{\langle a_\lambda, y_\lambda\rangle \mid \lambda < 2^\omega\}$. We have that $\text{St}^2(A, U) = X$. Let $B$ be an arbitrary countable subset of $A$. Take a $\lambda_0 < 2^\omega$ such that $\langle a_{\lambda_0}, y_{\lambda_0}\rangle \notin B$. Then we can show that $\langle a_{\lambda_0}, y_{\lambda_0}\rangle \notin \text{St}^2(B, U)$. Hence $X$ is not strictly 2-star-Lindelöf.

Let $R^*$ be the real line $\mathbb{R}$ with the topology

$$T_c = \{U \subset \mathbb{R} \mid \mathbb{R} \setminus U \text{ is a countable subset of } \mathbb{R}\}. $$

Then $R^*$ is not strictly starcompact because it is not countably compact. And clearly $R^*$ is strictly 1²-starcompact. Note that $R^*$ is a $T_1$-space which is not Hausdorff. We cannot
construct a strictly $1\frac{1}{2}$-starcompact Hausdorff space which is not strictly starcompact yet.

It is known that the Tychonoff plank $T = (\omega_1 + 1) \times (\omega + 1) \setminus \{ (\omega_1, \omega) \}$ is $1\frac{1}{2}$-starcompact ([1],[7]), whereas we show it is not even strictly 2-starcompact.

**Example 4.6.** The Tychonoff plank $T$ is $1\frac{1}{2}$-starcompact but not strictly 2-starcompact.

**Proof.** Define $U = \{(0, \alpha) \times (\omega + 1) \mid \alpha < \omega_1\} \cup \{(\omega_1 + 1) \times \{n\} \mid n < \omega\}$. Then $U$ is an open cover of $T$. For every $n < \omega$, we have $\text{St}((\omega_1, n), U) = (\omega_1 + 1) \times \{n\}$ since $(\omega_1 + 1) \times \{n\}$ is the only element of $U$ containing $(\omega_1, n)$. Then we have $\text{St}^2((\omega_1, n), U) = (\omega_1 \times (\omega + 1)) \cup \{(\omega_1, n)\}$. Hence, the subset $A = \{\omega_1\} \times \omega$ of $T$ satisfies $\text{St}(A, U) = T$.

Take a finite subset $F$ of $A$ arbitrarily. Then $\text{St}^2(F, U) \neq T$, and hence $T$ is not strictly 2-starcompact.

H. Ohta pointed out that the Tychonoff plank $T$ is not strictly star-Lindelöf. He proved the fact by showing that the Tychonoff plank $T$ contained the closed subspace $((\omega_1 + 1) \times \{0\}) \cup (\omega_1 \times \{\omega\})$ which is not strictly star-Lindelöf. We apply the idea to the following stronger result.

**Example 4.7.** The Tychonoff plank $T$ is not strictly 2-star-Lindelöf.

**Proof.** For each $\alpha < \omega_1$, set $U_\alpha = ([0, \alpha] \times [2, \omega]) \cup \{(\alpha+1, 1)\}$ and $V_\alpha = \{(\alpha+1, 0), (\alpha+1, 1)\}$. Let $A = (\omega_1 + 1) \times \{0\}$ and $W = (\omega_1 + 1) \times [1, \omega)$. Define $\mathcal{U} = \{U_\alpha \mid \alpha < \omega_1\} \cup \{V_\alpha \mid \alpha < \omega_1\} \cup \{A, W\}$. Then $\mathcal{U}$ is an open cover of $T$ and we have that $\text{St}^2(A, U) = T$.

Let $B$ be a countable subset of $A$. Set $\beta_0 = \sup\{\beta \mid (\beta, 0) \in B \setminus \{\omega_1\}\}$. Then we have $\text{St}^2((\beta_0 + 1, \omega), U) = W \cup \left( \bigcup \{V_\gamma \mid \beta_0 + 1 \leq \gamma < \omega_1\} \right) \cup \left( \bigcup \{U_\alpha \mid \alpha < \omega_1\} \right)$ because $\text{St}((\beta_0 + 1, \omega), U) = \bigcup \{U_\gamma \mid \beta_0 + 1 \leq \gamma < \omega_1\}$. Hence $\text{St}^2((\beta_0 + 1, \omega), U) \cap B = \emptyset$. Therefore $T$ is not strictly 2-star-Lindelöf.

Moreover, under $2^{\omega_1} = 2^\omega$, the Scott-type fat $\Psi$-space is a $2\frac{1}{2}$-star-Lindelöf completely regular space which is not 2-star-Lindelöf ([7]). Then this space is strictly $2\frac{1}{2}$-star-Lindelöf but not strictly 2-star-Lindelöf. On the other hand, under CH, the Scott-type fat $\Psi$-space is a $2\frac{1}{2}$-starcompact completely regular space which is not 2-starcompact ([1], [10]). Then this space is a strictly $2\frac{1}{2}$-starcompact space that is not strictly 2-starcompact.

Here, we give the following diagrams which illustrate relationships among star covering properties discussed above.

In the diagrams, the symbols $a \rightarrow b$ and $a \rightarrow b$ mean that $a$ implies $b$, and $a$ does not necessarily imply $b$, respectively. We list corresponding counterexamples by the side of the symbols $a \rightarrow b$. 
(ω₁-compact) \[\xrightarrow{\text{Ex. 4.2}}\] (strictly star-Lindelöf) \[\xrightarrow{\text{Ex. 4.1}}\] (Lindelöf)

(star-Lindelöf) \[\xrightarrow{\text{Ex. 4.3}}\] (strictly star-Lindelöf)

(1\frac{1}{2}-star-Lindelöf) \[\xrightarrow{\text{Ex. 4.4, 4.5, 4.7}}\] (strictly 1\frac{1}{2}-star-Lindelöf)

(2-star-Lindelöf) \[\xrightarrow{\text{Ex. 4.4, 4.5, 4.7}}\] (strictly 2-star-Lindelöf)

(2\frac{1}{2}-star-Lindelöf) \[\xrightarrow{T_3, (DCCC)}\] (strictly 2\frac{1}{2}-star-Lindelöf)

(\omega-star-Lindelöf) \[\xrightarrow{T_3}\] (strictly \omega-star-Lindelöf)

\[\text{Diagram 1}\]

(countably compact) \[\xrightarrow{\text{Ex. 4.2}}\] (strictly starcompact)

(\omega-starcompact) \[\xrightarrow{T_2}\] (strictly starcompact)

(\omega-starcompact) \[\xrightarrow{\text{Ex. 4.6}}\] (strictly starcompact)

(\omega-starcompact) \[\xrightarrow{\text{Ex. 4.6, 4.7}}\] (strictly starcompact)

(\omega-starcompact) \[\xrightarrow{T_3, (DFCC)}\] (strictly \omega-starcompact)

(\omega-starcompact) \[\xrightarrow{T_3}\] (strictly \omega-starcompact)

\[\text{Diagram 2}\]
At the last of this section, we give remarks on examples which are given above.

Remark 4.8. As to subspaces, closed subspaces of strictly $k$-star-Lindelöf spaces are not necessarily strictly $k$-star-Lindelöf for every $k \in \mathbb{N}$ with $k \geq 1\frac{1}{2}$. For, the $\Psi$-space, constructed from a m.a.d.family $\mathcal{A}$ of uncountably many infinite subsets of $\omega$, is $k$-star-Lindelöf for every $k \geq 1\frac{1}{2}$ by Example 4.3. However, the subspace $\mathcal{A}$ does not satisfy the DCCC since $\mathcal{A}$ is uncountable discrete and closed in $\Psi$. Therefore $\mathcal{A}$ is not strictly $k$-star-Lindelöf for every $k \in \mathbb{N}$ with $k \geq 1\frac{1}{2}$.

Remark 4.9. The idea in Ohta’s proof stated above also suggests that the topological sum $\omega_1 \oplus (\omega_1 + 1)$ is not strictly 1\frac{1}{2}-star-Lindelöf. Hence a topological sum of a strictly star-Lindelöf space and a compact space is not even strictly 1\frac{1}{2}-star-Lindelöf. Furthermore, the proof of Example 4.7 shows that $\omega_1 \oplus (\omega_1 + 1) \oplus (\omega_1 + 1)$ is not strictly 2-star-Lindelöf.

Remark 4.10. Concerning product spaces, Example 4.2 shows that a product of a strictly star-Lindelöf (respectively, strictly star-compact) space and a compact space need not be strictly star-Lindelöf (respectively, strictly star-compact). Moreover, Example 4.5 shows that a product of a strictly 1\frac{1}{2}-star-Lindelöf (respectively, strictly 2-star-Lindelöf) space and a compact space need not be strictly 1\frac{1}{2}-star-Lindelöf (respectively, strictly 2-star-Lindelöf).

Furthermore, by a similar argument in the proof of Example 4.7, we can show that neither $\omega_1 \times (\omega + 1)$ nor $\omega_1 \times (\omega_1 + 1)$ are strictly 2-star-Lindelöf strictly 1\frac{1}{2}-star-Lindelöf. Therefore, a product of a strictly star-Lindelöf space with a separable metric space is not even strictly 2-star-Lindelöf.

The following problems are not solved yet.

Problem 4.11. Does there exist a strictly 2-star-Lindelöf space which is not strictly 1\frac{1}{2}-star-Lindelöf?

Problem 4.12. Does there exist a strictly 2-star-compact space which is not strictly 1\frac{1}{2}-star-compact?

5. Concluding remarks. As we mentioned above, strict star-Lindelöfness is not necessarily preserved by taking topological sums. Accordingly, we introduce the following notion of spaces with possible properties of strictly star-Lindelöf spaces so as to be preserved by taking topological sums.

Definition 5.1. A space $X$ satisfies the condition $(\ast)$ if for each open cover $U$ of $X$, there exists an open refinement $V$ of $U$ such that every subset $A$ of $X$ satisfying $\text{St}(A, V) = X$ contains a countable subset $B$ of $A$ satisfying $\text{St}(B, V) = X$.

Then we easily have that every strictly star-Lindelöf space satisfies the condition $(\ast)$, and every space satisfying the condition $(\ast)$ is $\omega_1$-compact.

We can show that the condition $(\ast)$ is preserved by taking a topological sum of countably many spaces satisfying the condition $(\ast)$.

Proposition 5.2. If $\{X_n \mid n \in \mathbb{N}\}$ is a countable family of spaces satisfying the condition $(\ast)$, then $\bigoplus_{n \in \mathbb{N}} X_n$ also satisfies the condition $(\ast)$.

We conclude this paper with the following example.

Recall that the Tychonoff plank $T$ is not even strictly 2-star-Lindelöf (see Example 4.7).

Example 5.3. The Tychonoff plank $T$ satisfies the condition $(\ast)$. 


Proof. Let $\mathcal{U}$ be an open cover of $T$. For every $\langle \alpha, n \rangle \in T \setminus (\omega_1 \times \omega) \cup (\omega_1 \times \{\omega\})$, take a $U_{\alpha,n} \in \mathcal{U}$ so that $\langle \alpha, n \rangle \in U_{\alpha,n}$. Then we can choose a $\beta_\alpha < \alpha$ so that $\langle \alpha, n \rangle \in (\beta_\alpha, \alpha] \times \{n\} \subset U_{\alpha,n}$. Put $V_{\alpha,n} = (\beta_\alpha, \alpha] \times \{n\}$.

For every $n < \omega$, pick a $U_n \in \mathcal{U}$ so that $\langle \omega_1, n \rangle \in U_n$. We can take an $\alpha_n < \omega_1$ such that $\langle \omega_1, n \rangle \in (\alpha_n, \omega_1] \times \{n\} \subset U_n$. Put $V_n = (\alpha_n, \omega_1] \times \{n\}$ for each $n < \omega$. Now, define $\xi = \sup\{\alpha_n \mid n < \omega\}$. Then $\xi < \omega_1$.

For every $\alpha < \omega_1$, pick a $U_\alpha \in \mathcal{U}$ so that $\langle \alpha, \omega \rangle \in U_\alpha$. Then there exist a $\beta_\alpha < \alpha$ and an $n_\alpha < \omega$ such that $\langle \alpha, \omega \rangle \in (\beta_\alpha, \alpha) \times (n_\alpha, \omega] \subset U_\alpha$. For each $\alpha < \omega_1$, set $V_\alpha = (\beta_\alpha, \alpha) \times (n_\alpha, \omega]$. By the pressing-down lemma ([6]), there exist an $\alpha_0 < \omega_1$, an $m_0 < \omega$ and a cofinal subset $C$ of $\omega_1$ such that $\beta_\alpha = \alpha_0$ and $n_\alpha = m_0$ for every $\alpha \in C$. Define

$$V = \{V_{\alpha,n} \mid \alpha < \omega_1, n < \omega\} \cup \{V_n \mid n < \omega\} \cup \{V_\alpha \mid \alpha \in [0, \alpha_0] \cup (C \cap (\alpha_0, \omega_1))\}.$$

Then $V$ is an open refinement of $\mathcal{U}$. Let $A$ be a subset of $T$ such that $\text{St}(A, V) = T$. Note that $V_\gamma = (\alpha_0, \gamma] \times (m_0, \omega]$ for every $\gamma \in C \cap (\alpha_0, \omega_1)$.

For every $n < \omega$, we have that $V_n$ is the only member of $V$ containing $\langle \omega_1, n \rangle$. Hence for every $n < \omega$ there exists an $a_n \in A$ such that $a_n \in V_n$. Set $A_1 = \{a_n \mid n < \omega\}$. We have that $\text{St}(A_1, V) = (\xi, \omega_1] \times \omega$ for some $\xi < \omega_1$.

Here, for the $\alpha_0 + 1$ there exist a $\gamma_0 \in C \cap (\alpha_0, \omega_1)$ and an $x_0 \in A$ such that $\langle \alpha_0 + 1, \omega \rangle \in V_{\gamma_0}$ and $x_0 \in V_{\gamma_0}$. Then we have $\text{St}(x_0, V) = (\alpha_0, \omega_1] \times (m_0, \omega]$.

Let $\eta = \max\{\alpha_0, \xi\}$. Then $\eta < \omega_1$. Because $[0, \eta + 1] \times (\omega + 1)$ is a compact subset of $T$, there exists a finite subset $A_2$ of $A$ such that $\text{St}(A_2, V) \supset [0, \eta + 1] \times (\omega + 1)$.

Therefore we have $\text{St}(A_1 \cup A_2 \cup \{x_0\}, V) = T$, and hence $T$ satisfies the condition $(\ast)$.

References


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