ON STRICTLY STAR-LINDELÖF SPACES

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Abstract. In this paper, we introduce new properties of topological spaces defined by stars of coverings, which are called strict star-Lindelöfness and strict starcompactness. Following the fundamental studies on star covering properties by E.K.van Douwen, G.M.Reed, A.W.Roscoe and I.J.Tree ([1]), there have been several related studies (see M.V.Matveev [7], [8], [9], for survey). Star-Lindelöf spaces have many nice properties. However, star-Lindelöfness is not preserved by closed subspaces ([7]). We define strict star-Lindelöfness to modify this defect and still so as to keep possible properties of star-Lindelöfness. Furthermore, we investigate relationships among these covering properties and give various examples.

1. Introduction and preliminaries. In this paper, all spaces are assumed to be T_1.
For a cover $\mathcal{U}$ of a space X and a subset A of X,

$$\mathrm{St}(A,\mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid U \cap A \neq \emptyset\}$$

is called a *star* of A (with respect to $\mathcal{U}$). Define $\mathrm{St}^0(A,\mathcal{U}) = A$, $\mathrm{St}^1(A,\mathcal{U}) = \mathrm{St}(A,\mathcal{U})$ and $\mathrm{St}^{n+1}(A,\mathcal{U}) = \mathrm{St}(\mathrm{St}^n(A,\mathcal{U}),\mathcal{U})$ for $n \in \mathbb{N}$. For a singleton $A = \{x\}$, we usually write $\mathrm{St}(x,\mathcal{U})$ instead of $\mathrm{St}(\{x\},\mathcal{U})$.

W.M.Fleischman [3] defined the following notion of starcompact spaces and studied its properties.

Definition 1.1 ([3]). A space X is *starcompact* if for every open cover $\mathcal{U}$ of X, there exists a finite subset F of X such that $\mathrm{St}(F,\mathcal{U}) = X$.

He proved in [3] that starcompactness is equivalent to countable compactness in the class of regular spaces. It was informed in [3] that R.S.Houston afterwards showed the equivalence in the class of Hausdorff spaces.

The following notion of star-Lindelöf spaces is defined by S.Ikenaga [5] originally under the name of ω-star spaces, the present term was given by E.K.van Douwen, G.M.Reed, A.W.Roscoe and I.J.Tree [1].

Definition 1.2 ([1]). A space is *star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there exists a countable subset A of X such that $\mathrm{St}(A,\mathcal{U}) = X$.

We call such notions of topological spaces defined by taking stars of coverings *star covering properties*.

Later, E.K.van Douwen, G.M.Reed, A.W.Roscoe and I.J.Tree [1] established the fundamentals of star covering properties. Subsequently, there have been several related studies. M.V.Matveev [7], [8], [9] presented a nice exposition of star covering properties which contain many significant results. Among star covering properties, star-Lindelöfness has so far

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most appeared in the related studies. Many covering properties are preserved by closed subspaces, while star-Lindelöfness is not the case ([7]).

The aim of this paper is to introduce the following new notion of spaces, called strictly star-Lindelöf spaces, which will be shown to have possible properties of star-Lindelöf spaces so as to improve the above defect on closed heredity.

Definition 1.3. A space X is *strictly star-Lindelöf* if for every open cover $\mathcal{U}$ of X and for every subset A of X satisfying $\mathrm{St}(A,\mathcal{U}) = X$ there exists a countable subset B of A such that $\mathrm{St}(B,\mathcal{U}) = X$.

We also define the following notion, called strictly starcompact spaces, which is related to starcompact spaces.

Definition 1.4. A space X is *strictly starcompact* if for every open cover $\mathcal{U}$ of X and for every subset A of X satisfying $\mathrm{St}(A,\mathcal{U}) = X$ there exists a finite subset F of A such that $\mathrm{St}(F,\mathcal{U}) = X$.

We investigate properties of strictly star-Lindelöf spaces in Section 2., and strictly starcompact spaces in Section 3.. In Section 4. various examples on strictly star-Lindelöf spaces and other related spaces will be given.

We denote $\mathbb{N}_{\frac{1}{2}} = \{n + \frac{1}{2} \,|\, n \in \mathbb{N} \cup \{0\}\}$ and $\widetilde{\mathbb{N}} = \mathbb{N} \cup \mathbb{N}_{\frac{1}{2}} \cup \{0\}$ following [7], and other notations and terminology are as in [2].

2. Strictly star-Lindelöf spaces.

In this section, we consider strict star-Lindelöfness. To begin with, we recall the definition of k-star-Lindelöfness.

For $k \in \mathbb{N} \cup \{0\}$, a space X is *k-star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there exists a countable subset A of X such that $\mathrm{St}^k(A,\mathcal{U}) = X$. A space X is *$k\frac{1}{2}$-star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there exists a countable subcollection $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{St}^k(\bigcup\mathcal{V},\mathcal{U}) = X$. Moreover, a space X is *ω-star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there exist an $n \in \mathbb{N}$ and a countable subset A of X such that $\mathrm{St}^n(A,\mathcal{U}) = X$. Note that $\frac{1}{2}$-star-Lindelöfness and 1-star-Lindelöfness are precisely Lindelöfness and star-Lindelöfness, respectively ([7]).

Just like k-star-Lindelöfness, we define the following notions of spaces that are weaker than strict star-Lindelöfness.

Definition 2.1. Let X be a space and $k \in \mathbb{N} \cup \{0\}$.

(1) X is *strictly k-star-Lindelöf* if for every open cover $\mathcal{U}$ of X and every subset A of X satisfying $\mathrm{St}^k(A,\mathcal{U}) = X$, there exists a countable subset B of A such that $\mathrm{St}^k(B,\mathcal{U}) = X$.

(2) X is *strictly $k\frac{1}{2}$-star-Lindelöf* if for every open cover $\mathcal{U}$ of X and every subcollection $\mathcal{V}$ of $\mathcal{U}$ satisfying $\mathrm{St}^k(\bigcup\mathcal{V},\mathcal{U}) = X$, there exists a countable subcollection $\mathcal{W}$ of $\mathcal{V}$ such that $\mathrm{St}^k(\bigcup\mathcal{W},\mathcal{U}) = X$.

(3) X is *strictly ω-star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there exists an $n \in \mathbb{N}$ such that for every subset A of X satisfying $\mathrm{St}^n(A,\mathcal{U}) = X$, there exists a countable subset B of A such that $\mathrm{St}^n(B,\mathcal{U}) = X$.

Obviously, strict $\frac{1}{2}$-star-Lindelöfness and strict 1-star-Lindelöfness are precisely Lindelöfness and strict star-Lindelöfness, respectively.

By Definition 2.1, it is clear that every strictly k-star-Lindelöf space is k-star-Lindelöf for every $k \in \widetilde{\mathbb{N}}$, and every strictly ω-star-Lindelöf space is ω-star-Lindelöf.

Furthermore, for every $k \in \mathbb{N}$ we have that every strictly k-star-Lindelöf space is strictly $k\frac{1}{2}$-star-Lindelöf, and every strictly $k\frac{1}{2}$-star-Lindelöf space is strictly $(k+1)$-star-Lindelöf (see Diagram 1 in Section 4.).

Now, recall that a space X satisfies the *discrete countable chain condition* (DCCC, for short) if every discrete collection of non-empty open sets in X is countable ([1]).

Here, we consider properties of strictly k-star-Lindelöf spaces. At first, similarly to the case of k-star-Lindelöfness ([7]), we have the following relations between strict k-star-Lindelöfness and the DCCC.

Theorem 2.2. *For a regular space X, the following are equivalent.*
(a) *X is strictly $2\frac{1}{2}$-star-Lindelöf.*
(b) *X is strictly k-star-Lindelöf for every $k \in \widetilde{\mathbb{N}}$ with $k \geq 3$.*
(c) *X is strictly ω-star-Lindelöf.*
(d) *X satisfies the* DCCC.

Proof. The implications $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ are trivial.

$(c) \Rightarrow (d)$: This follows from the fact that every ω-star-Lindelöf regular space satisfies the DCCC ([7]).

$(d) \Rightarrow (a)$: Suppose that X is regular but not strictly $2\frac{1}{2}$-star-Lindelöf. Then there exist an open cover $\mathcal{U}$ of X and a subcollection $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{St}^2(\bigcup\mathcal{V},\mathcal{U}) = X$ and $\mathrm{St}^2(\bigcup\mathcal{A},\mathcal{U}) \neq X$ for any countable subcollection $\mathcal{A}$ of $\mathcal{V}$.

Let $\alpha < \omega_1$. Suppose that a subset $\{x_\beta \,|\, \beta < \alpha\}$ of X and a subcollection $\{U_\beta \,|\, \beta < \alpha\}$ of $\mathcal{V}$ satisfying $x_\beta \in X \setminus \mathrm{St}^2(\bigcup_{\gamma<\beta} U_\gamma,\mathcal{U})$ are given for every $\beta < \alpha$. By the assumption above, we can take $x_\alpha \in X \setminus \mathrm{St}^2(\bigcup_{\beta<\alpha} U_\beta,\mathcal{U})$. Then there exists a $U_\alpha \in \mathcal{V}$ such that $x_\alpha \in \mathrm{St}^2(U_\alpha,\mathcal{U})$. Set $\mathcal{W} = \{U_\alpha \,|\, \alpha < \omega_1\}$. Then one can show that $\mathcal{W}$ is an uncountable discrete collection consisting of non-empty open sets in X. Therefore X does not satisfy the DCCC. □

Now, a space X is ω_1-*compact* if every uncountable subset of X has an accumulation point. Every countably compact space and every Lindelöf space are ω_1-compact, and every ω_1-compact space is star-Lindelöf.

The following theorem shows that strictly star-Lindelöf spaces are located between Lindelöf spaces and ω_1-compact spaces.

Theorem 2.3. *Every strictly star-Lindelöf space is ω_1-compact.*

Proof. Suppose that X is not ω_1-compact. Then there exists an uncountable subset A of X with no accumulation points. For each $a \in A$, take a neighborhood U_a of a so that $U_a \cap A = \{a\}$, and define $\mathcal{U} = \{U_a \,|\, a \in A\} \cup \{X \setminus A\}$. Then $\mathcal{U}$ is an open cover of X. Pick $x_0 \in X \setminus A$ arbitrarily. We have $\mathrm{St}(A \cup \{x_0\},\mathcal{U}) = X$, but no countable subset B of A satisfies $\mathrm{St}(B \cup \{x_0\},\mathcal{U}) = X$. Thus X is not strictly star-Lindelöf. □

The converse of Theorem 2.3 need not be true (see Example 4.2 below).

It is known that every separable space is star-Lindelöf ([7]). Here we have the following theorem in the case of strictly k-star-Lindelöf spaces.

Theorem 2.4. *Every separable space is strictly $1\frac{1}{2}$-star-Lindelöf.*

Proof. Let X be a separable space and D a countable dense subset of X. Let $\mathcal{U}$ be an open cover of X and $\mathcal{V}$ a subcollection of $\mathcal{U}$ satisfying $\mathrm{St}(\bigcup\mathcal{V},\mathcal{U}) = X$. For each $x \in X$, there exist a $V_x \in \mathcal{V}$ and a $U_x \in \mathcal{U}$ such that $x \in U_x$ and $V_x \cap U_x \neq \emptyset$. Then we can take $d_x \in D \cap (V_x \cap U_x)$ for every $x \in X$. Put $D' = \{d_x \,|\, x \in X\}$. Then D' is a countable subset of D satisfying $\mathrm{St}(D',\mathcal{U}) = X$. Denote $D' = \{d_n \,|\, n \in \mathbb{N}\}$. For each $d_n \in D'$, choose a $V_n \in \{V_x \,|\, x \in X\}$ so that $d_n \in V_n$. Define $\mathcal{W} = \{V_n \,|\, n \in \mathbb{N}\}$. Then $\mathcal{W}$ is a countable subcollection of $\mathcal{V}$ satisfying $\mathrm{St}(\bigcup\mathcal{W},\mathcal{U}) = X$. Therefore, X is strictly $1\frac{1}{2}$-star-Lindelöf. □

On the other hand, we cannot conclude that every separable space is strictly star-Lindelöf (see Example 4.3).

It is known that every open F_σ-set of a star-Lindelöf space is star-Lindelöf ([7]). Concerning strictly star-Lindelöf spaces, we have

Theorem 2.5. *Every F_σ-set of a strictly star-Lindelöf space is strictly star-Lindelöf.*

Proof. Let X be a strictly star-Lindelöf space and $Y = \bigcup_{n\in\mathbb{N}} H_n$ an F_σ-set of X, where H_n is a closed set in X for every $n \in \mathbb{N}$.

Let $\mathcal{U}$ be an open cover of Y and A a subset of Y satisfying $\mathrm{St}(A,\mathcal{U}) = Y$. For each $U \in \mathcal{U}$, take an open set V_U in X so that $V_U \cap Y = U$. Set $\mathcal{V} = \{V_U \mid U \in \mathcal{U},\ U \cap A \neq \emptyset\}$.

If $\bigcup \mathcal{V} = X$, notice that $\mathrm{St}(A, \mathcal{V}) = X$. Since X is strictly star-Lindelöf, there exists a countable subset B of A such that $\mathrm{St}(B, \mathcal{V}) = X$. Then, we have $\mathrm{St}(B,\mathcal{U}) \supset \mathrm{St}(B, \mathcal{V}) \cap Y = Y$.

Suppose $\bigcup \mathcal{V} \neq X$. Then, fix $n \in \mathbb{N}$. Note that $\mathcal{V}_n = \mathcal{V} \cup \{X \setminus H_n\}$ is an open cover of X. Choose $x_0 \in X \setminus \bigcup \mathcal{V}$ and put $A' = A \cup \{x_0\}$. Then A' satisfies $\mathrm{St}(A', \mathcal{V}_n) = X$. Since X is strictly star-Lindelöf, there exists a countable subset B'_n of A' such that $\mathrm{St}(B'_n, \mathcal{V}_n) = X$. Let $B_n = B'_n \setminus \{x_0\}$. Then B_n is a countable subset of A satisfying $\mathrm{St}(B_n,\mathcal{U}) \supset \mathrm{St}(B_n, \mathcal{V}) \cap Y \supset H_n$.

Let us set $B = \bigcup_{n\in\mathbb{N}} B_n$. Then B is a countable subset of A, and we have

$$\mathrm{St}(B,\mathcal{U}) = \bigcup_{n\in\mathbb{N}} \mathrm{St}(B_n,\mathcal{U}) \supset \bigcup_{n\in\mathbb{N}} H_n = Y.$$

Therefore Y is strictly star-Lindelöf. □

As opposed to star-Lindelöfness, we have

Corollary 2.6. *Every closed subspace of a strictly star-Lindelöf space is also strictly star-Lindelöf.*

On the other hand, strict k-star-Lindelöfness is not necessarily preserved by closed subspaces for every $k \in \widetilde{\mathbb{N}}$ with $k \geq 1\frac{1}{2}$ (see Section 4.).

A subset A of a space X is called a *cozero-set* if there is a continuous function $f : X \to \mathbb{R}$ such that $A = \{x \in X \mid f(x) \neq 0\}$.

Corollary 2.7. *Every cozero-set of a strictly star-Lindelöf space is also strictly star-Lindelöf.*

It is also known that every continuous image of a star-Lindelöf space is star-Lindelöf ([5],[7]). Likewise, we have

Theorem 2.8. *Every continuous image of a strictly star-Lindelöf space is also strictly star-Lindelöf.*

Proof. Let X be a strictly star-Lindelöf space, Y a space and $f : X \to Y$ a continuous mapping from X onto Y.

Let $\mathcal{U}$ be an open cover of Y and A a subset of Y satisfying $\mathrm{St}(A,\mathcal{U}) = Y$. Put $\mathcal{V} = \{f^{-1}(U) \mid U \in \mathcal{U}\}$. Then $\mathcal{V}$ is an open cover of X satisfying $\mathrm{St}(f^{-1}(A), \mathcal{V}) = X$. Since X is strictly star-Lindelöf, there is a countable subset B of $f^{-1}(A)$ such that $\mathrm{St}(B, \mathcal{V}) = X$. Then $f(B)$ is countable and $f(B) \subset A$. We have $\mathrm{St}(f(B),\mathcal{U}) = Y$. Thus Y is strictly star-Lindelöf. □

3. Strictly starcompact spaces. Next, we consider properties of strictly starcompact spaces. We recall the definition of k-starcompactness.

For $k \in \mathbb{N} \cup \{0\}$, a space X is *k-starcompact* if for each open cover $\mathcal{U}$ of X, there exists a finite subset F of X such that $\mathrm{St}^k(F,\mathcal{U}) = X$. A space X is *$k\frac{1}{2}$-starcompact* if for each open cover $\mathcal{U}$ of X, there exists a finite subcollection $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{St}^k(\bigcup\mathcal{V},\mathcal{U}) = X$. Moreover, a space X is *ω-starcompact* if for every open cover $\mathcal{U}$ of X, there exist an $n \in \mathbb{N}$ and a finite subset A of X such that $\mathrm{St}^n(A,\mathcal{U}) = X$. Note that $\frac{1}{2}$-starcompactness and 1-starcompactness are precisely compactness and starcompactness, respectively ([7]).

Now, we define the following notions of spaces that are related to strict starcompactness.

Definition 3.1. Let X be a space and $k \in \mathbb{N} \cup \{0\}$.

(1) X is *strictly k-starcompact* if for every open cover $\mathcal{U}$ of X and every subset A of X satisfying $\mathrm{St}^k(A,\mathcal{U}) = X$, there exists a finite subset F of A such that $\mathrm{St}^k(F,\mathcal{U}) = X$.

(2) X is *strictly $k\frac{1}{2}$-starcompact* if for every open cover $\mathcal{U}$ of X and every subcollection $\mathcal{V}$ of $\mathcal{U}$ satisfying $\mathrm{St}^k(\bigcup\mathcal{V},\mathcal{U}) = X$, there exists a finite subcollection $\mathcal{A}$ of $\mathcal{V}$ such that $\mathrm{St}^k(\bigcup\mathcal{A},\mathcal{U}) = X$.

(3) X is *strictly ω-starcompact* if for every open cover $\mathcal{U}$ of X, there exists an $n \in \mathbb{N}$ such that for every subset A of X satisfying $\mathrm{St}^n(A,\mathcal{U}) = X$, there exists a finite subset F of A such that $\mathrm{St}^n(F,\mathcal{U}) = X$.

In particular, strict $\frac{1}{2}$-starcompactness and strict 1-starcompactness are precisely compactness and strict starcompactness, respectively.

It follows from Definition 3.1 that every strictly k-starcompact space is k-starcompact for every $k \in \widetilde{\mathbb{N}}$, and every strictly ω-starcompact space is ω-starcompact.

In addition, every strictly k-starcompact space is clearly strictly k-star-Lindelöf, and every strictly ω-starcompact space is strictly ω-star-Lindelöf.

Furthermore, for every $k \in \mathbb{N}$ we have that every strictly k-starcompact space is strictly $k\frac{1}{2}$-starcompact, and every strictly $k\frac{1}{2}$-starcompact space is strictly $(k+1)$-starcompact (see Diagram 2 in Section 4.).

Now, recall that a space X satisfies the *discrete finite chain condition* (DFCC, for short) if every discrete collection of non-empty open sets in X is finite ([1]). Similarly to the case of k-starcompact spaces ([7]), we have the following relations between strict k-starcompactness and the DFCC.

Theorem 3.2. *For a regular space X, the following are equivalent.*
(a) *X is strictly $2\frac{1}{2}$-starcompact.*
(b) *X is strictly k-starcompact for every $k \in \widetilde{\mathbb{N}}$ with $k \geq 3$.*
(c) *X is strictly ω-starcompact.*
(d) *X satisfies the* DFCC.

It is known that every countably compact Lindelöf space is compact. The following result seems to be interesting in itself; the proof are easy and omitted.

Theorem 3.3. *A space X is strictly starcompact if and only if X is countably compact and strictly star-Lindelöf.*

Strictly starcompact spaces have the following properties similar to strictly star-Lindelöf spaces. Proofs are similar to the case of strictly star-Lindelöf spaces.

Theorem 3.4. *Every closed subspace of a strictly starcompact space is also strictly starcompact.*

Theorem 3.5. *Every continuous image of a strictly starcompact space is also strictly starcompact.*

4. Examples. In this section, we list various examples on strictly k-star-Lindelöf spaces and strictly k-starcompact spaces.

Let an infinite ordinal τ have the order topology. The symbol βX is the Stone-Čech compactification of a completely regular space X.

To begin with, we give an example of a strictly star-Lindelöf space which is not Lindelöf. It shows the gap between strict $\frac{1}{2}$-star-Lindelöfness and strict 1-star-Lindelöfness.

Example 4.1. The space ω_1 is strictly starcompact (and hence strictly star-Lindelöf).

Proof. Let $\mathcal{U}$ be an open cover of ω_1 and A a subset of ω_1 satisfying $\mathrm{St}(A,\mathcal{U}) = \omega_1$. For each $\alpha \in \omega_1$, there exist a $U_\alpha \in \mathcal{U}$ and a $\gamma_\alpha < \omega_1$ such that $(\gamma_\alpha, \alpha] \subset U_\alpha$. If A is not cofinal in ω_1, A itself is countable. Hence we can assume that A is cofinal in ω_1. By the pressing-down lemma ([6]), there exist an $\alpha_0 < \omega_1$ and a cofinal subset C of ω_1 such that $\gamma_\alpha < \alpha_0$ for every $\alpha \in C$.

Because A is cofinal in ω_1, there is a $\xi \in A$ with $\alpha_0 < \xi$ such that $\gamma_\alpha < \alpha_0 < \xi$ for any $\alpha \in C \cap (\xi, \omega_1)$. Then for every $\beta \in (\xi, \omega_1)$, we can take an $\eta \in C$ so that $\xi < \beta < \eta$. Thus there exists a $U \in \mathcal{U}$ such that $\beta, \xi \in (\gamma_\eta, \eta] \subset U$. Hence $\beta \in \mathrm{St}(\xi,\mathcal{U})$. Therefore, we have $\mathrm{St}(\xi,\mathcal{U}) \supset (\xi, \omega_1)$.

Moreover, for each $\gamma \leq \xi$ there is an $a_\gamma \in A$ such that $\gamma \in \mathrm{St}(a_\gamma,\mathcal{U})$. Then $\{\mathrm{St}(a_\gamma,\mathcal{U}) \mid \gamma \leq \xi\}$ is an open cover of $[0, \xi]$. Since $[0, \xi]$ is compact, we can take finitely many $\gamma_1, \ldots, \gamma_n \leq \xi$ so that $\{\mathrm{St}(a_{\gamma_i},\mathcal{U}) \mid i = 1, \cdots, n\}$ covers $[0, \xi]$. Let $F = \{a_{\gamma_i} \mid i = 1, \ldots, n\} \cup \{\xi\}$. Then F is a finite subset of A satisfying $\mathrm{St}(F,\mathcal{U}) = \omega_1$. Hence ω_1 is strictly starcompact. □

Therefore, the space ω_1 is also an example of a non-compact strictly starcompact space.

Example 4.2. The space $\omega_1 \times (\omega_1 + 1)$ is ω_1-compact but not strictly star-Lindelöf.

Proof. Define

$$\mathcal{U} = \{[0, \alpha] \times (\alpha, \omega_1] \mid \alpha < \omega_1\} \cup \{\omega_1 \times \omega_1\}$$

and

$$A = \{\langle \alpha, \beta \rangle \in \omega_1 \times (\omega_1 + 1) \mid \alpha < \beta < \omega_1\}.$$

Then $\mathcal{U}$ is an open cover of $\omega_1 \times (\omega_1 + 1)$ and we have $\mathrm{St}(A,\mathcal{U}) = \omega_1 \times (\omega_1 + 1)$.

Let C be an arbitrary countable subset of A. Define

$$\alpha_0 = \sup\{\alpha \mid \langle \alpha, \beta \rangle \in C \text{ for some } \beta\} \text{ and } \beta_0 = \sup\{\beta \mid \langle \alpha, \beta \rangle \in C \text{ for some } \alpha\}.$$

Then $\alpha_0 < \beta_0 < \omega_1$. If $\gamma > \beta_0$, then we have

$$\{U \in \mathcal{U} \mid (\gamma, \omega_1) \in U\} = \{[0, \alpha] \times (\alpha, \omega_1] \mid \alpha \geq \gamma\}.$$

For any $\xi \geq \gamma$, $[0, \xi] \times (\xi, \omega_1]$ contains no points of C. Hence $\langle \gamma, \omega_1 \rangle \notin \mathrm{St}(C,\mathcal{U})$. □

Hence, the space $\omega_1 \times (\omega_1 + 1)$ is a countably compact but not strictly starcompact. In addition, we also have that $\omega_1 \times (\omega_1 + 1)$ is not strictly 2-star-Lindelöf (see Remark 4.10 below).

Moreover we obtain the following example.

Example 4.3. There exists a star-Lindelöf completely regular space that is strictly $1\frac{1}{2}$-star-Lindelöf but not ω_1-compact.

Proof. Let ω be a countable discrete space and $\mathcal{A}$ be a maximal almost disjoint family (m.a.d.family, for short) of infinite subsets of ω. Put $\Psi = \omega \cup \mathcal{A}$. Topologize Ψ by letting ω be an open subspace of Ψ and defining a local base $\mathcal{N}(x)$ at each $x \in \mathcal{A}$ by $\mathcal{N}(x) = \{\{x\} \cup (x \setminus F) \mid F \in [x]^{<\omega}\}$. This space is called a Ψ*-space* ([4]). Then it is known that Ψ is a separable completely regular space which is 2-starcompact but neither $1\frac{1}{2}$-starcompact nor ω_1-compact ([1],[7]).

Since Ψ is separable, Ψ is star-Lindelöf. Moreover, Ψ is strictly $1\frac{1}{2}$-star-Lindelöf by Theorem 2.4. $\square$

Now, we construct the following spaces so as to obtain a $1\frac{1}{2}$-star-Lindelöf space that is not strictly 2-star-Lindelöf.

For a completely regular space X and an infinite cardinal τ with $\mathrm{cf}(\tau) > \omega$, the space

$$N_\tau X = ((\tau + 1) \times \beta X) \setminus (\{\tau\} \times (\beta X \setminus X))$$

is called the *Noble plank*. By [7], $N_\tau X$ is 2-starcompact, and furthermore, $N_\tau X$ is $1\frac{1}{2}$-starcompact if $\tau > \mathrm{cf}(\tau) > \ell(X) > \omega$.

Example 4.4. There exists a $1\frac{1}{2}$-star-Lindelöf completely regular space that is neither star-Lindelöf nor strictly 2-star-Lindelöf.

Proof. Let D be a discrete space of size ω_1. Then the Noble plank $N_{\omega_2} D$ is $1\frac{1}{2}$-star-Lindelöf but not star-Lindelöf ([7]). We show that the Noble plank $N_{\omega_2} D$ is not strictly 2-star-Lindelöf.

Define $\mathcal{U} = \{[0, \alpha) \times \beta D \mid \alpha < \omega_2\} \cup \{(\omega_2 + 1) \times \{d\} \mid d \in D\}$ and $A = \{\omega_2\} \times D$. Then $\mathcal{U}$ is an open cover of $N_{\omega_2} D$ and we have $\mathrm{St}^2(A, \mathcal{U}) = N_{\omega_2} D$. However, no countable subset B of A satisfies $\mathrm{St}^2(B, \mathcal{U}) = N_{\omega_2} D$. Hence $N_{\omega_2} D$ is not strictly 2-star-Lindelöf. $\square$

For later use, we also have the another example (see Remark 4.10).

Example 4.5. There exists a $1\frac{1}{2}$-star-Lindelöf completely regular space that is not strictly 2-star-Lindelöf.

Proof. Let $\Psi = \omega \cup \mathcal{A}$ be the Ψ-space constructed form a m.a.d.family $\mathcal{A} = \{a_\lambda \mid \lambda < 2^\omega\}$ of infinite subsets of ω. Let D be a discrete space of size 2^ω. Denote $D = \{y_\lambda \mid \lambda < 2^\omega\}$. Define $X = \Psi \times A(D)$, where $A(D)$ is the one-point compactification of D.

At first, it is easy to see that X is $1\frac{1}{2}$-star-Lindelöf since $\omega \times A(D)$ is Lindelöf and dense in X.

Next, we prove that X is not strictly 2-star-Lindelöf. Define an open cover of X by

$$\mathcal{U} = \big\{\Psi \times \{y_\lambda\} \mid \lambda < 2^\omega\big\} \cup \big\{\{n\} \times A(D) \mid n \in \omega\big\} \\ \cup \big\{(\{a_\lambda\} \cup a_\lambda) \times (A(D) \setminus \{y_\lambda\}) \mid \lambda < 2^\omega\big\}.$$

Put $A = \{\langle a_\lambda, y_\lambda \rangle \mid \lambda < 2^\omega\}$. We have that $\mathrm{St}^2(A, \mathcal{U}) = X$. Let B be an arbitrary countable subset of A. Take a $\lambda_0 < 2^\omega$ such that $\langle a_{\lambda_0}, y_{\lambda_0} \rangle \notin B$. Then we can show that $\langle a_{\lambda_0}, y_{\lambda_0} \rangle \notin \mathrm{St}^2(B, \mathcal{U})$. Hence X is not strictly 2-star-Lindelöf. $\square$

Let $\mathbb{R}^*$ be the real line $\mathbb{R}$ with the topology

$$\mathcal{T}_c = \{U \subset \mathbb{R} \mid \mathbb{R} \setminus U \text{ is a countable subset of } \mathbb{R}\}.$$

Then $\mathbb{R}^*$ is not strictly starcompact because it is not countably compact. And clearly $\mathbb{R}^*$ is strictly $1\frac{1}{2}$-starcompact. Note that $\mathbb{R}^*$ is a T_1-space which is not Hausdorff. We cannot construct a strictly $1\frac{1}{2}$-starcompact Hausdorff space which is not strictly starcompact yet.

It is known that the Tychonoff plank $T = (\omega_1+1)\times(\omega+1)\setminus\{\langle\omega_1,\omega\rangle\}$ is $1\frac{1}{2}$-starcompact ([1],[7]), whereas we show it is not even strictly 2-starcompact.

Example 4.6. The Tychonoff plank T is $1\frac{1}{2}$-starcompact but not strictly 2-starcompact.

Proof. Define $\mathcal{U} = \{[0,\alpha)\times(\omega+1) \mid \alpha < \omega_1\} \cup \{(\omega_1+1)\times\{n\} \mid n < \omega\}$. Then $\mathcal{U}$ is an open cover of T. For every $n < \omega$, we have $\mathrm{St}((\omega_1, n), \mathcal{U}) = (\omega_1+1)\times\{n\}$ since $(\omega_1+1)\times\{n\}$ is the only element of $\mathcal{U}$ containing $\langle\omega_1, n\rangle$. Then we have $\mathrm{St}^2(\langle\omega_1, n\rangle, \mathcal{U}) = (\omega_1\times(\omega+1)) \cup \{\langle\omega_1, n\rangle\}$. Hence, the subset $A = \{\omega_1\}\times\omega$ of T satisfies $\mathrm{St}(A,\mathcal{U}) = T$.

Take a finite subset F of A arbitrarily. Then $\mathrm{St}^2(F,\mathcal{U}) \neq T$, and hence T is not strictly 2-starcompact. $\square$

H.Ohta pointed out that the Tychonoff plank T is not strictly star-Lindelöf. He proved the fact by showing that the Tychonoff plank T contained the closed subspace $((\omega_1+1)\times\{0\}) \cup (\omega_1\times\{\omega\})$ which is not strictly star-Lindelöf. We apply the idea to the following stronger result.

Example 4.7. The Tychonoff plank T is not strictly 2-star-Lindelöf.

Proof. For each $\alpha < \omega_1$, set $U_\alpha = ([0,\alpha]\times[2,\omega]) \cup \{\langle\alpha+1, 1\rangle\}$ and $V_\alpha = \{\langle\alpha+1, 0\rangle, \langle\alpha+1, 1\rangle\}$. Let $A = (\omega_1+1)\times\{0\}$ and $W = (\omega_1+1)\times[1,\omega)$. Define $\mathcal{U} = \{U_\alpha \mid \alpha < \omega_1\} \cup \{V_\alpha \mid \alpha < \omega_1\} \cup \{A, W\}$. Then $\mathcal{U}$ is an open cover of T and we have that $\mathrm{St}^2(A,\mathcal{U}) = T$.

Let B be a countable subset of A. Set $\beta_0 = \sup\{\beta \mid \langle\beta, 0\rangle \in B\setminus\{\omega_1\}\}$. Then we have $\mathrm{St}^2(\langle\beta_0+1, \omega\rangle,\mathcal{U}) = W \cup \left(\bigcup\{V_\gamma \mid \beta_0+1 \leq \gamma < \omega_1\}\right) \cup \left(\bigcup\{U_\alpha \mid \alpha < \omega_1\}\right)$ because $\mathrm{St}(\langle\beta_0+1, \omega\rangle,\mathcal{U}) = \bigcup\{U_\gamma \mid \beta_0+1 \leq \gamma < \omega_1\}$. Hence $\mathrm{St}^2(\langle\beta_0+1, \omega\rangle,\mathcal{U}) \cap B = \emptyset$. Therefore T is not strictly 2-star-Lindelöf. $\square$

Moreover, under $2^{\omega_1} = 2^{\omega}$, the Scott-type fat Ψ-space is a $2\frac{1}{2}$-star-Lindelöf completely regular space which is not 2-star-Lindelöf ([7]). Then this space is strictly $2\frac{1}{2}$-star-Lindelöf but not strictly 2-star-Lindelöf. On the other hand, under CH, the Scott-type fat Ψ-space is a $2\frac{1}{2}$-starcompact completely regular space which is not 2-starcompact ([1], [10]). Then this space is a strictly $2\frac{1}{2}$-starcompact space that is not strictly 2-starcompact.

Here, we give the following diagrams which illustrate relationships among star covering properties discussed above.

In the diagrams, the symbols $a \to b$ and $a \nrightarrow b$ mean that a implies b, and a does not necessarily imply b, respectively. We list corresponding counterexamples by the side of the symbols $a \nrightarrow b$.

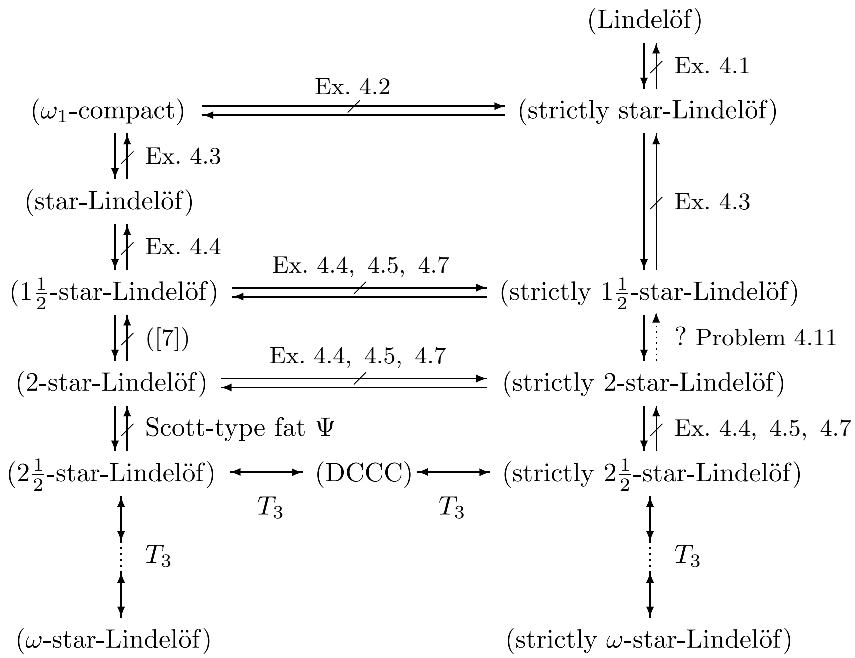

Diagram 1

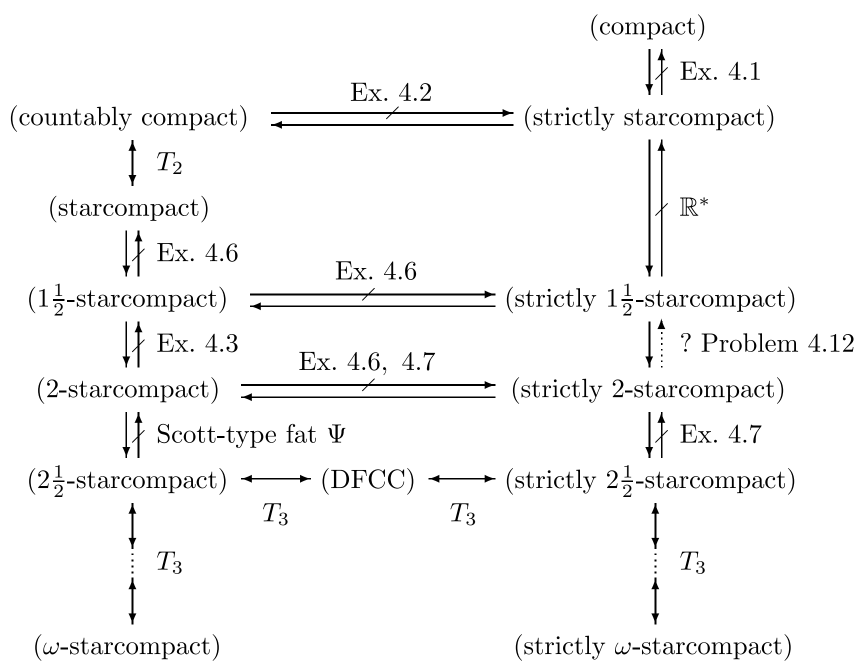

Diagram 2

At the last of this section, we give remarks on examples which are given above.

Remark 4.8. As to subspaces, closed subspaces of strictly k-star-Lindelöf spaces are not necessarily strictly k-star-Lindelöf for every $k \in \widetilde{\mathbb{N}}$ with $k \geq 1\frac{1}{2}$. For, the Ψ-space, constructed from a m.a.d.family $\mathcal{A}$ of uncountably many infinite subsets of ω, is k-star-Lindelöf for every $k \geq 1\frac{1}{2}$ by Example 4.3. However, the subspace $\mathcal{A}$ does not satisfy the DCCC since $\mathcal{A}$ is uncountable discrete and closed in Ψ. Therefore $\mathcal{A}$ is not strictly k-star-Lindelöf for every $k \in \widetilde{\mathbb{N}}$ with $k \geq 1\frac{1}{2}$.

Remark 4.9. The idea in Ohta's proof stated above also suggests that the topological sum $\omega_1 \oplus (\omega_1 + 1)$ is not strictly $1\frac{1}{2}$-star-Lindelöf. Hence a topological sum of a strictly star-Lindelöf space and a compact space is not even strictly $1\frac{1}{2}$-star-Lindelöf. Furthermore, the proof of Example 4.7 shows that $\omega_1 \oplus (\omega_1 + 1) \oplus (\omega_1 + 1)$ is not strictly 2-star-Lindelöf. Therefore a topological sum of a strictly star-Lindelöf space and a compact space is not even strictly 2-star-Lindelöf.

Remark 4.10. Concerning product spaces, Example 4.2 shows that a product of a strictly star-Lindelöf (respectively, strictly starcompact) space and a compact space need not be strictly star-Lindelöf (respectively, strictly starcompact). Moreover, Example 4.5 shows that a product of a strictly $1\frac{1}{2}$-star-Lindelöf (respectively, strictly 2-star-Lindelöf) space and a compact space need not be strictly $1\frac{1}{2}$-star-Lindelöf (respectively, strictly 2-star-Lindelöf).

Furthermore, by a similar argument in the proof of Example 4.7, we can show that neither $\omega_1 \times (\omega + 1)$ nor $\omega_1 \times (\omega_1 + 1)$ are strictly 2-star-Lindelöf strictly $1\frac{1}{2}$-star-Lindelöf. Therefore, a product of a strictly star-Lindelöf space with a separable metric space is not even strictly 2-star-Lindelöf.

The following problems are not solved yet.

Problem 4.11. Does there exist a strictly 2-star-Lindelöf space which is not strictly $1\frac{1}{2}$-star-Lindelöf?

Problem 4.12. Does there exist a strictly 2-starcompact space which is not strictly $1\frac{1}{2}$-starcompact?

5. Concluding remarks. As we mentioned above, strict star-Lindelöfness is not necessarily preserved by taking topological sums. Accordingly, we introduce the following notion of spaces with possible properties of strictly star-Lindelöf spaces so as to be preserved by taking topological sums.

Definition 5.1. A space X satisfies the condition $(*)$ if for each open cover $\mathcal{U}$ of X, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that every subset A of X satisfying $\mathrm{St}(A, \mathcal{V}) = X$ contains a countable subset B of A satisfying $\mathrm{St}(B, \mathcal{V}) = X$.

Then we easily have that every strictly star-Lindelöf space satisfies the condition $(*)$, and every space satisfying the condition $(*)$ is ω_1-compact.

We can show that the condition $(*)$ is preserved by taking a topological sum of countably many spaces satisfying the condition $(*)$.

Proposition 5.2. *If $\{X_n \,|\, n \in \mathbb{N}\}$ is a countable family of spaces satisfying the condition $(*)$, then $\bigoplus_{n\in\mathbb{N}} X_n$ also satisfies the condition $(*)$.*

We conclude this paper with the following example.

Recall that the Tychonoff plank T is not even strictly 2-star-Lindelöf (see Example 4.7).

Example 5.3. The Tychonoff plank T satisfies the condition $(*)$.

Proof. Let $\mathcal{U}$ be an open cover of T. For every $\langle \alpha, n\rangle \in T \setminus ((\{\omega_1\} \times \omega) \cup (\omega_1 \times \{\omega\}))$, take a $U_{\alpha,n} \in \mathcal{U}$ so that $\langle \alpha, n\rangle \in U_{\alpha,n}$. Then we can choose a $\beta_\alpha < \alpha$ so that $\langle \alpha, n\rangle \in (\beta_\alpha, \alpha] \times \{n\} \subset U_{\alpha,n}$. Put $V_{\alpha,n} = (\beta_\alpha, \alpha] \times \{n\}$.

For every $n < \omega$, pick a $U_n \in \mathcal{U}$ so that $\langle \omega_1, n\rangle \in U_n$. We can take an $\alpha_n < \omega_1$ such that $\langle \omega_1, n\rangle \in (\alpha_n, \omega_1] \times \{n\} \subset U_n$. Put $V_n = (\alpha_n, \omega_1] \times \{n\}$ for each $n < \omega$. Now, define $\xi = \sup\{\alpha_n \,|\, n < \omega\}$. Then $\xi < \omega_1$.

For every $\alpha < \omega_1$, pick a $U_\alpha \in \mathcal{U}$ so that $\langle \alpha, \omega\rangle \in U_\alpha$. Then there exist a $\beta_\alpha < \alpha$ and an $n_\alpha < \omega$ such that $\langle \alpha, \omega\rangle \in (\beta_\alpha, \alpha] \times (n_\alpha, \omega] \subset U_\alpha$. For each $\alpha < \omega_1$, set $V_\alpha = (\beta_\alpha, \alpha] \times (n_\alpha, \omega]$. By the pressing-down lemma ([6]), there exist an $\alpha_0 < \omega_1$, an $m_0 < \omega$ and a cofinal subset C of ω_1 such that $\beta_\alpha = \alpha_0$ and $n_\alpha = m_0$ for every $\alpha \in C$. Define

$$\mathcal{V} = \big\{V_{\alpha,n} \,|\, \alpha < \omega_1,\ n < \omega\big\} \cup \big\{V_n \,|\, n < \omega\big\} \cup \big\{V_\alpha \,|\, \alpha \in [0, \alpha_0] \cup (C \cap (\alpha_0, \omega_1))\big\}.$$

Then $\mathcal{V}$ is an open refinement of $\mathcal{U}$. Let A be a subset of T such that $\mathrm{St}(A, \mathcal{V}) = T$. Note that $V_\gamma = (\alpha_0, \gamma] \times (m_0, \omega]$ for every $\gamma \in C \cap (\alpha_0, \omega_1)$.

For every $n < \omega$, we have that V_n is the only member of $\mathcal{V}$ containing $\langle \omega_1, n\rangle$. Hence for every $n < \omega$ there exists an $a_n \in A$ such that $a_n \in V_n$. Set $A_1 = \{a_n \,|\, n < \omega\}$. We have that $\mathrm{St}(A_1, \mathcal{V}) \supset (\xi, \omega_1] \times \omega$ for some $\xi < \omega_1$.

Here, for the $\alpha_0 + 1$ there exist a $\gamma_0 \in C \cap (\alpha_0, \omega_1)$ and an $x_0 \in A$ such that $\langle \alpha_0 + 1, \omega\rangle \in V_{\gamma_0}$ and $x_0 \in V_{\gamma_0}$. Then we have $\mathrm{St}(x_0, \mathcal{V}) \supset (\alpha_0, \omega_1) \times (m_0, \omega]$.

Let $\eta = \max\{\alpha_0, \xi\}$. Then $\eta < \omega_1$. Because $[0, \eta + 1] \times (\omega + 1)$ is a compact subset of T, there exists a finite subset A_2 of A such that $\mathrm{St}(A_2, \mathcal{V}) \supset [0, \eta + 1] \times (\omega + 1)$.

Therefore we have $\mathrm{St}(A_1 \cup A_2 \cup \{x_0\}, \mathcal{V}) = T$, and hence T satisfies the condition $(*)$. □

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